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Research Article

Reduction Formulae of a Binomial Expansion via a 2F1 Hypergeometric Function and its Derivative

To my Prof. Kevin Clement, a phenomenal teacher, scholar, researcher, friend, the best Master Thesis academic advisor who I have ever known in my life.[i]

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Abstract: The reduction formulae of a binomial expansion of that leads to a weighted sum of the hypergeometric function and its derivative is provided. We have employed three well-known definitions of the ordinary or Gaussian hypergeometric functions the Gauss power series expansion, the Euler integral, and the Euler transformation. We started out to prove two identities; however, by making use of the Euler transformation we produce two more identities. Furthermore, we were able to generalize the identity to include more general cases which have led to a total of six identities. A couple of identities are proved by exploiting the total mathematical induction principle. Qualitative and quantitative assessment of a few cases is provided in the journal paper.
Index Terms— ordinary hypergeometric functions, hypergeometric function derivative, binomial expansion, Gauss power series, Euler integral, Euler transformation.

1 Introduction

The ordinary or Gaussian 2F1 hypergeometric function (OHF or GHF) is a special function represented by the hypergeometric series; it contains many other special functions as specific or limiting cases.

The first serious work on the GHF and its partial derivatives I performed in Progri (2016, [2]). In this paper I mentioned that Euler was the first to have studied the integral representation; I also mentioned the Gauss Power Series and the Taylor Series Expansion; however, I failed to provide more details about many early contributions. For that reason I will perform a brief review of all the fundamental concepts of algebra that are the main contributions and the main contributors.

1.1 The Genesis of Algebraic Power

The concept of algebraic power, as in raising a quantity to a power, was developed and refined by several mathematicians throughout history. While Al-Khwarizmi (c. 780 – c. 850) is often considered the "father of algebra" for his contributions to solving equations, it was Al-Karaji (c. 953 – c. 1029) who further developed algebra by incorporating integral powers and roots into his work. Al-Karaji's treatise, Al-Fakhri, detailed rules for manipulating polynomials and introduced the concept of algebraic calculus, paving the way for the modern algebraic notation we use today.

Al-Khwarizmi: His work focused on solving linear and quadratic equations, providing the foundation for algebra. While he did not explicitly use the term *power*, his methods laid the groundwork for later developments.

Al-Karaji: He extended Al-Khwarizmi's work by incorporating powers and roots into his algebraic system, moving beyond geometric interpretations of algebra to a more arithmetical approach. His work laid the foundation for understanding polynomials and their manipulation.

Other contributors: Later mathematicians, such as Diophantus (c. 200 – c. 284 AD) and Pierre de Fermat (1601 – 1665), further developed the theory of algebraic expressions and equations. Francois Viète (1540 – 1603) is credited with introducing a more systematic notation for algebraic expressions.

1.2 The Genesis of the Factorial Formula

The *factorial formula* $(n!)$ was not discovered by a single individual. The concept of counting permutations using factorials dates back to the 1200s, and the notation $n!$ was introduced by Christian Kramp in 1808. Daniel Bernoulli and Leonhard Euler later extended the factorial function to complex numbers.

Early Use: Factorials were used to count permutations as early as the 1200s.

Christian Kramp: In 1808, Kramp introduced the notation $n!$ to represent the product of all positive integers less than or equal to n.

Daniel Bernoulli and *Leonhard Euler*: They developed a continuous extension of the factorial function, known as the gamma function, which applies to complex numbers. This allowed factorials to be used in a wider range of mathematical contexts.

Other Notable Contributions: Stirling's approximation (late 18th - early 19th century) provides an accurate approximation of factorials for large numbers. Legendre's formula describes the exponents of prime numbers in the prime factorization of factorials.

1.3 The Discovery of the Binomial Theorem

The *Binomial Theorem* was discovered by Sir Isaac Newton, a renowned English mathematician and physicist. He developed the theorem around 1665, allowing for the expansion of binomial expressions to any real-number exponent.

Newton's *Binomial Theorem* is a fundamental concept in algebra, providing a formula for expanding expressions of the form $(a+b)^n$, where n can be any real number.

While Newton is credited with the general form of the theorem, early understandings of the binomial expansion existed before him, particularly in combinations.

He showed how to expand expressions of the form $(1+x)^r$ for any real exponent r, replacing the finite sum of the traditional binomial theorem with an infinite series when r is not a non-negative integer.

1.4 The Discovery of the Hypergeometric Function

The term *hypergeometric series* appeared in the mid-17th century, with John Wallis being one of the first to discuss them in his 1655 book, *Arithmetica Infinitorum*.

Euler developed the hypergeometric function integral formula in 1748, as part of his book *Introductio in Analysin Infinitorum*. This formula, along with his transformations and those of Pfaff, forms a cornerstone of the theory of hypergeometric functions.

Euler first introduced the hypergeometric differential equation in 1769, as found in his Opera Omnia Ser. 1, 11–13. While the hypergeometric function itself was not explicitly defined until later by Gauss, Euler's work laid the foundation for its understanding and study. The first systematic treatment of the hypergeometric series was given by Gauss in 1813.

Carl Friedrich Gauss made significant contributions to the understanding of the hypergeometric function and its power series expansion in 1812. While earlier work had been done by others like Euler and Wallis, Gauss is credited with the first systematic study and publication of the series.

Gauss provided the first systematic and comprehensive treatment of the hypergeometric series in his work "Disquisitiones generales circa seriem infinitam."

1.5 The Discovery of the Pochhammer Symbol

The Pochhammer symbol, also known as the shifted factorial, was first introduced in the context of the hypergeometric function and associated series by Carl Friedrich Gauss in 1813. While the symbol itself was later named by P.E. Appell in 1880, referencing the research of L.A. Pochhammer, the fundamental concept was integral to Gauss's work on hypergeometric functions.

1.6 The Discovery of the Euler Transformation Formula

Leonhard Euler's work with integral transforms, including the Euler transformation, was primarily developed between 1763 and 1769. He introduced and used integral transforms in the context of second-order differential equations. His formal presentation and publication of these ideas occurred in a fragment published in 1763 and in a chapter of his book Institutiones Calculi Integralis (1769).

1.7 The Discovery of the Pfaff Transformation Formula

Pfaff, in 1797, was the first to discover a transformation formula for the hypergeometric function. This transformation was initially found by Pfaff and was later rediscovered by Saalschütz in 1890.

1.8 The Computation of the GHF Derivative

The derivative of the GHF, a crucial result in its study, was not explicitly computed by a single individual. However, the development of techniques for finding derivatives and the subsequent application to the hypergeometric function were contributed to by several mathematicians.

In addition to the works of John Wallis, Leonhard Euler, Carl Friedrich Gauss, *Bernhard Riemann*, Ernst Kummer, and Hermann Schwarz.

Bernhard Riemann: Riemann (1857) characterized the hypergeometric function by the differential equation it satisfies.

Ernst Kummer: Kummer (1836) made important contributions to the understanding of hypergeometric functions.

Hermann Schwarz: Schwarz found cases where the solutions are algebraic functions. Hermann Schwarz found the fifteen cases where solutions to hypergeometric functions are algebraic in 1873. He published his findings in his work titled "Schwarz's list." This list identifies the specific parameters that lead to hypergeometric equations having a finite monodromy group, and consequently, two independent solutions that can be expressed as algebraic functions.

1.9 The Computation of the Beta Function

The beta function was first studied by Leonhard Euler and Adrien-Marie Legendre. The name *beta function* was given to it by Jacques Binet.

Therefore, the hypergeometric function expansion was not known when Newton invented his binomial theorem in 1665, as the full systematic treatment by Gauss occurred significantly later. Wallis's earlier work in 1655 laid some groundwork but predates Newton's work on the generalized binomial theorem.

The hypergeometric function expansion is considered a generalization of Newton's binomial theorem, especially when the exponent is a non-negative integer.

Even with all of the discoveries that occurred I believe that more was needed to come up with the reduction formulae of a binomial expansion of the GHF and its derivative.

In Progri (2018, [3]) I raised the possibility of computing the Kampé de Fériet or the (or double hypergeometric series) in the same manner as the MATLAB built-in-function (BIF) *hypergeom*(a,b,z). This was a truly turning point in the building of Giftet Indoor Geolocation Systems computational library.

Both the Kampé de Fériet MATLAB function and the Progri implementation of the *hypergeom*(a,b,z), *phypergeom*(a,b,z,op), and the implementation of the Kampé de Fériet in *kamdefer*(a,b,z,op) known as became a reality a year later in Progri (2019, [4]).

In Progri (2020, [5]) both *kamdefer*(a,b,z,op) and *phypergeom*(a,b,z,op) were used in the computation of a 2F2 hypergeometric function.

A rather fundamental discussion and further improvements of the *phypergeom*(a,b,z,op) in relation to the singularities of the confluent hypergeometric functions were performed in Progri (2021, [6]).

In Progri (2022, [7], [8]) further landmark improvements of both Giftet *kamdefer*(a,b,z,op) and *hypergeom*(a,b,z) were performed.

A rather remarkable recursive implementation of a fundamental 2F2 hypergeometric function was accomplished in Progri (2023, [9]).

Hence, by the time Progri (2024, [10]) was written, Giftet possesses the fastest and the most accurate MATLAB library in the computation of the hypergeometric functions.

Is it possible that the Reduction Formulae of a Binomial Expansion via a 2F1 Hypergeometric Function and its Derivative would have been known before the publication of this journal paper?

I do not know the answer to that question. However, I do know that should this reduction formula that was first discovered in Progri (2024, [10]), that would have been published in several textbooks like Bateman (1953, [11]), Arfken, Weber (1995, [12]), Gradshteyn, Ryzhik (2007, [13]). Any of the publications that might exist on this subject do not take anything away from the originality, novelty, innovation, and creativity of this journal paper.

This journal paper is organized as follows: in Sect. 2 GHF derivative fundamentals are discussed. In Sect. 3, first identity of the Binomial expansion via GHF and its derivative is presented. Second Identity of the Binomial Expansion via GHF and its Derivative in detailed in Sect. 4. In Sect. 5 several numerical examples are considered. Conclusion is provided in Sect. 6 along with Acknowledgments in Sect. 7, and a list of references in Sect. 8. In Appen. A: The GHF symmetry property is discussed in detail in Sect. 9. Finally, in Appen. B: The Beta function recursive relation is discussed in Sect. 10.

2 GHF Derivative Fundamentals

In Progri (2016, [2] (1)) a detailed discussion of the notation of the ordinary or Gauss hypergeometric function (OHF or GHF) is provided.

First, the GHF derivative computation via the Euler integral is given in Sect. 2.1. Second, the GHF derivative computation via the Euler transformation is deliberated in Sect. 2.2. Third, we provide the GHF derivative computation via the Gauss Power Series (GPS) in Sect. 2.4.

2.1 GHF Derivative via Euler Integral

In Progri (2016, [2] (19)) the GHF via Euler Integral is provided.

$$F\begin{bmatrix}a,b;\\c;\end{bmatrix}z = \frac{\int_0^1 \frac{x^{b-1}(1-x)^{c-b-1}}{(1-zx)^a}dx}{\mathrm{B}(b,c-b)} \tag{1}$$

In Appen. A Sect. 9, the symmetry property is discussed.

