Giftet Inc.
Journal of Geolocation, Geo-information, and Geo-intelligence
Volume 2024, article ID 2024071604, 24 pages
https://doi.org/10.18610/JG3.2024.071604

Research Article

On the Computation of the Cumulative Probability Distribution Function of the Exponential Generalized Beta Generalized Family of Distributions

To my Prof. Dennise Nicoletti, a phenomenal teacher, scholar, researcher, the best female professor who I have ever known in my life.[i]

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Received May 8, 2024; Revised May 19, 2024-June 17, 2024, Accepted July 16, 2024; Published November 1, 2024.

Editor-in-Chief/Senior Scientific Editor: Ilir F. Progri

This is a continuation of the research on the computation of the cumulative probability distribution function (cdf) of the exponential generalized Beta (EGB) also known as 5-paramater EGB[ii] (or EGB3) generalized family of distributions. This paper is based on the recent research that is published in 2024 on the subject of the EGB3 family of distributions; it is the third paper on the subject of EGB3 distribution and the second paper on the subject of the EGB3 cdf. A complete derivation of the EGB3 cdf is provided in the paper. A complete, comprehensive analysis of the generalized EGB cdfs such as generalized Gumbel (GGu) of Type 1, generalized Logistics of Types 1 through 4 (GL1), (GL2), (GL3), and (GL4) are derived in the paper in fine details. A detailed and unique proof of monotonic nature of the EGB3 cdf that leads to an original, novel, and innovative closed form expression (CFE) of a *Binomial Expansion* as the weighted sum of two 2F1 hypermetric functions is deliberated in the paper. Two original, novel, and innovative CFE of the same binomial expansion as a weighted sum of two 2F1 hypermetric functions are derived when performing the derivative of the cdf of the GL3 and GL4. We perform *analytical* and *conceptual* assessments to gain a comprehensive understanding of the EGB3 cdf family of distributions, by exploring both the *what* (*analytical*) and the *why* (*conceptual*) behind it, leading to profound understanding of the power and flexibility that comes from the analysis, modeling, and simulation of the EGB3 cdf.
Index Terms—Exponential generalized Beta, probability density function, cumulative distribution function, generalized Gumbel, generalized Logistics, binomial expansion, limit computation, monotonicity analysis.

1 Introduction

This journal paper builds on the research of Progri (2016, [1]) on the exponential generalized Beta (EGB3) distribution, also known as the EGB of type 3 (EGB3[iii]). This journal paper makes use of the description of the EGB3 probability density function (pdf), cumulative probability distribution function (cdf), initially derived in Progri (2016, [1]).

This journal paper also utilizes the technique of computation of the partial derivatives of the hypergeometric functions from Progri (2016, [2]) and builds on the analysis of the EGB pdf and cdf.

The computations of the EGB3 cdf and a couple of generalized logistic cdfs are computed based on the research performed for the first time in Progri (2019, [4]), a superfast implementation of the computation of the hypergeometric function.

In Progri (2019, [3], [5]) I developed a rather unique, original, novel, innovative approach or model of the 4-paramater ionospheric model base on understanding that came from Progri (2016, [1], [2]); however, I did not take it very far and any further.

In Progri (2022, [6]) I performed a study of the computation of the exponential function for both the positive and negative exponents. The wisdom, knowledge, and understanding that was produced from Progri (2022, [6]) will be employed throughout Progri (2024, [7], [8]) and in this journal paper.

This journal paper builds on the solid foundational, computational research in Progri (2024, [7], [8]). The second journal paper Progri (2024, [8]) a rather solid paper on the fundamental properties of the cdf and a rather detailed description of the nongeneralized members of the EGB3 family of distributions.

This journal paper aims to ameliorate the content related to the EGB3 cdf generalized family of cdfs. However, before we dive into a deeper discussion on the main theme of the journal paper I will continue with the introduction of the main contributors and their contributions that were left off in Progri (2024, [7], [8]).

Contributions from Prentice (1976, [10]) on the Generalized Logistic Distribution of the type IV, Balakrishnan, Leung (1988, [11]) on the Generalized Logistic distribution of type I, and Johnson et al. (1995, [12]) on the Generalized Logistic distribution of type III are included in the paper.

Additional contributions from McDonald, Xu (1995, [13]) not included in Progri (2024, [7], [8]) are mentioned in this journal paper.

In McDonald, Xu (1996, [14]) a comparison of semi-parametric and partially adaptive estimators of the censored regression model[iv] with possibly skewed and leptokurtic error distributions is discussed. The algorithm depicted by McDonald, Xu (1996, [14]) is a semi-parametric and partially adaptive estimators because it involves relationships between variables when the dependent variable is partially or completely unknown (censored) above or below a certain threshold.