Taking the derivative of both sides of (1) by applying Leibnitz's Rule we obtain the following:

$$\begin{aligned}
\frac{\partial}{\partial z}F\begin{bmatrix}a,b;\\c;\end{bmatrix}z &= \frac{\partial}{\partial z}\frac{\int_0^1 \frac{x^{b-1}(1-x)^{c-b-1}}{(1-zx)^a}dx}{\mathrm{B}(b,c-b)}\\
&= \frac{\int_0^1 \frac{\partial}{\partial z}\frac{x^{b-1}(1-x)^{c-b-1}}{(1-zx)^a}dx}{\mathrm{B}(b,c-b)}\\
&= \frac{\int_0^1 x^{b-1}(1-x)^{c-b-1}\frac{\partial}{\partial z}\frac{1}{(1-zx)^a}dx}{\mathrm{B}(b,c-b)}\\
&= \frac{\int_0^1 x^{b-1}(1-x)^{c-b-1}\frac{(ax)}{(1-zx)^{a+1}}dx}{\mathrm{B}(b,c-b)}\\
&= \frac{ab}{c}\frac{\int_0^1 \frac{x^{b+1-1}(1-x)^{c+1-b-1-1}}{(1-zx)^{a+1}}dx}{\mathrm{B}(b,c-b)}\\
&= \frac{ab}{c}\frac{\int_0^1 \frac{x^{b+1-1}(1-x)^{c+1-b-1-1}}{(1-zx)^{a+1}}dx}{\frac{\Gamma(b+1)\Gamma(c+1-b-1)}{\Gamma(c+1)}}\\
&= \frac{ab}{c}\frac{\int_0^1 \frac{x^{b+1-1}(1-x)^{c+1-b-1-1}}{(1-zx)^{a+1}}dx}{\mathrm{B}(b+1,c+1-b-1)}\\
&= \frac{ab}{c}F\begin{bmatrix}a+1,b+1;\\c+1;\end{bmatrix}z
\end{aligned} \tag{2}$$

The derivative of the GHF (2) is consistent with the results provided in the literature [16].

2.2 Derivative of the GHF Euler's Transformation

The definition of the GHF via Euler's Transformation, Gradshteyn, Ryzhik (2007, [13] 9.131 1. pg. 1008), as follows:

$$F\begin{bmatrix}a,b;\\c;\end{bmatrix}z = (1-z)^{c-a-b}F\begin{bmatrix}c-a,c-b;\\c;\end{bmatrix}z \tag{3}$$

Is taking the derivative of the Euler Transformation of the GHF the same as applying the Euler Transformation of the derivative of the GHF?

No, they are not generally the same. Taking the derivative of the Euler's transformation of the GHF and applying the Euler's transformation to the derivative of the GHF result in different expressions.

Euler's Transformation: This transformation relates two hypergeometric functions with different arguments. For instance, Euler's transformation for the GHF transforms the function at argument z to the function at argument $1-z$.

Differentiating a Transformation: If you differentiate the Euler's transformed function, you are applying the chain rule and potentially also differentiate the function itself, which will be related to the original hypergeometric function but also involve derivatives of the transformed argument.

In summary, differentiating a transformed function and applying a transformation to a differentiated function are different operations that result in distinct hypergeometric functions.

In general, taking the derivative of an integral transform is not the same as applying the integral transform to the derivative of the original function. The relationship between these two operations depends on the specific integral transform and the conditions under which the differentiation and integration can be interchanged.

To differentiate the Euler integral transformation with respect to a parameter (say z), you would use the Leibniz integral rule.

This involves differentiating the integrand with respect to z inside the integral.

On the other hand, to apply the Euler integral transformation to the derivative of the hypergeometric function, you would first find the derivative of the hypergeometric function with respect to the desired variable (e.g., z) and then apply the Euler integral formula to that derivative.

Commutation of Differentiation and Integration: Whether these two operations yield the same result depends on whether the differentiation and integration operations commute. For the Euler integral transformation, this means you need to examine the conditions under which you can interchange the order of differentiation with respect to z and integration with respect to t.

In summary: The following can be concluded

1. *Taking the derivative of the Euler Integral Transformation*: You differentiate the integral with respect to z using the Leibniz rule, which involves differentiating the integrand.
2. *Applying the Euler Integral Transformation to the derivative*: You first differentiate the hypergeometric function with respect to z and then apply the integral transformation formula to the resulting expression.

These two procedures are *not generally equivalent*. The conditions under which the differentiation and integration can be interchanged need to be considered. The specific forms of the derivative of the hypergeometric function and the Euler integral transformation highlight that the operations are generally distinct.

Because this is such an involved deliberation, it will be the subject to a separate journal paper. However, in the context of this journal paper, we are first taking the derivative of the GHF and then applying the Euler's Integral Transformation to the differentiated function.

2.3 Euler's Transformation of the GHF Derivative

Applying Euler's Transformation to the Derivative: If you first take the derivative of the GHF and then apply Euler's transformation, you are still applying the transformation, but to a function that is already differentiated. This will result in a different GHF with a different parameter set and argument.

Applying the Euler's Integral Transformation (3) to the derivative of the GHF (2) produces:

$$\frac{\partial}{\partial z} F\left[\begin{matrix} a, b; \\ c; \end{matrix} z\right] = \frac{ab}{c} F\left[\begin{matrix} a+1, b+1; \\ c+1; \end{matrix} z\right]$$

$$= \frac{ab}{c}(1-z)^{c-a-b-1} F\left[\begin{matrix} c-a, c-b; \\ c+1; \end{matrix} z\right] \tag{4}$$

By observing (4) we see that the application of the Euler's Integral Transformation to the derivative of the GHF leads to a product of a Binomial expression and another GHF. Equations (3) and (4) are employed later on as part of creating new identities of the reduction formulae of the binomial expression as a sum of a GHF and its derivative.

2.4 GHF Derivative via Gauss Power Series

From the well-known definition of the F GHF via the *Gauss Power Series* (GPS) Progri (2016, [2]) we have

$$F(a, b; c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

$$= 1 + \sum_{k+1=1}^{\infty} \frac{(a)_{k+1} (b)_{k+1}}{(c)_{k+1}} \frac{z^{k+1}}{(k+1)!}$$

$$= 1 + \frac{ab}{c} \sum_{k=0}^{\infty} \frac{(a+1)_k (b+1)_k}{(c+1)_k} \frac{z^{k+1}}{(k+1)!} \tag{5}$$

Well known reduction formula for the Pochhammer symbol (or rising factorial [17]) that was employed in (5) is from the definition of the Gamma function [18] as follows Progri (2023, [9]),

$$(q)_{n+1} = q \frac{\Gamma(q+1+n)}{q\Gamma(q)} = q \frac{\Gamma(q+1+n)}{\Gamma(q+1)} \ ; \ q = \{a, b\}$$

$$= q(q+1)_n \tag{6}$$

Taking the derivative of (5) we have the following:

$$\frac{\partial}{\partial z} F(a, b; c|z) = \frac{\partial}{\partial z} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

$$= \frac{\partial}{\partial z}\left[1 + \sum_{k+1=1}^{\infty} \frac{(a)_{k+1} (b)_{k+1}}{(c)_{k+1}} \frac{z^{k+1}}{(k+1)!}\right]$$

$$= \frac{ab}{c} \sum_{k=0}^{\infty} \frac{(a+1)_k (b+1)_k}{(c+1)_k} \frac{\partial}{\partial z} \frac{z^{k+1}}{(k+1)!}$$

$$= \frac{ab}{c} \sum_{k=0}^{\infty} \frac{(a+1)_k (b+1)_k}{(c+1)_k} \frac{z^k}{k!}$$

$$= \frac{ab}{c} F(a+1, b+1; c+1|z) \tag{7}$$

The derivative of the GHF (7) is equivalent with (2). This concludes the discussion on the GHF derivative via GPS expansion.

3 First Identity of Binomial Expansion via GHF and its Derivative

In order to prove the first identity we consider from Gradshteyn, Ryzhik (2007, [13] 1.110) the definition of binomial expansion as follows:

$$(1-z)^{d-1} = \sum_{n=0}^{\infty}\binom{d-1}{n}(-z)^n$$
$$= \sum_{n=0}^{\infty}\frac{(d-1)(d-2)\cdots(d-n)}{n!}(-z)^n$$
$$= \sum_{n=0}^{\infty}\frac{(d-1)^{\underline{n}}}{n!}(-z)^n$$
$$= \left|1-(d-1)z+\frac{(d-1)(d-2)}{2}z^2-\frac{(d-1)(d-2)(d-3)}{6}z^3+\cdots\right.$$
$$= \left|1+(1-d)z+\frac{(1-d)(2-d)}{2}z^2+\frac{(1-d)(2-d)(3-d)}{6}z^3+\cdots\right.$$
$$= (1+z)^{1-d} \quad (8)$$

In Progri (2024, [10] (51)) a binomial expansion of the GHF and its derivative is provided which is the first identity that will prove in this journal letter as follows:

$$F_1(d,z)+\frac{(1-d)z}{d+1}F_2(d,z) = (1-z)^{d-1} \quad (9)$$

where

$$F_1(d,z) \equiv {}_2F_1\left[\begin{matrix}d,1-d;\\ d+1;\end{matrix} z\right] \quad (10)$$

$$F_2(d,z) \equiv {}_2F_1\left[\begin{matrix}d+1,2-d;\\ d+2;\end{matrix} z\right] \quad (11)$$

The question that was posed in Progri (2024, [10] (51)) was as follows: if we started from the left side of (9) can we arrive at the right side of (9)?

First, the first identity proof via the Euler Integral is given in Sect. 3.1. Second, the first identity proof via the Euler Transformation is deliberated in Sect. 3.2. Third, we provide the first identity proof via the Gauss Power Series in Sect. 3.3.

3.1 First Identity Proof via Euler Integral

In this Sect. we will prove (9) via the definition of the GHF via Euler's Integral (see Progri (2016, [2])) as follows:

$$F\left[\begin{matrix}d,1-d;\\ d+1;\end{matrix} z\right] = \frac{\int_0^1 \frac{x^{1-d-1}(1-x)^{d+1-1+d-1}}{(1-zx)^d}dx}{\mathrm{B}(1-d,d+1-1+d)}$$
$$= \frac{\int_0^1 \frac{x^{-d}(1-x)^{2d-1}}{(1-zx)^d}dx}{\mathrm{B}(1-d,2d)} \quad (12)$$

$$F\left[\begin{matrix}d+1,2-d;\\ d+2;\end{matrix} z\right] = \frac{\int_0^1 \frac{x^{2-d-1}(1-x)^{d+2-2+d-1}}{(1-zx)^{d+1}}dx}{\mathrm{B}(2-d,d+2-2+d)}$$
$$= \frac{\int_0^1 \frac{x^{1-d}(1-x)^{2d-1}}{(1-zx)^{d+1}}dx}{\mathrm{B}(2-d,2d)} \quad (13)$$

$$\frac{(1-d)z}{d+1}F\left[\begin{matrix}d+1,2-d;\\ d+2;\end{matrix} z\right] = \frac{(1-d)z}{d+1}\frac{\int_0^1 \frac{x^{1-d}(1-x)^{2d-1}}{(1-zx)^{d+1}}dx}{\frac{\Gamma(2-d)\Gamma(2d)}{\Gamma(2+d)}}$$
$$= \frac{\int_0^1 \frac{x^{1-d}(1-x)^{2d-1}z}{(1-zx)^{d+1}}dx}{\frac{\Gamma(1-d)\Gamma(2d)}{\Gamma(1+d)}}$$
$$= \frac{\int_0^1 \frac{x^{1-d}(1-x)^{2d-1}z}{(1-zx)^{d+1}}dx}{\mathrm{B}(1-d,2d)} \quad (14)$$

Next, add (12) and (14) yields,

$$F_1(d,z)+\frac{(1-d)zF_2(d,z)}{d+1} = \frac{\int_0^1 \frac{x^{-d}(1-x)^{2d-1}}{(1-zx)^d}dx}{\mathrm{B}(1-d,2d)}+\frac{\int_0^1 \frac{x^{1-d}(1-x)^{2d-1}z}{(1-zx)^{d+1}}dx}{\mathrm{B}(1-d,2d)}$$
$$= \frac{\int_0^1 \frac{x^{-d}(1-x)^{2d-1}}{(1-zx)^d}\left[1+\frac{xz}{(1-zx)}\right]dx}{\mathrm{B}(1-d,2d)}$$
$$= \frac{\int_0^1 \frac{x^{-d}(1-x)^{2d-1}}{(1-zx)^{d+1}}dx}{\mathrm{B}(1-d,2d)} \quad (15)$$

Next, we perform the binomial expansion of the following term:

$$(1-zx)^{-d-1} = \sum_{n=0}^{\infty}\binom{-d-1}{n}(-zx)^n$$
$$= \sum_{n=0}^{\infty}\frac{(-d-1)(-d-2)\cdots(-d-n)}{n!}(-zx)^n$$
$$= \sum_{n=0}^{\infty}\frac{(-d-1)^{\underline{n}}}{n!}(-zx)^n$$
$$= \left|1-(-d-1)zx+\frac{(-d-1)(-d-2)}{2}z^2x^2-\frac{(-d-1)(-d-2)(-d-3)}{6}z^3x^3+\cdots\right.$$
$$= \left|1+(d+1)zx+\frac{(d+1)(d+2)}{2}z^2x^2+\frac{(d+1)(d+2)(d+3)}{6}z^3x^3+\cdots\right.$$
$$= \sum_{n=0}^{\infty}\binom{d+1}{n}z^nx^n \quad (16)$$

Next, substitute (16) into (15) produces:

$$\int_0^1 \frac{x^{-d}(1-x)^{2d-1}}{(1-zx)^{d+1}}dx = \int_0^1 \frac{(1-x)^{2d-1}\sum_{n=0}^{\infty}\binom{d+1}{n}z^nx^n}{x^d}dx$$
$$= \sum_{n=0}^{\infty}\binom{d+1}{n}z^n\int_0^1 x^{n-d}(1-x)^{2d-1}dx$$
$$= \sum_{n=0}^{\infty}\binom{d+1}{n}z^n\mathrm{B}(n+1-d,2d) \quad (17)$$

Next, we make use of the Beta function recursive relation (96) in Appen. B, Sect. 10:

$$x = 1-d \quad (18)$$
$$y = 2d \quad (19)$$
$$\mathrm{B}(n+1-d,2d) = \mathrm{B}(1-d,2d)\frac{(1-d)_{(n)}}{(1+d)_{(n)}} \quad (20)$$

Next, substitute (20) into (17) produces:

$$\int_0^1 \frac{x^{-d}(1-x)^{2d-1}}{(1-zx)^{d+1}}dx = \sum_{n=0}^{\infty}\binom{d+1}{n} z^n \mathrm{B}(1-d,2d)\frac{(1-d)_{(n)}}{(1+d)_{(n)}}$$

$$= \mathrm{B}(1-d,2d)\sum_{n=0}^{\infty}\frac{(1+d)_{(n)}}{n!} z^n \frac{(1-d)_{(n)}}{(1+d)_{(n)}}$$

$$= \mathrm{B}(1-d,2d)\sum_{n=0}^{\infty}\frac{(1-d)_{(n)}}{n!} z^n \quad \text{(see (8))}$$

$$= \mathrm{B}(1-d,2d)(1-z)^{d-1} \tag{21}$$

Finaly, we substitute (21) into (15) yields the desired result (9).

This completes the discussion on the first identity proof via Euler integral.

3.2 *First Identity Proof via Euler Transformation*

In this Sect. we will prove (9) via the definition of the GHF via Euler Transformation, Gradshteyn, Ryzhik (2007, [13] 9.131 1. pg. 1008), as follows:

$$F\left[\begin{matrix} d, 1-d; \\ d+1; \end{matrix} z\right] = (1-z)^d F\left[\begin{matrix} 1,2d; \\ d+1; \end{matrix} z\right] \tag{22}$$

$$F\left[\begin{matrix} d+1, 2-d; \\ d+2; \end{matrix} z\right] = (1-z)^{d-1} F\left[\begin{matrix} 1,2d; \\ d+2; \end{matrix} z\right] \text{[ii]} \tag{23}$$

Next, add (22) and (23) produces:

$$F_1(d,z) + \frac{(1-d)z}{d+1}F_2(d,z) = \left| \begin{array}{l} (1-z)^d F\left[\begin{matrix} 1,2d; \\ d+1; \end{matrix} z\right] + \\ (1-z)^{d-1}\frac{(1-d)z}{(d+1)} F\left[\begin{matrix} 1,2d; \\ d+2; \end{matrix} z\right] \end{array} \right.$$

$$= \frac{\left| \begin{array}{l} (1-z)F\left[\begin{matrix} 1,2d; \\ d+1; \end{matrix} z\right] \\ + \frac{(1-d)z}{(d+1)}F\left[\begin{matrix} 1,2d; \\ d+2; \end{matrix} z\right] \end{array} \right.}{(1-z)^{1-d}} \tag{24}$$

To prove (9) is equivalent of proving the following:

$$(1-z)F\left[\begin{matrix} 1,2d; \\ d+1; \end{matrix} z\right] + \frac{(1-d)z}{(d+1)} F\left[\begin{matrix} 1,2d; \\ d+2; \end{matrix} z\right] = 1 \tag{25}$$

First, the proof of (25) can be given using the GPS expansion as follows:

$$F\left[\begin{matrix} 1,2d; \\ d+1; \end{matrix} z\right] = \sum_{n=0}^{\infty}\frac{(1)_n (2d)_n}{(d+1)_n}\frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(2d)_n}{(d+1)_n} z^n \tag{26}$$

$$F\left[\begin{matrix} 1,2d; \\ d+2; \end{matrix} z\right] = \sum_{n=0}^{\infty}\frac{(1)_n (2d)_n}{(d+2)_n}\frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(2d)_n}{(d+2)_n} z^n \tag{27}$$

Next, substitute (25) and (26) into the left side of (24) yields,

$$(1-z)\sum_{n=0}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n + \frac{(1-d)z}{(d+1)}\sum_{n=0}^{\infty}\frac{(2d)_n}{(d+2)_n}z^n =$$

$$= \left| \begin{array}{l} \sum_{n=0}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n - z\sum_{n=0}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n \\ + \frac{(1-d)z}{(d+1)}\sum_{n=0}^{\infty}\frac{(2d)_n}{(d+2)_n}z^n \end{array} \right.$$

$$= \sum_{n=0}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n - z\sum_{n=0}^{\infty}\frac{(2d)_n}{(d+1)_n}\left[\frac{2d+n}{d+1+n}\right]z^n$$

$$= \sum_{n=0}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n - z\sum_{n=0}^{\infty}\frac{(2d+1)_n}{(d+2)_n}z^n$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n - z\sum_{n=0}^{\infty}\frac{(2d+1)_n}{(d+2)_n}z^n$$

$$= 1 + \sum_{n=0}^{\infty}\frac{(2d)_{n+1}}{(d+1)_{n+1}}z^{n+1} - z\sum_{n=0}^{\infty}\frac{(2d+1)_n}{(d+2)_n}z^n$$

$$= 1 + z\sum_{n=0}^{\infty}\frac{(2d+1)_n}{(d+2)_n}z^n - z\sum_{n=0}^{\infty}\frac{(2d+1)_n}{(d+2)_n}z^n$$

$$= 1 \tag{28}$$

We could have shown that

$$(1-z)\sum_{n=0}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n + \frac{(1-d)z}{(d+1)}\sum_{n=0}^{\infty}\frac{(2d)_n}{(d+2)_n}z^n =$$

$$= \sum_{n=0}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n - z\sum_{n=0}^{\infty}\frac{(2d+1)_n}{(d+2)_n}z^n$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n - \sum_{n=0}^{\infty}\frac{(2d+1)_n}{(d+2)_n}z^{n+1}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n - \sum_{n=0}^{\infty}\frac{(2d)_{n+1}}{(d+1)_{n+1}}z^{n+1}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n - \sum_{n=1}^{\infty}\frac{(2d)_n}{(d+1)_n}z^n$$

$$= 1 \tag{29}$$

Equations (28) and (29) prove that (25) is true via the GPS expansion.

Next, we will try to prove the same employing the Euler Integral formula as follows:

$$F\left[\begin{matrix} 1,2d; \\ d+1; \end{matrix} z\right] = \frac{\int_0^1 \frac{x^{2d-1}(1-x)^{d+1-2d-1}}{(1-zx)}dx}{\mathrm{B}(2d,d+1-2d)}$$

$$= \frac{\int_0^1 \frac{x^{2d-1}(1-x)^{-d}}{(1-zx)}dx}{\mathrm{B}(2d,1-d)}$$

$$= \frac{\int_0^1 x^{2d-1}(1-x)^{-d}\sum_{n=0}^{\infty}(-1)^n (zx)^n dx}{\mathrm{B}(2d,1-d)}$$

$$= \frac{\sum_{n=0}^{\infty}(-1)^n z^n \int_0^1 x^{2d+n-1}(1-x)^{-d}dx}{\mathrm{B}(2d,1-d)}$$

$$= \frac{\sum_{n=0}^{\infty}(-1)^n z^n \mathrm{B}(2d+n,1-d)}{\mathrm{B}(2d,1-d)}$$

$$= \frac{\sum_{n=0}^{\infty}(-1)^n z^n \mathrm{B}(2d,1-d)\frac{(2d)_{(n)}}{(1+d)_{(n)}}}{\mathrm{B}(2d,1-d)}$$

$$= \sum_{n=0}^{\infty}(-1)^n z^n \frac{(2d)_{(n)}}{(1+d)_{(n)}} \tag{30}$$

Next, we make use of the Beta function recursive relation (96) in Appen. B, Sect. 10:

$$x = 2d \tag{31}$$

$$y = 1-d \tag{32}$$

$$\mathrm{B}(2d+n,1-d)=\mathrm{B}(2d,1-d)\frac{(2d)_{(n)}}{(1+d)_{(n)}} \quad (33)$$