A random-coefficient flexible parametric specification which nests probit and logit models [v] is introduced in McDonald (1996, [15]).

Wang et al (1997, [16]), Wang (1997, [17]), Wang et al. (1998, [18]) introduced a more general GARCH model, based on the EGB family of distributions, which can accommodate most nonnormal characteristics of data, including leptokurtosis, skewness, and high peakedness [vi], and yet remains tractable for estimation.

In Hwang (1998, [19]) it is claimed that for kinds of data, on stock returns and exchange rates, EGB is a better fit for the models with both asymmetry and long memory.

Ebens (1998, [20]) one density that allows for skewness and kurtosis is the EGB McDonald, Xu (1995, [13]). We used EGB density to estimate the ARFIMAX [vii] specification discussed below to model variances and standard deviations directly.

Barniv et al. (1999, [21]) EGB2 (EGB of the second kind) model provides significantly better classifications and predictions than logit[viii].

The paper by Miravete (1999, [23]) help explain the widespread use of tariff discounts embodied into tariff options in several monopolistic and competitive markets making the assumptions that the distribution is EGB2.

Mcdonald et al. (2000, [24]) address the problem of heteroskedasticity[ix] in comparison to some flexible parametric and semiparametric qualitative-response models.

According to Goss et al. (2000, [25]) EGB2 is less costly to misclassification of insurance insolvency than Discriminant Analysis[x] model.

I found the paper by Fischer (2000, [26]) to contain a wealth of EGB family of distribution models. According to Fischer (2000, [26]) the generalized logistic of type IV is also known as the EGB2 distribution. The 4-paramater EGB2 fails in modeling skewness greater than two and kurtosis greater than nine. To overcome this shortcoming Fischer (2000, [26], [27]) introduces an additional parameter that leads to the generalized EGB2 or FEGB2 or generalized logistic of type V.

The EGB3 distribution is a useful tool in engineering analysis, modeling, and simulation, particularly when dealing with variables that are bound and may exhibit skewed or non-normal distributions. Its flexibility stems from its ability to capture both symmetric and skewed data forms through its shape parameters, and it can serve as a robust alternative to the more commonly used exponential or beta distributions[xi].

The EGB3 model is useful in GPS analysis, modeling, and simulation due to its flexibility and ability to capture various data shapes. It is an extension of the beta distribution, offering more versatility in modeling compared to simpler distributions[xii].

The EGB3 distribution can be useful for modeling ionospheric distortion, particularly for modeling the behavior of random variables within a finite range (see Progri (2019, [3], [5])).

The EGB3 distribution can be useful for modeling tropospheric distortion, offering flexibility in capturing various shapes of distortion patterns. It can be particularly helpful when the data exhibits skewness or leptokurtosis, common in tropospheric models[xiii].

Other contributions coming from Kalbfleisch, Prentice (2002, [28]) on the role of Generalized Logistics of Type IV are included in the paper.

The EGB3 distribution can be useful for modeling multipath distributions Progri (2003, [29]; 2006, [31]; 2010, [33]) and Progri et al. (2004, [30]; 2007, [32]) distortions[xiv], particularly in scenarios where the data exhibits *skewness* and *non-standard* tail behavior. The EGB distribution offers flexibility in capturing various shapes and is a generalization of the standard Beta distribution, which is often used for modeling proportions and probabilities.

This journal publication offers a rather *unique, innovative, novel, and original approach of the utilization of the techniques* that originated in Progri (2003, [29]; 2006, [31]; 2010, [33]) Progri et al. (2004, [30]; 2007, [32]) were developed in Progri (2016, [2]; 2019, [3], [5]) which has led to the creation of *three original, novel, unique, and innovative reduction formulae of a Binomial expansion as the weighted sum of the two 2F1 hypergeometric functions* that perhaps did not exist before in the technical literature.

This paper is organized as follows: in Sect. 2 the computation of the EGB3 cdf is presented. In Sect. 3 generalized cdfs derived from EGB3 cdf are depicted. Analytical, conceptual results are deliberated in Section 4. Conclusion is provided in Sect. 5, Acknowledgment in Sect. 6, along with a list of references in Sect. 7. The derivation of the EGB3 cdf is provided in Appen. A in Sect. 8. In Appen. B: special cases of EGB CDF: EGB1 and EGB2 cdfs are displayed in Sect. 9. Appendix C is dedicated to the origin of EGB2 in Sect. 10.