Second, we show that

$$F\begin{bmatrix}1,2d; \\ d+2; \end{bmatrix} z = \frac{\int_0^1 \frac{x^{2d-1}(1-x)^{d+2-2d-1}}{(1-zx)}dx}{\mathrm{B}(2d,d+2-2d)}$$

$$= \frac{\int_0^1 \frac{x^{2d-1}(1-x)^{1-d}}{(1-zx)}dx}{\mathrm{B}(2d,2-d)}$$

$$= \frac{\int_0^1 x^{2d-1}(1-x)^{1-d}\sum_{n=0}^{\infty}(-1)^n(zx)^n dx}{\mathrm{B}(2d,2-d)}$$

$$= \frac{\sum_{n=0}^{\infty}(-1)^n z^n \int_0^1 x^{2d+n-1}(1-x)^{1-d}dx}{\mathrm{B}(2d,2-d)}$$

$$= \frac{\sum_{n=0}^{\infty}(-1)^n z^n \mathrm{B}(2d+n,2-d)}{\mathrm{B}(2d,2-d)}$$

$$= \frac{\sum_{n=0}^{\infty}(-1)^n z^n \mathrm{B}(2d,2-d)\frac{(2d)_{(n)}}{(2+d)_{(n)}}}{\mathrm{B}(2d,2-d)}$$

$$= \sum_{n=0}^{\infty}(-1)^n z^n \frac{(2d)_{(n)}}{(2+d)_{(n)}} \quad (34)$$

Next, we make use of the Beta function recursive relation (96) in Appen. B, Sect. 10:

$$x=2d \quad (35)$$

$$y=2-d \quad (36)$$

$$\mathrm{B}(2d+n,2-d)=\mathrm{B}(2d,2-d)\frac{(2d)_{(n)}}{(2+d)_{(n)}} \quad (37)$$

Next, we have

$$(1-z)F\begin{bmatrix}1,2d; \\ d+1; \end{bmatrix} z = \left| \begin{array}{l} \sum_{n=0}^{\infty}(-1)^n z^n \frac{(2d)_{(n)}}{(1+d)_{(n)}} - \\ z\sum_{n=0}^{\infty}(-1)^n z^n \frac{(2d)_{(n)}}{(1+d)_{(n)}} \end{array}\right. \quad (38)$$

$$\frac{(1-d)z}{(d+1)}F\begin{bmatrix}1,2d; \\ d+2; \end{bmatrix} z = \frac{(1-d)z}{(d+1)}\sum_{n=0}^{\infty}(-1)^n z^n \frac{(2d)_{(n)}}{(2+d)_{(n)}}$$

$$= (1-d)z\sum_{n=0}^{\infty}\frac{(2d)_{(n)}(-1)^n z^n}{(1+d)_{(n)}(1+d+n)} \quad (39)$$

Next, we add (38) and (39) we obtain:

$$(1-z)F\begin{bmatrix}1,2d; \\ d+1; \end{bmatrix} z + \frac{(1-d)z}{(d+1)}F\begin{bmatrix}1,2d; \\ d+2; \end{bmatrix} z =$$

$$= \left| \begin{array}{l} \sum_{n=0}^{\infty}(-1)^n z^n \frac{(2d)_{(n)}}{(1+d)_{(n)}} + \\ z\sum_{n=0}^{\infty}\frac{(2d)_{(n)}(-1)^n z^n}{(1+d)_{(n)}}\left(\frac{2d+n}{1+d+n}\right) \end{array}\right.$$

$$= \left| \begin{array}{l} 1+\sum_{n=1}^{\infty}(-1)^n z^n \frac{(2d)_{(n)}}{(1+d)_{(n)}} + \\ \sum_{n=0}^{\infty}\frac{(2d+1)_{(n)}(-1)^n z^{n+1}}{(2+d)_{(n)}} \end{array}\right.$$

$$= \left| \begin{array}{l} 1+\sum_{n=1}^{\infty}(-1)^n z^n \frac{(2d)_{(n)}}{(1+d)_{(n)}} - \\ \sum_{n=0}^{\infty}\frac{(2d)_{(n+1)}(-1)^{n+1} z^{n+1}}{(1+d)_{(n+1)}} \end{array}\right.$$

$$= \left| \begin{array}{l} 1+\sum_{n=1}^{\infty}(-1)^n z^n \frac{(2d)_{(n)}}{(1+d)_{(n)}} - \\ \sum_{n=1}^{\infty}(-1)^n z^n \frac{(2d)_{(n)}}{(1+d)_{(n)}} \end{array}\right.$$

$$= 1 \quad (40)$$

This completes the proof first identity via Euler Transformation and also it demonstrates that the analytical or conceptual or theoretical assessment the first identity is rock solid; i.e., it does not exhibit any weaknesses or pitfalls or drawbacks.

3.3 First Identity Proof via Gauss Power Series

GPS, an acronym for Gauss Power Series, is first discussed in great detail in Progri (2016, [2]), second deliberation in Progri (2020, [5]), third investigation in Progri (2021, [6]), fourth detailed analysis in Progri (2023, [9]), and finally in this journal paper in Sect. 2.4.

First, writing (10) and (11) in GPS expansion we have the following:

$$F\begin{bmatrix}d,1-d; \\ d+1; \end{bmatrix} z = \sum_{n=0}^{\infty}\frac{(d)_n(1-d)_n}{(d+1)_n}\frac{z^n}{n!}$$

$$= 1+\frac{d(1-d)}{d+1}z+\frac{d(d+1)(1-d)(2-d)}{(d+1)(d+2)}\frac{z^2}{2}+\cdots$$

$$= 1+\frac{d(1-d)}{d+1}z+\frac{d(1-d)(2-d)}{d+2}\frac{z^2}{2}+\cdots \quad (41)$$

$$F\begin{bmatrix}d+1,2-d; \\ d+2; \end{bmatrix} z = \sum_{n=0}^{\infty}\frac{(d+1)_n(2-d)_n}{(d+2)_n}\frac{z^n}{n!}$$

$$= 1+\frac{(d+1)(2-d)}{d+2}z+\frac{(d+1)(d+2)(2-d)(3-d)}{(d+2)(d+3)}\frac{z^2}{2}+\cdots$$

$$= 1+\frac{(d+1)(2-d)}{d+2}z+\frac{(d+1)(2-d)(3-d)}{d+3}\frac{z^2}{2}+\cdots \quad (42)$$

Next, let us compute the following:

$$\frac{(1-d)z}{d+1}F_2(d,z) = \frac{(1-d)z}{d+1}\sum_{n=0}^{\infty}\frac{(d+1)_n(2-d)_n}{(d+2)_n}\frac{z^n}{n!}$$

$$= \left| \begin{array}{l} \frac{(1-d)z}{d+1}+\frac{(1-d)z}{d+1}\frac{(d+1)(2-d)}{d+2}z+ \\ \frac{(1-d)z}{d+1}\frac{(d+1)(d+2)(2-d)(3-d)}{(d+2)(d+3)}\frac{z^2}{2}+\cdots \end{array}\right.$$

$$= \left| \begin{array}{l} \frac{(1-d)z}{d+1}+\frac{(1-d)(2-d)}{d+2}z^2+ \\ \frac{(1-d)(2-d)(3-d)}{d+3}\frac{z^3}{2}+\cdots \end{array}\right. \quad (43)$$

Next, add (41) and (43) produces:

$$F_1(d,z)+\frac{(1-d)zF_2(d,z)}{d+1} = \left| \begin{array}{l} 1+\frac{d(1-d)}{d+1}z+\frac{d(1-d)(2-d)}{d+2}\frac{z^2}{2}+\cdots \\ \frac{(1-d)z}{d+1}+\frac{(1-d)(2-d)}{d+2}z^2+ \\ \frac{(1-d)(2-d)(3-d)}{d+3}\frac{z^3}{2}+\cdots \end{array}\right.$$

$$= \left| \begin{array}{l} 1 + (1-d)z + \frac{(1-d)(2-d)}{2} z^2 + \\ \frac{(1-d)(2-d)(3-d)}{3} \frac{z^3}{2} + \cdots \end{array} \right.$$

$$= \left| \begin{array}{l} 1 + (1-d)z + \frac{(1-d)(2-d)}{2} z^2 + \\ \frac{(1-d)(2-d)(3-d)}{6} z^3 + \cdots \end{array} \right. \quad (44)$$

If we compare and contrast the first four terms of (44) and (8) they are identical. However, the complete proof of whether (44) and (8) are an identity; i.e., (9) an identity can only be given by means of total mathematical induction[iiiiv].

The principle of total mathematical induction has three steps: *base case*, *inductive hypothesis*, and *inductive step*.

We have already proved the *base case* by showing that the first four terms of both (8) and (44) are identical.

Inductive hypothesis: We assume that (8) and (44) are identical up to the *k*th term; i.e.,

$$\left| \begin{array}{l} \sum_{n=0}^{k} \frac{(d)_n (1-d)_n}{(d+1)_n} \frac{z^n}{n!} + \\ \frac{(1-d)z}{d+1} \sum_{n=0}^{k-1} \frac{(d+1)_n (2-d)_n}{(d+2)_n} \frac{z^n}{n!} \end{array} \right. =$$

$$= \sum_{n=0}^{k} \frac{(d-1)(d-2)\cdots(d-n)}{n!} (-z)^n \quad (45)$$

We do not need to prove the *inductive hypothesis step* because this is assumed to be true. We will use the *base case* and the *inductive hypothesis step* to prove the inductive step which is given below:

Inductive Step: If (8) and (44) are identical up to the *k*th term, they must also be for the *k*th plus one term; i.e., we must prove that if (8), (44), and (45) are true then

$$\left| \begin{array}{l} \sum_{n=0}^{k+1} \frac{(d)_n (1-d)_n}{(d+1)_n} \frac{z^n}{n!} + \\ \frac{(1-d)z}{d+1} \sum_{n=0}^{k} \frac{(d+1)_n (2-d)_n}{(d+2)_n} \frac{z^n}{n!} \end{array} \right. = \sum_{n=0}^{k+1} \frac{(d-1)(d-2)\cdots(d-n)(-z)^n}{n!} \quad (46)$$

Starting from the left-hand side we can write (46) as follows:

$$\left| \begin{array}{l} \sum_{n=0}^{k+1} \frac{(d)_n (1-d)_n}{(d+1)_n} \frac{z^n}{n!} + \\ \frac{(1-d)z}{d+1} \sum_{n=0}^{k} \frac{(d+1)_n (2-d)_n}{(d+2)_n} \frac{z^n}{n!} \end{array} \right. =$$

$$= \left| \begin{array}{l} \sum_{n=0}^{k} \frac{(d)_n (1-d)_n}{(d+1)_n} \frac{z^n}{n!} + \\ \frac{(1-d)z}{d+1} \sum_{n=0}^{k-1} \frac{(d+1)_n (2-d)_n}{(d+2)_n} \frac{z^n}{n!} + \\ \frac{(d)_{k+1} (1-d)_{k+1}}{(d+1)_{k+1}} \frac{z^{k+1}}{(k+1)!} + \\ \frac{(1-d)z}{d+1} \frac{(d+1)_k (2-d)_k}{(d+2)_k} \frac{z^k}{k!} \end{array} \right.$$

$$= \left| \begin{array}{l} \sum_{n=0}^{k} \frac{(d-1)(d-2)\cdots(d-n)}{n!} (-z)^n + \\ \frac{(d)_{k+1} (1-d)_{k+1}}{(d+1)_{k+1}} \frac{z^{k+1}}{(k+1)!} + \\ \frac{(1-d)z}{d+1} \frac{(d+1)_k (2-d)_k}{(d+2)_k} \frac{z^k}{k!} \end{array} \right.$$

$$= \left| \begin{array}{l} \sum_{n=0}^{k} \frac{(d-1)(d-2)\cdots(d-n)}{n!} (-z)^n + \\ \frac{(1-d)_{k+1}}{(d+k+1)} \left[\frac{d}{k+1} + 1 \right] \frac{z^{k+1}}{k!} \end{array} \right.$$

$$= \left| \begin{array}{l} \sum_{n=0}^{k} \frac{(d-1)(d-2)\cdots(d-n)(-z)^n}{n!} + \\ \frac{(1-d)_{k+1} z^{k+1}}{(k+1)k!} \end{array} \right.$$

$$= \left| \begin{array}{l} \sum_{n=0}^{k} \frac{(d-1)(d-2)\cdots(d-n)(-z)^n}{n!} + \\ \frac{(d-1)_{k+1} (-z)^{k+1}}{(k+1)!} \end{array} \right.$$

$$= \sum_{n=0}^{k+1} \frac{(d-1)(d-2)\cdots(d-n)}{n!} (-z)^n \quad (47)$$

Since (47) is identical to (46) therefore, we were able to prove the *inductive step*. Because we were able to prove the inductive step, it postulates that (46) extents to all the integers and therefore it extends to infinity and finally completes the proof of (8) based on the GPS expansion.