2 The EGB3 CDF

In Progri (2024, [7]) the connection of the EGB3 pdf is provided in great detail. In Progri (2024, [8]) a detailed description of the nongeneralized cdfs connected with the EGB3 cdf was deliberated. In this section the computation of the EGBD cdf is performed. The EGB3 cdf in Sect. 2.1 and special cases of the EGB3 cdf: EGB1 and EGB2 in Sect. 2.2.

2.1 The EGB3 CDF

The complete derivation of the EGB3 [39] cdf [38] is provided in Appen. A (79).

$$F_{\mathrm{EGB}}(x;a,b,d,p,q)=\begin{vmatrix}F(x), & -\infty<x\le b' \\ 1, & b'<x<\infty\end{vmatrix} \tag{1}$$

where the function $F(x)$ is the integral of the pdf function $f(x)$.

Based on the explanations provided in Wang et al. (1996, [16]), Wang (1997, [17]), Wang et al. (1998, [18]), the parameter a reflects the peakedness of the density function, b is a defined as the location parameter that affects the mean of the distribution, d and p are shape parameters that together have a major influence in the *skewness* and *kurtosis* of the distribution, and the parameter q is greater than or equal to zero and less than or equal to one.

From Appen. A (78) we have obtained the derivations of the function $F(x)$ as the regularized incomplete beta function $\mathrm{I}_z(d,p)$, where z is given by Appen. A (75)

$$F(x)=\frac{\mathrm{B}(d,p;z)}{\mathrm{B}(d,p)}$$
$$=\mathrm{I}_z(d,p) \tag{2}$$

Next, substitute (2) into (1) yields

$$F_{\mathrm{EGB}}(x;a,b,d,p,q)=\begin{vmatrix}\mathrm{I}_z(d,p), & -\infty<x\le b' \\ 1, & b'<x<\infty\end{vmatrix} \tag{3}$$

Equation (3) is the general CFE of the cdf [38] of the EGB [39].

To show that this function goes from zero to one as the main variable x goes from minus infinity to plus infinity; i.e., it is *monotonically non-decreasing* function from zero to one is pretty straight forward from the definition of the incomplete Beta function as depicted in Appen. A.

2.2 Special Cases of the EGB3 CDF: EGB1 and EGB2

Of special interest are the EGB3 distributions of the first and second kind, EGB1 and EGB2, respectively, which correspond to the limiting values of q equal to zero and q equal to one and are alternative representations of the generalized exponential and generalized logistic distributions, respectively Wang et al (1996, [16]), Wang (1997, [17]), Wang et al (1998, [18]).

In Appen. B a brief discussion of the EGB1 and EGB2 is provided.

3 Generalized CDFs Derived from EGB CDF

The concept of generalized distributions[xv], particularly within the realm of generalized functions or distributions in mathematics, is primarily attributed to Laurent Schwartz, who formalized the theory in the late 1940s. However, the origins of these ideas can be traced back to earlier work in partial differential equations by Sergei Sobolev in the 1930s [48].

In Progri (2024, [7]) the connection of the generalized pdfs from the EGB3 pdf family of distribution is shown. And in Progri (2024, [8]) the nongeneralized cdfs from the EGB3 cdf family of distributions is perovided.

The EGB3 distribution includes also a number of generalized cdfs derived from the EGB3 cdf such as the generalized Gumbel of type 1 CDF in Sect. 3.1; the generalized Logistics of type 1 CDF in Sect. 3.2; The generalized Logistics of type 2 CDF in Sect. 3.3; The generalized Logistics of type 3 CDF in Sect. 3.4; and The Generalized Logistics of Type 4 CDF in Sect. 3.5.

3.1 The Generalized Gumbel of Type 1 CDF

The generalized Gumbel[xvi] (or type I Gumbel GGu [43]) cdf [38] can be obtained from substituting $x \to -x$ into the EW [40] cdf [38] as follows:

$$\begin{aligned} F_{\text{GGu}}(x; a, c) &= e^{-e^{-ax}e^{ba}}; a, b > 0 \\ &= e^{-ce^{-ax}}, a, c > 0 \\ &= e^{-ce^{-ax}}, a, c > 0 \\ &= [F_{\text{Gum}}(x; a)]^c; a, c > 0 \end{aligned} \tag{4}$$

The GGu [43] cdf [38] is an analytical CFE that involves the computation of only the exponential function for both positive and negative exponents. For an exhaustive deliberation on the computation of the exponential function the reader may refer to Progri (2022, [6]).