In summary, we have completely shown that Progri (2024, [10] (51)) is true based on (8), (44), (45) through (47).

4 Second Identity of the Binomial Expansion via GHF and its Derivative

In order to prove the second identity via the GPS expansion we consider from Gradshteyn, Ryzhik (2007, [13] 1.110) the definition of the binomial expansion as follows:

$$1_{-z}{}^{p_{-1}} = \sum_{n=0}^{\infty} \binom{p_{-1}}{n} (-z)^n$$

$$= 1 + 1_{-p} z + \frac{1_{-p} 2_{-p}}{2} z^2 + \frac{1_{-p} 2_{-p} 3_{-p}}{6} z^3 + \cdots$$

$$= 1_{+z}{}^{1-p} \quad (48)$$

In Progri (2024, [10] (65)) a binomial expansion of the GHF and its derivative is provided which is the second identity that we will prove in this journal letter as follows:

$$F_1(d,p,z) + \frac{1_{-p} z}{d_{+1}} F_2(d,p,z) = 1_{-z}{}^{p_{-1}} \quad (49)$$

[v]

where

$$F_1(d,p,z) \equiv {}_2F_1 \left[\begin{matrix} d, 1_{-p}; \\ d_{+1}; \end{matrix} z \right] \quad (50)$$

$$F_2(d,p,z) \equiv {}_2F_1 \left[\begin{matrix} d_{+1}, 2_{-p}; \\ d_{+2}; \end{matrix} z \right] \quad (51)$$

The question that was posed in Progri (2024, [10] (65)) was as follows: if we started from the left side of (49) can we arrive at the right side of (49)?

First, we provide the second identity proof via the Euler Integral is given in Sect. 4.1. Second, the second identity proof via the Euler Transformation is deliberated in Sect. 4.2. Third, the second identity proof via the Gauss Power Series in Sect. 4.3.

4.1 Second Identity Proof via Euler Integral

In this Sect. we will prove (49) via Euler's Integral definition, Progri (2016, [2]), of the GHF as follows:

$$F\begin{bmatrix} d, 1_{-p}; \\ d_{+1}; \end{bmatrix} z = \frac{\int_0^1 \frac{x^{1-p-1} {1_{-x}}^{d_{+1}-1-p-1}}{(1-zx)^d} dx}{\mathrm{B}(1_{-p}, d_{+1}-1_{-p})} = \frac{\int_0^1 \frac{x^{-p} {1_{-x}}^{d+p-1}}{(1-zx)^d} dx}{\mathrm{B}(1_{-p}, d_{+p})} \tag{52}$$

$$F\begin{bmatrix} d_{+1}, 2_{-p}; \\ d_{+2}; \end{bmatrix} z = \frac{\int_0^1 \frac{x^{2-p-1} {1_{-x}}^{d_{+2}-2-p-1}}{(1-zx)^{d+1}} dx}{\mathrm{B}(2_{-p}, d_{+2}-2_{-p})} = \frac{\int_0^1 \frac{x^{1-p} {1_{-x}}^{d+p-1}}{(1-zx)^{d+1}} dx}{\mathrm{B}(2_{-p}, d_{+p})} \tag{53}$$

$$\frac{1_{-p}z}{d_{+1}} F\begin{bmatrix} d_{+1}, 2_{-p}; \\ d_{+2}; \end{bmatrix} z = \frac{1_{-p}z}{d_{+1}} \frac{\int_0^1 \frac{x^{1-p} (1-x)^{d+p-1}}{(1-zx)^{d+1}} dx}{\frac{\Gamma(2-p)\Gamma(d_{+p})}{\Gamma(d_{+2})}}$$

$$= \frac{\int_0^1 \frac{x^{1-p} {1_{-x}}^{d+p-1} z}{(1-zx)^{d+1}} dx}{\frac{\Gamma(1-p)\Gamma(d_{+p})}{\Gamma(d_{+1})}} = \frac{\int_0^1 \frac{x^{1-p} {1_{-x}}^{d+p-1} z}{(1-zx)^{d+1}} dx}{\mathrm{B}(1_{-p}, d_{+p})} \tag{54}$$

Next, add (53) and (54) yields,

$$F_1(d,p,z) + \frac{1_{-p}zF_2(d,p,z)}{d_{+1}} = \frac{\int_0^1 \frac{x^{-p} {1_{-x}}^{d+p-1}}{(1-zx)^d} dx}{\mathrm{B}(1_{-p}, d_{+p})} + \frac{\int_0^1 \frac{x^{1-p} {1_{-x}}^{d+p-1} z}{(1-zx)^{d+1}} dx}{\mathrm{B}(1_{-p}, d_{+p})}$$

$$= \frac{\int_0^1 \frac{x^{-p} {1_{-x}}^{d+p-1}}{(1-zx)^d} \left(1+\frac{xz}{1-zx}\right) dx}{\mathrm{B}(1_{-p}, d_{+p})} = \frac{\int_0^1 \frac{x^{-p} {1_{-x}}^{d+p-1}}{(1-zx)^{d+1}} dx}{\mathrm{B}(1_{-p} d_{+p})} \tag{55}$$

Next, substitute (16) into the integral or numerator of (55) produces:

$$\int_0^1 \frac{x^{-p} {1_{-x}}^{d+p-1}}{(1-zx)^{d+1}} dx = \int_0^1 \frac{{1_{-x}}^{d+p-1} \sum_{n=0}^{\infty} \binom{d_{+1}}{n} z^n x^n}{x^p} dx$$

$$= \sum_{n=0}^{\infty} \binom{d_{+1}}{n} z^n \int_0^1 x^{n-p} {1_{-x}}^{d+p-1} dx$$

$$= \sum_{n=0}^{\infty} \binom{d_{+1}}{n} z^n \mathrm{B}(n+1_{-p}, d_{+p}) \tag{56}$$

Next, we make use of the Beta function recursive relation (96) in Appen. B, Sect. 10:

$$x = 1_{-p} \tag{57}$$

$$y = d_{+p} \tag{58}$$

$$\mathrm{B}(n+1_{-p}, d_{+p}) = \mathrm{B}(1_{-p}, d_{+p}) \frac{(1_{-p})_{(n)}}{(d_{+1})_{(n)}} \tag{59}$$

Next, substitute (59) into (56) produces:

$$\int_0^1 \frac{x^{-p} {1_{-x}}^{d+p-1}}{(1-zx)^{d+1}} dx = \sum_{n=0}^{\infty} \binom{d_{+1}}{n} z^n \mathrm{B}(1_{-p}, d_{+p}) \frac{(1_{-p})_{(n)}}{(d_{+1})_{(n)}}$$

$$= \mathrm{B}(1_{-p}, d_{+p}) \sum_{n=0}^{\infty} \frac{(d_{+1})_{(n)}}{n!} z^n \frac{(1_{-p})_{(n)}}{(d_{+1})_{(n)}}$$

$$= \mathrm{B}(1_{-p}, d_{+p}) \sum_{n=0}^{\infty} \frac{(1_{-p})_{(n)}}{n!} z^n \text{ (see (9))}$$

$$= \mathrm{B}(1_{-p}, d_{+p}) {1_{-z}}^{p-1} \tag{60}$$

Finaly, we substitute (60) into (55) yields the desired result (49).

This completes the discussion on the second identity proof via Euler integral.

4.2 Second Identity Proof via Euler's Transformation

In this Sect. we will prove (49) via the definition of the GHF via Euler's Transformation, Gradshteyn, Ryzhik (2007, [13] 9.131 1. pg. 1008), as follows:

$$F\begin{bmatrix} d, 1_{-p}; \\ d_{+1}; \end{bmatrix} z = {1_{-z}}^p F\begin{bmatrix} 1, d_{+p}; \\ d_{+1}; \end{bmatrix} z \tag{61}$$

$$F\begin{bmatrix} d_{+1}, 2_{-p}; \\ d_{+2}; \end{bmatrix} z = {1_{-z}}^{p-1} F\begin{bmatrix} 1, d_{+p}; \\ d_{+2}; \end{bmatrix} z \text{ vi} \tag{62}$$

Next, add (61) and (62) it produces:

$$F_1(d,p,z) + \frac{1_{-p}z}{d_{+1}} F_2(d,p,z) = \left| \begin{matrix} {1_{-z}}^p F\begin{bmatrix} 1, d_{+p}; \\ d_{+1}; \end{bmatrix} z + \\ {1_{-z}}^{p-1} \frac{1_{-p}z}{d_{+1}} F\begin{bmatrix} 1, 2d; \\ d_{+2}; \end{bmatrix} z \end{matrix} \right.$$

$$= \frac{1_{-z} F\begin{bmatrix} 1, d_{+p}; \\ d_{+1}; \end{bmatrix} z + \frac{1_{-p}z}{d_{+1}} F\begin{bmatrix} 1, d_{+p}; \\ d_{+2}; \end{bmatrix} z}{{1_{-z}}^{1-d}} \tag{63}$$

To prove (49) is equivalent of proving the following:

$$1_{-z} F\begin{bmatrix} 1, d_{+p}; \\ d_{+1}; \end{bmatrix} z + \frac{1_{-p}z}{d_{+1}} F\begin{bmatrix} 1, d_{+p}; \\ d_{+2}; \end{bmatrix} z = 1 \tag{64}$$

First, the proof of (64) can be given using the GPS expansion as follows:

$$F\begin{bmatrix} 1, d_{+p}; \\ d_{+1}; \end{bmatrix} z = \sum_{n=0}^{\infty} \frac{(1)_n (d_{+p})_n}{(d_{+1})_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n \tag{65}$$

$$F\begin{bmatrix} 1, d_{+p}; \\ d_{+2}; \end{bmatrix} z = \sum_{n=0}^{\infty} \frac{(1)_n (d_{+p})_n}{(d_{+2})_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+2})_n} z^n \tag{66}$$

Next, substitute (66) and (65) into the left side of (64) yields,

$$1_{-z} \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n + \frac{1_{-p}z}{d_{+1}} \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+2})_n} z^n =$$

$$= \left| \begin{array}{l} \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n - z \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n \\ + \frac{1_{-p}z}{d_{+1}} \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+2})_n} z^n \end{array} \right.$$

$$= \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n - z \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} \left[\frac{d_{+p}+n}{d_{+1}+n}\right] z^n$$

$$= \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n - z \sum_{n=0}^{\infty} \frac{(d_{+p}+1)_n}{(d_{+2})_n} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n - z \sum_{n=0}^{\infty} \frac{(d_{+p}+1)_n}{(d_{+2})_n} z^n$$

$$= 1 + \sum_{n=0}^{\infty} \frac{(d_{+p})_{n+1}}{(d_{+1})_{n+1}} z^{n+1} - z \sum_{n=0}^{\infty} \frac{(d_{+p}+1)_n}{(d_{+2})_n} z^n$$

$$= 1 + z \sum_{n=0}^{\infty} \frac{(d_{+p}+1)_n}{(d_{+2})_n} z^n - z \sum_{n=0}^{\infty} \frac{(d_{+p}+1)_n}{(d_{+2})_n} z^n$$

$$= 1 \tag{67}$$

We could have shown that

$$1_{-z} \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n + \frac{1_{-p}z}{d_{+1}} \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+2})_n} z^n =$$

$$= \sum_{n=0}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n - z \sum_{n=0}^{\infty} \frac{(d_{+p}+1)_n}{(d_{+2})_n} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n - \sum_{n=0}^{\infty} \frac{(d_{+p}+1)_n}{(d_{+2})_n} z^{n+1}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n - \sum_{n=0}^{\infty} \frac{(d_{+p})_{n+1}}{(d_{+1})_{n+1}} z^{n+1}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n - \sum_{n=1}^{\infty} \frac{(d_{+p})_n}{(d_{+1})_n} z^n$$

$$= 1 \tag{68}$$

Equations (67) and (68) prove that (64) is true via the GPS expansion.

Next, we will try to prove the same employing the Euler Integral formula as follows:

$$F\begin{bmatrix} 1, d_{+p}; \\ d_{+1}; \end{bmatrix} = \frac{\int_0^1 \frac{x^{d+p-1} 1_{-x}{}^{d+1-d+p-1}}{(1-zx)} dx}{\mathrm{B}(d_{+p}, d_{+1} - d_{+p})}$$

$$= \frac{\int_0^1 \frac{x^{d+p-1} 1_{-x}{}^{-p}}{(1-zx)} dx}{\mathrm{B}(d_{+p}, 1_{-p})}$$

$$= \frac{\int_0^1 x^{d+p-1} 1_{-x}{}^{-p} \sum_{n=0}^{\infty} (-1)^n (zx)^n dx}{\mathrm{B}(d_{+p}, 1_{-p})}$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n z^n \int_0^1 x^{d+p+n-1} 1_{-x}{}^{-p} dx}{\mathrm{B}(d_{+p}, 1_{-p})}$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n z^n \mathrm{B}(d_{+p}+n, 1_{-p})}{\mathrm{B}(d_{+p}, 1_{-p})}$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n z^n \mathrm{B}(d_{+p}, 1_{-p}) \frac{(d_{+p})_{(n)}}{(d_{+1})_{(n)}}}{\mathrm{B}(d_{+p}, 1_{-p})}$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n \frac{(d_{+p})_{(n)}}{(d_{+1})_{(n)}} \tag{69}$$

Next, we make use of the Beta function recursive relation (96) in Appen. B, Sect. 10:

$$x = d_{+p} \tag{70}$$

$$y = 1_{-p} \tag{71}$$

$$\mathrm{B}(d_{+p} + n, 1_{-p}) = \mathrm{B}(d_{+p}, 1_{-p}) \frac{(d_{+p})_{(n)}}{(d_{+1})_{(n)}} \tag{72}$$

Second, we show that

$$F\begin{bmatrix} 1, d_{+p}; \\ d_{+2}; \end{bmatrix} = \frac{\int_0^1 \frac{x^{d+p-1} 1_{-x}{}^{d+2-d+p-1}}{(1-zx)} dx}{\mathrm{B}(d_{+p}, d_{+p} - d_{+p})}$$

$$= \frac{\int_0^1 \frac{x^{d+p-1} 1_{-x}{}^{1-p}}{(1-zx)} dx}{\mathrm{B}(d_{+p}, 2_{-p})}$$

$$= \frac{\int_0^1 x^{d+p-1} 1_{-x}{}^{1-p} \sum_{n=0}^{\infty} (-1)^n (zx)^n dx}{\mathrm{B}(d_{+p}, 2_{-p})}$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n z^n \int_0^1 x^{2d+n-1} 1_{-x}{}^{1-p} dx}{\mathrm{B}(d_{+p}, 2_{-p})}$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n z^n \mathrm{B}(d_{+p}+n, 2_{-p})}{\mathrm{B}(d_{+p}, 2_{-p})}$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n z^n \mathrm{B}(d_{+p}, 2_{-p}) \frac{(d_{+p})_{(n)}}{(d_{+2})_{(n)}}}{\mathrm{B}(d_{+p}, 2_{-p})}$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n \frac{(d_{+p})_{(n)}}{(d_{+2})_{(n)}} \tag{73}$$

Next, we make use of the Beta function recursive relation (96) in Appen. B, Sect. 10:

$$x = d_{+p} \tag{74}$$

$$y = 2_{-p} \tag{75}$$

$$\mathrm{B}(d_{+p} + n, 2_{-p}) = \mathrm{B}(d_{+p}, 2_{-p}) \frac{(d_{+p})_{(n)}}{(d_{+2})_{(n)}} \tag{76}$$

Next, we have

$$1_{-z} F\begin{bmatrix} 1, d_{+p}; \\ d_{+1}; \end{bmatrix} = \left| \begin{array}{l} \sum_{n=0}^{\infty} (-1)^n z^n \frac{(d_{+p})_{(n)}}{(d_{+1})_{(n)}} - \\ z \sum_{n=0}^{\infty} (-1)^n z^n \frac{(d_{+p})_{(n)}}{(d_{+1})_{(n)}} \end{array} \right. \tag{77}$$

$$\frac{1_{-p}z}{d_{+1}} F\begin{bmatrix} 1, d_{+p}; \\ d_{+2}; \end{bmatrix} = \frac{1_{-p}z}{d_{+1}} \sum_{n=0}^{\infty} (-1)^n z^n \frac{(d_{+p})_{(n)}}{(d_{+2})_{(n)}}$$

$$= 1_{-p} z \sum_{n=0}^{\infty} \frac{(d_{+p})_{(n)} (-1)^n z^n}{(d_{+1})_{(n)} (d_{+1}+n)} \tag{78}$$

Next, we add (77) and (78) we obtain:

$$1_{-z} F\begin{bmatrix} 1, d_{+p}; \\ d_{+1}; \end{bmatrix} + \frac{1_{-p}z}{d_{+1}} F\begin{bmatrix} 1, d_{+p}; \\ d_{+2}; \end{bmatrix} =$$

$$= \left| \begin{array}{l} \sum_{n=0}^{\infty} (-1)^n z^n \frac{(d_{+p})_{(n)}}{(d_{+1})_{(n)}} + \\ z \sum_{n=0}^{\infty} \frac{(d_{+p})_{(n)} (-1)^n z^n}{(d_{+1})_{(n)}} \left(\frac{d_{+p}+n}{d_{+1}+n}\right) \end{array} \right.$$

$$= \left| \begin{matrix} 1 + \sum_{n=1}^{\infty} (-1)^n z^n \frac{(d_{+p})_{(n)}}{(d_{+1})_{(n)}} + \\ \sum_{n=0}^{\infty} \frac{(d_{+p}+1)_{(n)} (-1)^n z^{n+1}}{(d_{+2})_{(n)}} \end{matrix} \right.$$

$$= \left| \begin{matrix} 1 + \sum_{n=1}^{\infty} (-1)^n z^n \frac{(d_{+p})_{(n)}}{(d_{+1})_{(n)}} - \\ \sum_{n=0}^{\infty} \frac{(d_{+p})_{(n+1)} (-1)^{n+1} z^{n+1}}{(d_{+1})_{(n+1)}} \end{matrix} \right.$$

$$= \left| \begin{matrix} 1 + \sum_{n=1}^{\infty} (-1)^n z^n \frac{(d_{+p})_{(n)}}{(d_{+1})_{(n)}} - \\ \sum_{n=1}^{\infty} (-1)^n z^n \frac{(d_{+p})_{(n)}}{(d_{+1})_{(n)}} \end{matrix} \right.$$

$$= 1 \tag{79}$$

This completes the proof of the second identity via Euler's Transformation and also it demonstrates that the analytical or conceptual or theoretical assessment of the second identity is rock solid; i.e., it does not exhibit any weaknesses or pitfalls or drawbacks.

4.3 Second Identity Proof via Gauss Power Series

GPS, an acronym for Gauss Power Series, is first discussed in great detail in Progri (2016, [2]), second deliberation in Progri (2020, [5]), third investigation in Progri (2021, [6]), fourth detailed analysis in Progri (2023, [9]), and finally in this journal paper in Sects. 2.4 and 3.3.

First, writing (50) and (51) in GPS expansion we have the following:

$$F\begin{bmatrix} d, 1_{-p}; \\ d_{+1}; \end{bmatrix} z = \sum_{n=0}^{\infty} \frac{(d)_n (1_{-p})_n}{(d_{+1})_n} \frac{z^n}{n!}$$

$$= 1 + \frac{d 1_{-p}}{d_{+1}} z + \frac{d d_{+1} 1_{-p} 2_{-p}}{d_{+1} d_{+2}} \frac{z^2}{2} + \cdots$$

$$= 1 + \frac{d 1_{-p}}{d_{+1}} z + \frac{d 1_{-p} 2_{-p}}{d_{+2}} \frac{z^2}{2} + \cdots \tag{80}$$

$$F\begin{bmatrix} d_{+1}, 2_{-p}; \\ d_{+2}; \end{bmatrix} z = \sum_{n=0}^{\infty} \frac{(d_{+1})_n (2_{-p})_n}{(d_{+2})_n} \frac{z^n}{n!}$$

$$= 1 + \frac{d_{+1} 2_{-p}}{d_{+2}} z + \frac{d_{+1} d_{+2} 2_{-p} 3_{-p}}{d_{+2} d_{+3}} \frac{z^2}{2} + \cdots$$

$$= 1 + \frac{d_{+1} 2_{-p}}{d_{+2}} z + \frac{d_{+1} 2_{-p} 3_{-p}}{d_{+3}} \frac{z^2}{2} + \cdots \tag{81}$$

Next, let us compute the following:

$$\frac{1_{-p} z}{d_{+1}} F_2(d,p,z) = \frac{1_{-p} z}{d_{+1}} \sum_{n=0}^{\infty} \frac{(d_{+1})_n (2_{-p})_n}{(d_{+2})_n} \frac{z^n}{n!}$$

$$= \left| \begin{matrix} \frac{1_{-p} z}{d_{+1}} + \frac{1_{-p} z}{d_{+1}} \frac{d_{+1} 2_{-p}}{d_{+2}} z + \\ \frac{1_{-p} z}{d_{+1}} \frac{d_{+1} d_{+2} 2_{-p} 3_{-p}}{d_{+2} d_{+3}} \frac{z^2}{2} + \cdots \end{matrix} \right.$$

$$= \frac{1_{-p} z}{d_{+1}} + \frac{1_{-p} 2_{-p}}{d_{+2}} z^2 + \frac{1_{-p} 2_{-p} 3_{-p}}{d_{+3}} \frac{z^3}{2} + \cdots \tag{82}$$

Next, add (81) and (82) produces:

$$\left| \begin{matrix} F_1(d,p,z) + \\ \frac{1_{-p} z F_2(d,p,z)}{d_{+1}} \end{matrix} \right. = \left| \begin{matrix} 1 + \frac{d 1_{-p}}{d_{+1}} z + \frac{d 1_{-p} 2_{-p}}{d_{+2}} \frac{z^2}{2} + \cdots \\ \frac{1_{-p} z}{d_{+1}} + \frac{1_{-p} 2_{-p}}{d_{+2}} z^2 + \\ \frac{1_{-p} 2_{-p} 3_{-p}}{d_{+3}} \frac{z^3}{2} + \cdots \end{matrix} \right.$$

$$= \left| \begin{matrix} 1 + 1_{-p} z + \frac{1_{-p} 2_{-p}}{2} z^2 + \\ \frac{1_{-p} 2_{-p} 3_{-p}}{3} \frac{z^3}{2} + \cdots \end{matrix} \right.$$

$$= 1 + 1_{-p} z + \frac{1_{-p} 2_{-p} z^2}{2} + \frac{1_{-p} 2_{-p} 3_{-p} z^3}{6} + \cdots \tag{83}$$

If we compare and contrast the first four terms of (83) and (48) they are identical. However, the complete proof of whether (83) and (48) are an identity; i.e., (49) an identity can only be given by means of total mathematical induction[vii][viii].