Bounded Property or Inequality[xvii]: Is GGu [43] cdf [38] a valid cdf? First, it is clear that

$$F_{\text{GGu}}(x; a, c) = [F_{\text{Gum}}(x; a)]^c \geq 0, -\infty < x < \infty, \left| \begin{matrix} a > 0 \\ c > 0 \end{matrix} \right. \tag{5}$$

Ditto comments on Progri (2024, [8] (3) and (36)).

$$F_{\text{GGu}}(x; a, c) = [F_{\text{Gum}}(x; a)]^c \leq 1, -\infty < x < \infty, \left| \begin{matrix} a > 0 \\ c > 0 \end{matrix} \right. \tag{6}$$

Ditto comments on Progri (2024, [8] (4) and (36)).

The proof of (5) and (6) is obtained easily if we consider Progri (2024, [8] (3), (4), (36), and (39))

Calculating Limits: Let us examine what happens when $x \to -\infty$ then we have

$$\begin{aligned} F_{\text{GGu}}(x \to -\infty; a, c) &= \lim_{x \to -\infty} [F_{\text{Gum}}(x; a)]^c \\ &= \left[\lim_{x \to -\infty} F_{\text{Gum}}(x; a)\right]^c \\ &= 0^c \\ &= 0 \end{aligned} \tag{7}$$

Ditto comments on Progri (2024, [8] (5) and (33)).

Next, we examine the following limit when $x \to \infty$ then we have

$$\begin{aligned} F_{\text{GGu}}(x \to \infty; a, c) &= \lim_{x \to \infty} [F_{\text{Gum}}(x; a)]^c \\ &= \left[\lim_{x \to \infty} F_{\text{Gum}}(x; a)\right]^c \\ &= 1^c \\ &= 1 \end{aligned} \tag{8}$$

Ditto comments on Progri (2024, [8] (6) and (34)).

The GGu [43] cdf [38] is also right continuous meaning

$$\begin{aligned} \lim_{x \to x^+} F_{\text{GGu}}(x; a, c) &= \left[\lim_{x \to x^+} F_{\text{Gum}}(x; a)\right]^c, \left| \begin{matrix} a > 0 \\ c > 0 \end{matrix} \right. \\ &= [F_{\text{Gum}}(x; a)]^c, \left| \begin{matrix} a > 0 \\ c > 0 \end{matrix} \right. \end{aligned} \tag{9}$$

Ditto comments on Progri (2024, [8] (7) and (35)).

Non-Decreasing Monotonicity: Next, for any Progri (2024, [8] (8)) we can show that since $F_{\text{Gum}}(x; a)$ is a *monotonically non-decreasing* function; i.e., Progri (2024, [8] (9)) holds; hence, we can obtain,

$$0 \leq [F_{\text{Gum}}(x_1; a)]^c \leq [F_{\text{Gum}}(x_2; a)]^c \leq 1 \tag{10}$$

Thus proving that

$$0 \leq F_{\text{GGu}}(x_1; a, c) \leq F_{\text{GGu}}(x_1; a, c) \leq 1 \tag{11}$$

We could have used a different approach to prove the monotonic nature of the GGu [43] cdf [38] by taking the derivative of (4) and showing that the derivative is equal to the GGu [43] pdf [37] Progri (2024, [7] (32)) always greater or equal to zero as shown in Progri (2024, [7] (33)).

In summary on the GGu [43] cdf [38] I have shown the CFE (4), bounding inequalities (5) and (6), the limits in (7), (8) and (9), and the monotonic nature in (10) ans (11), and I have shown the connection with Progri (2024, [7] (32), (33)) and Progri (2024, [8] (8), (9)).

This concludes the explicit discussion on the GGu [43] cdf [38]. Next, we continue with the description of the Generalized Logistic distribution of type I cdf.

3.2 The Generalized Logistics of Type 1 CDF

The Generalized Logistic distribution of type I (LDI) was introduced by Narayanan Balakrishnan and Patrick Leung (1988, [11]). They proposed and studied two generalized classes of logistic distributions, including LDI and another type, LDII.

The research paper by Balakrishnan and Leung (1988, [11]) in the journal Statistica Neerlandica introduced the LDI and LDII distributions. Their work aimed to generalize the standard logistic distribution, providing more flexibility for modeling various data shapes. The LDI (or we use the notation GL1) distribution, along with LDII, offers researchers and practitioners a wider range of options for modeling and analyzing data that may not perfectly fit the standard logistic distribution.