As stated earlier the principle of total mathematical induction has three steps: *base case*, *inductive hypothesis*, and *inductive step*.

We have already proved the *base case* by showing that the first four terms of both (49) and (83) are identical.

Inductive hypothesis: We assume that (49) and (83) are identical up to the *k*th term; i.e.,

$$\left| \begin{matrix} \sum_{n=0}^{k} \frac{(d)_n (1_{-p})_n}{(d_{+1})_n} \frac{z^n}{n!} + \\ \frac{1_{-p} z}{d_{+1}} \sum_{n=0}^{k-1} \frac{(d_{+1})_n (2_{-p})_n}{(d_{+2})_n} \frac{z^n}{n!} \end{matrix} \right. = \sum_{n=0}^{k} \frac{p_{-1} p_{-2} \cdots p_{-n}}{n!} (-z)^n \tag{84}$$

We do not need to prove the *inductive hypothesis step* because this is assumed to be true. We will use the *base case* and the *inductive hypothesis step* to prove the inductive step which is given below:

Inductive Step: If (49) and (83) are identical up to the *k*th term, they must also be for the *k*th plus one term; i.e., we must prove that if (49), (83), and (84) are true then

$$\left| \begin{matrix} \sum_{n=0}^{k+1} \frac{(d)_n (1_{-p})_n}{(d_{+1})_n} \frac{z^n}{n!} + \\ \frac{1_{-p} z}{d_{+1}} \sum_{n=0}^{k} \frac{(d_{+1})_n (2_{-p})_n}{(d_{+2})_n} \frac{z^n}{n!} \end{matrix} \right. = \sum_{n=0}^{k+1} \frac{p_{-1} p_{-2} \cdots p_{-n}}{n!} (-z)^n \tag{85}$$

Starting from the left-hand side we can write (85) as follows:

$$\left| \begin{matrix} \sum_{n=0}^{k+1} \frac{(d)_n (1_{-p})_n}{(d_{+1})_n} \frac{z^n}{n!} + \\ \frac{1_{-p} z \sum_{n=0}^{k} \frac{(d_{+1})_n (2_{-p})_n z^n}{(d+2)_n \, n!}}{d_{+1}} \end{matrix} \right. = \left| \begin{matrix} \sum_{n=0}^{k} \frac{(d)_n (1_{-p})_n}{(d_{+1})_n} \frac{z^n}{n!} + \\ \frac{1_{-p} z}{d_{+1}} \sum_{n=0}^{k-1} \frac{(d_{+1})_n (2_{-p})_n}{(d+2)_n} \frac{z^n}{n!} + \\ \frac{(d)_{k+1} (1_{-p})_{k+1}}{(d_{+1})_{k+1}} \frac{z^{k+1}}{(k+1)!} + \\ \frac{1_{-p} z}{d_{+1}} \frac{(d_{+1})_k (2-d)_k}{(d+2)_k} \frac{z^k}{k!} \end{matrix} \right.$$

TABLE I: DIRECT COMPUTATION

Eq. (9)	Left of (9)	Right of (9)
$d = 2,\ z = 0.5$	0.500000000000000	0.500000000000000
$d = 2.5,\ z = 0.5$	0.353553390593274	0.353553390593274
$d = 2,\ z = 0.5$	0.500000000000000	0.500000000000000
$d = 2.5,\ z = 0.5$	0.353553390593274	0.353553390593274

TABLE II: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

Eq. (9)	Left $-$ Right of (9)	tsim (ms)
$d = 2,\ z = 0.5$	0	26.09
$d = 2.5,\ z = 0.5$	0	25.79
$d = 2,\ z = 0.5$	0	6.81
$d = 2.5,\ z = 0.5$	-2.220446049250313e-16	8.27

TABLE III: DIRECT COMPUTATION

Eq. (49)	Left of (49)	Right of (9)
$p = 4,\ z = 0.5$	0.125000000000000	0.125000000000000
$p = 4.5,\ z = 0.5$	0.088388347648318	0.088388347648318
$p = 4,\ z = 0.5$	0.125000000000000	0.125000000000000
$p = 4.5,\ z = 0.5$	0.088388347648318	0.088388347648318

TABLE IV: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

Eq. (49)	Left – Right of (49)	tsim (ms)
$p = 4,\ z = 0.5$	0	28.64
$p = 4.5,\ z = 0.5$	0	25.81
$p = 4,\ z = 0.5$	0	7.44
$p = 4.5,\ z = 0.5$	-4.996003610813204e-16	8.43

$$= \left| \begin{array}{l} \sum_{n=0}^{k} \frac{p_{-1}p_{-2}\cdots p_{-n}}{n!}(-z)^n + \\ \frac{(d)_{k+1}(1_{-p})_{k+1}}{(d_{+1})_{k+1}} \frac{z^{k+1}}{(k+1)!} + \\ \frac{1_{-p}z}{d_{+1}} \frac{(d_{+1})_k(2-d)_k}{(d+2)_k} \frac{z^k}{k!} \end{array} \right.$$

$$= \left| \begin{array}{l} \sum_{n=0}^{k} \frac{p_{-1}p_{-2}\cdots p_{-n}}{n!}(-z)^n + \\ \frac{(1_{-p})_{k+1}}{(d_{+1}+k)} \left[\frac{d}{k+1} + 1\right] \frac{z^{k+1}}{k!} \end{array} \right.$$

$$= \sum_{n=0}^{k} \frac{p_{-1}\cdots p_{-n}(-z)^n}{n!} + \frac{(1_{-p})_{k+1} z^{k+1}}{(k+1)k!}$$

$$= \left| \begin{array}{l} \sum_{n=0}^{k} \frac{p_{-1}p_{-2}\cdots p_{-n}(-z)^n}{n!} + \\ \frac{(p_{-1})_{k+1}(-z)^{k+1}}{(k+1)!} \end{array} \right.$$

$$= \sum_{n=0}^{k+1} \frac{p_{-1}p_{-2}\cdots p_{-n}}{n!}(-z)^n \tag{86}$$

Since (86) is identical to (85) therefore, we were able to prove the *inductive step*. Because we were able to prove the inductive step, it postulates that (86) extents to all the integers and

TABLE V: DIRECT COMPUTATION

Eq. (25)	Left of (25)	Right of (25)
$d = 2,\ z = 0.5$	1	1
$d = 2.5,\ z = 0.5$	1	1
$d = 2,\ z = 0.5$	1	1
$d = 2.5,\ z = 0.5$	0.999999999999999	1

TABLE VI: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

Eq. (25)	Left $-$ Right of (25)	tsim (ms)
$d = 2,\ z = 0.5$	0	29.10
$d = 2.5,\ z = 0.5$	0	33.18
$d = 2,\ z = 0.5$	0	6.30
$d = 2.5,\ z = 0.5$	-8.881784197001252e-16	5.87

TABLE VII: DIRECT COMPUTATION

Eq. (49)	Left of (49)	Right of (9)
$p = 4,\ z = 0.5$	1.000000000000000	1
$p = 4.5,\ z = 0.5$	1	1
$p = 4,\ z = 0.5$	1.000000000000000	1
$p = 4.5,\ z = 0.5$	0.999999999999999	1

TABLE VIII: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

Eq. (49)	Left – Right of (49)	tsim (ms)
$p = 4,\ z = 0.5$	-2.220446049250313e-16	24.08
$p = 4.5,\ z = 0.5$	0	26.97
$p = 4,\ z = 0.5$	2.220446049250313e-16	6.22
$p = 4.5,\ z = 0.5$	-8.881784197001252e-16	6.83

therefore it extends to infinity and finally completes the proof of (49) based on the GPS expansion.

In summary, we have completely shown that Progri (2024, [10] (65)) is true based on (48), (49), (82) through (86).

5 Numerical Examples

Let us consider several numerical examples similar to the examples in Progri (2023, [9]). For the simulation parameters the reader may refer to Progri (2024, [10]).

Example 1: The first numerical example considers the computation of the first identity $F[a,b;c;z] + b/c\,F[e,f;g;z] = (1-z)^{d-1}$ based on (9) for $a = d$, $b = 1-d$, $c = d+1$, $e = d+1$, $f = 2-d$, $g = d+2$ and $z = 0.5$. The results of the direct computation are shown in Tab. I and of the absolute error of the direct computation are shown in Tab. II.

The first two rows of Tab. I correspond to the MATLAB built-in-function (BIF) *hypergeom*(a,b,z) and the second last two rows correspond to Giftet implementation of the hypergeometric function *phypergeom*(a,b,z) (see Progri (2019, [4]). The same can be said about Tab. II. The implementation of both functions leads to amazingly accurate and fast computations. Based on the results of Tab. I and II we can conclude that (9) is in fact in identity.

Example 2: The second numerical example considers the computation of the second identity $F[a,b;c;z]+b/c\,F[e,f;g;z]=(1-z)^{p-1}$ based on (49) for $a=d$, $b=1-p$, $c=d+1$, $e=d+1$, $f=2-p$, $g=d+2$ and $z=0.5$. The results of the direct computation are shown in Tab. III and of the absolute error of the direct computation are shown in Tab. IV.

The same comments that were made about Tab. I and II can be made for Tab. III and IV. Based on the results of Tab. III and IV we can conclude that (49) is in fact in identity.

Example 3: The third numerical example considers the computation of the first identity $(1-z)F[a,b;c;z]+(1-d)/c\,F[e,f;g;z]=1$ based on (25) for $a=1$, $b=2d$, $c=d+1$, $e=1$, $f=2d$, $g=d+2$ and $z=0.5$. The results of the direct computation are shown in Tab. V and of the absolute error of the direct computation are shown in Tab. VI.

The same comments that were made about Tab. III and IV can be made for Tab. V and VI. Based on the results of Tab. V and VI we can conclude that (25) is in fact in identity.

Example 4: The third numerical example considers the computation of the first identity $(1-z)F[a,b;c;z]+(1-p)/c\,F[e,f;g;z]=1$ based on (64) for $a=1$, $b=d+p$, $c=d+1$, $e=1$, $f=d+p$, $g=d+2$ and $z=0.5$. The results of the direct computation are shown in Tab. VII and of the absolute error of the direct computation are shown in Tab. VIII.

The same comments that were made about Tab. V and VI can be made for Tab. VII and VIII. Based on the results of Tab. VII and VIII we can conclude that (64) is in fact in identity.

6 Conclusions

In conclusion, I have produced another outstanding journal paper that shows a remarkable relationship of binomial expansion as a weighted sum of a 2F1 GHF and its derivative.

From Progri (2024, [10]) we started with trying to prove two identities Progri (2024, [10] (51) and (65)) and based on the symmetry properties of GHF Appen. A, Sect. 9 and on the Euler's Integral Transformation, we are able to produce as many as sixteen identities.

Analytically, we used three well-known methods to prove the identities: Euler's Integral, Euler's Identity Transformation, and GPS, an acronym that stands for Gauss Power Series expansion. We used all of these methods two prove both identities in Progri (2024, [10] (51) and (65)) are in fact true. In this journal paper we also made use of the total mathematic induction principle.

Numerically, we build four main examples each one of which with four runs and a total of sixteen data. A total of sixty-four data points were produced from these four examples thirty-two corresponding to the first identity and the other thirty-two corresponding to the second identity.

The content of this paper exceeded my initial expectations of producing a journal letter. The content became so big that it exceeded the requirements of the journal letter; therefore, I could no longer call it a journal letter as I did in Progri (2024, [10]).

A future journal paper will discuss the commutative property Sect. 2.2 in greater detail and the more generalized identity that will be a generalization of the identities discussed in this journal paper.

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I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [21] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications*. *Vol. I* (Not yet available in print) [14].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [19] reminds me of my grandfather who I loved him very much.

Finally, I am also dedicating this journal paper to our most beloved President Ronald Reagan, who reminds me of my

grandfather, [20] who spoke from the heart, who spoke the truth, who was a staunch supporter of personal freedom, the greatest leader of the free world, and perhaps the greatest critic of the wasteful government control and spending[ix].

"*Government is not the answer, government is the problem*."—Ronald Reagan

I would like to express my deepest gratitude to my mother Lumturi Progri, my sister Adriana Dine, and Ms. Elizabeth Demir for their support, loyally, and dedication to me during some of the most difficult times in my life as a boy, man, and scholar.

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9 Appendix A: The GHF Symmetry Property

Before taking the derivative of (1) we have to ask the question. Has the GHF a certain symmetry property? I.e., is the following true if we use definition (1)?

First, we asked AI overview the following question?

Is the hypergeometric function $F[a,b;c;z]$ the same as $F[b,a;c;z]$?

The AI overview answer was as follows: No, the hypergeometric function $F[a,b;c;z]$ is generally not the same as $F[b,a;c;z]$. While the hypergeometric function has some symmetric properties, swapping a and b can change the resulting value.

AI General Case: In the general case, $F[a,b;c;z]$ and $F[b,a;c;z]$ are not equal. They are different hypergeometric functions with different parameter values, and their values will vary depending on the values of a, b, c, and z.

AI Symmetric Properties: While the hypergeometric function has some symmetric properties, such as being symmetric with respect to the interchange of a and b in certain contexts, this doesn't imply that $F[a,b;c;z]$ is always the same as $F[b,a;c;z]$.

AI Specific Examples: For example, when $a=1$ and $b=c$, the series reduces into a geometric series, but the value can still vary depending on other parameter values.

In summary, swapping a and b in the hypergeometric function generally leads to a different function, not an equivalent one.

This Google AI response was wrong. When asked then to dive dipper into the AI then we get a different answer.

$$F\begin{bmatrix} a,b; \\ c; \end{bmatrix} z =?\, F\begin{bmatrix} b,a; \\ c; \end{bmatrix} z \tag{87}$$

$$F\begin{bmatrix} b,a; \\ c; \end{bmatrix} z = \frac{\int_0^1 \frac{x^{a-1}(1-x)^{c-a-1}}{(1-zx)^b} dx}{\mathrm{B}(a,c-a)} \tag{88}$$

In order to prove that (1) is identical to (87) we will have to expand the Binomial expression on both (1) and (88) and then eliminate the Beta function which leads to the Gauss Power Series expansion of the GHF.

$$(1-zx)^{-a} = \sum_{n=0}^{\infty} \binom{-a}{n} (-zx)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-a)_{-n}}{n!} (-1)^n z^n x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (a)_n}{n!} (-1)^n z^n x^n$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n x^n \tag{89}$$

Similarly,

$$(1-zx)^{-b} = \sum_{n=0}^{\infty} \binom{-b}{n} (-zx)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-b)_{-n}}{n!} (-1)^n z^n x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (b)_n}{n!} (-1)^n z^n x^n$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n}{n!} z^n x^n \tag{90}$$

If we apply (90) into (1) we have

$$F\begin{bmatrix} a,b; \\ c; \end{bmatrix} z = \frac{\int_0^1 x^{b-1}(1-x)^{c-b-1} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n x^n dx}{\mathrm{B}(b,c-b)}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \int_0^1 x^{b+n-1}(1-x)^{c-b-1} dx}{\mathrm{B}(b,c-b)}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \mathrm{B}(b+n,c-b)}{\mathrm{B}(b,c-b)}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \mathrm{B}(b,c-b) \frac{(b)_n}{(c)_n}}{\mathrm{B}(b,c-b)} \text{ (see Appen. B)}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{91}$$

Similarly, apply (91) into (88)

$$F\begin{bmatrix} b,a; \\ c; \end{bmatrix} z = \frac{\int_0^1 \frac{x^{a-1}(1-x)^{c-a-1} \sum_{n=0}^{\infty} \frac{(b)_n}{n!} z^n x^n}{(1-zx)^b} dx}{\mathrm{B}(a,c-a)}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{(b)_n}{n!} z^n \int_0^1 \frac{x^{a+n-1}(1-x)^{c-a-1}}{(1-zx)^b} dx}{\mathrm{B}(a,c-a)}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{(b)_n}{n!} z^n \mathrm{B}(a+n,c-a)}{\mathrm{B}(a,c-a)}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{(b)_n}{n!} z^n \mathrm{B}(a,c-a) \frac{(a)_n}{(c)_n}}{\mathrm{B}(a,c-a)} \text{ (see Appen. B)}$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n (a)_n}{(c)_n} \frac{z^n}{n!} \tag{92}$$

Equations (92) and (91) are identical since the order of multiplications of $(a)_n(b)_n$ does not change the result $(b)_n(a)_n$. This is an example in which the AI overview was wrong while diving deeper into AI was right. So, the right answer is yes. Swapping the parameters a and b does not change the values of the GHF $F[a,b;c;z]$.

10 Appendix B: The Beta Function Recursive Relation

A recursive relation of the Beta function is used throughout this journal letter.

We start with the well-known Beta function relation as follows

$$\mathrm{B}(x+n,y)=\mathrm{B}(x,y),\ n=0,\ n \text{ non-negative integer} \quad (93)$$

$$\mathrm{B}(x+1,y)=\mathrm{B}(x,y)\frac{x}{x+y},\ n=1$$

$$=\mathrm{B}(x,y)\frac{(x)_1}{(x+y)_1},\ n=1 \quad (94)$$

$$\mathrm{B}(x+2,1-d)=\mathrm{B}(x+1,y)\frac{x+1}{x+y+1},\ n=2$$

$$=\mathrm{B}(x,y)\frac{x}{x+y}\frac{x+1}{x+y+1},\ n=2$$

$$=\mathrm{B}(x,y)\frac{(x)_2}{(x+y)_2},\ n=2 \quad (95)$$

Based on the principle of total mathematical induction we can prove that

$$\mathrm{B}(x+n,y)=\mathrm{B}(x,y)\frac{(x)_n}{(x+y)_n} \quad (96)$$

This completes the discussion on the recursive relation of the Beta function Arfken, Weber (1995, [12]), Gradshteyn, Ryzhik (2007, [13]).

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[i] I am dedicating this journal-paper to my Master Science Academic Advisor Prof. Kevin Clements. I wrote on my 1997 Master Thesis Progri (1997, [1]): "Last but not at least, I send my appreciation to Prof. Clements, who has been an inspiring and motivating teacher and helpful advisor." Prof. Clements was assigned to ma as my Master Science academic advisor, at the Electrical and Computer Engineering Department, Worcester Polytechnic Institute (WPI), Worcester, Massachusetts. He was an American born. He spoke with no accent. More importantly, Prof. Clements was very passionate about math, mathematical models, mathematical equations, numerical methods, numerical linear algebra, etc. He was working on a book when I took a graduate course with him on Power Systems Analysis. He was very organized, very methodical. Prof, Clements and my Master Thesis Advisor Prof. Alex Emanuel got along very well. They were both highly respected and accomplished scholars, teachers, researchers, and amazing people. Last time I spoke with Prof. Clements was at Prof. Emanuel's memorial services. I had a very pleasant conversation with him. I felt extremely motivated by him and Prof. Alex Emanuel during my Master Thesis studies and I cannot be more thankful of having him as my Master Science academic advisor and Prof. Alex Amanuel as my Master Thesis Advisor and having both of them in my Master Thesis committee.

[ii] We are taking the derivative first and then applying the Euler's Transformation Integral.

[iii] The principle of total mathematical induction, also known as the principle of mathematical induction, is a method for proving statements about natural numbers (integers greater than or equal to zero or one). It works by proving a base case (a starting point, like zero or one) and then demonstrating that if the statement holds for any number, it also holds for the next number in the sequence.
Base Case: This is the starting point of the proof. It involves proving that the statement is true for the initial value (usually zero or one).
Inductive Hypothesis: This is the assumption that the statement is true for an arbitrary natural number, typically denoted as *k* or *n*.
Inductive Step: This is where you prove that if the statement is true for *k* (or *n*), it is also true for *k*+1 (or *n*+1).
In essence: If you can prove the base case and the inductive step, then the statement is true for all natural numbers greater than or equal to the starting point of the base case.

[iv] The total principle of mathematical induction, also known as the complete induction or strong induction, is not attributed to a single inventor, but rather developed and formalized by mathematicians working independently in the 19th and early 20th centuries. Key figures involved in its development include Giuseppe Peano and Richard Dedekind, who laid the groundwork for the modern understanding of mathematical induction.
Peano and Dedekind: Peano refined a set of axioms for positive integers, including the principle of mathematical induction as one of his five axioms for arithmetic. Dedekind also developed a set of axioms describing the positive integers, contributing to the foundation for mathematical induction.
Early Use: While the principle of induction had been used in various forms earlier, the modern, rigorous formulation emerged in the late 19th century.
Giovanni Vacca's Contribution: Giovanni Vacca, a student of Peano, was credited with identifying Francesco Maurolico as an early user of a form of induction in 1575.
Henri Poincaré's Role: Henri Poincaré made significant contributions to understanding the philosophical implications of mathematical induction, particularly its role as a foundation for mathematical reasoning.
Mathematical Intuitionism: Poincaré's ideas influenced the school of mathematical intuitionism, which views mathematical induction as an ultimate foundation of mathematical thought.

[v] We will use the shorthand notation $a_{\pm b} \equiv a \pm b$.

[vi] Ditto ii.

[vii] Ditto iii.

[viii] Ditto iv.

[ix] Ditto Progri(2020, [5]) EN vii.