According to Fischer (2000, [26]), GL1 [47] cdf [38] can be obtained from the EGB3 [39] cdf [38] If we set the parameters a equal to one, b equal to zero, d equal to d, p equal to one, and q equal to one as follows:

$$\begin{aligned} F_{\mathrm{GL1}}(x;d) &= F_{\mathrm{EGB}}(x;a=1,b=0,d=d,p=1,q=1) \\ &= \int_{-\infty}^{x} \frac{e^{du}}{\mathrm{B}(d,1)(1+e^{u})^{d+1}}du \\ &= \int_{-\infty}^{x} \frac{\Gamma(d+1)e^{du}}{\Gamma(d)\Gamma(1)(1+e^{u})^{d+1}}du \\ &= \int_{-\infty}^{x} \frac{de^{du}}{(1+e^{u})^{d+1}}du \\ &= \frac{1}{(1+e^{-x})^{d}}, d>0 \end{aligned} \tag{12}$$

We can show the same thing in a slightly different form

$$\begin{aligned} F_{\mathrm{GL1}}(x;p) &= F_{\mathrm{EGB}}(x;a=1,b=0,d=d,p=1,q=1) \\ &= \frac{\int_{0}^{x'\equiv e^{-x}} v^{d-1}(1+e^{v})^{d+1}dv}{\mathrm{B}(d,1)} \\ &= \frac{\mathrm{B}\left(1,d;z\equiv\frac{x'}{1+x'}\right)}{\mathrm{B}(1,d)} \\ &= \frac{1}{(1+x')^{d}} \\ &= \frac{1}{(1+e^{-x})^{d}} \end{aligned} \tag{13}$$

The GL1 [47] cdf [38] is an analytical CFE (12) and (13) that involves the computation of only the exponential function for both positive and negative exponents. For a precise analysis on the computation of the exponential function the reader may refer to Progri (2022, [6]).

Bounded Property or Inequality[xviii]: Is GL1 [47] cdf [38] a valid cdf? First, it is clear that

$$F_{\mathrm{GL1}}(x;d) = \frac{1}{(1+e^{-x})^{d}} \geq 0, -\infty < x < \infty, d>0 \tag{14}$$

Ditto comments on Progri (2024, [8] (3) and (47)).

$$F_{\mathrm{GL1}}(x;d) = \frac{1}{(1+e^{-x})^{d}} \leq 1, -\infty < x < \infty, d>0 \tag{15}$$

Ditto comments on Progri (2024, [8] (4) and (47)).

Calculating Limits: Second, if we take the limit when $x \to \pm\infty$

$$\begin{aligned} \lim_{x\to-\infty} F_{\mathrm{GL1}}(x;d) &= \lim_{x\to-\infty} \frac{1}{(1+e^{-x})^{d}} \\ &= \frac{1}{\lim_{x\to-\infty}(1+e^{-x})^{d}} \\ &= \frac{1}{\infty} \\ &= 0 \end{aligned} \tag{16}$$

Ditto comments on Progri (2024, [8] (5) and (41)).

Next, we examine the following limit when $x \to \infty$ then we have

$$\begin{aligned} \lim_{x\to+\infty} F_{\mathrm{GL1}}(x;d) &= \lim_{x\to+\infty} \frac{1}{(1+e^{-x})^{d}} \\ &= \frac{1}{\lim_{x\to+\infty}(1+e^{-x})^{d}} \\ &= \frac{1}{1} \\ &= 1 \end{aligned} \tag{17}$$

Ditto comments on Progri (2024, [8] (6) and (42)).

The GL1 [47] cdf [38] is also right continuous meaning

$$\begin{aligned} \lim_{x\to x^{+}} F_{\mathrm{GL1}}(x;d) &= \lim_{x\to x^{+}} (1+e^{-x})^{-d}, d>0 \\ &= (1+e^{-x})^{-d}, d>0 \end{aligned} \tag{18}$$

Ditto comments on Progri (2024, [8] (7) and (43)).

Non-Decreasing Monotonicity: Third, we need to prove that

$F_{\mathrm{GL1}}(x;d)$ is a *monotonically non-decreasing* function. For that we assume that for any Progri (2024, [8] (8)) we have Progri (2024, [8] (9)) for a equal to one

$$\infty > e^{-x_1} \geq e^{-x_2} \geq 0, a > 0 \tag{19}$$

Adding one on both sides of (19) produces

$$\infty > 1 + e^{-x_1} \geq 1 + e^{-x_2} > 1 \tag{20}$$

Raising into power of *d* (20) yields

$$\infty > (1 + e^{-x_1})^d \geq (1 + e^{-x_2})^d > 1, d > 0 \tag{21}$$

After taking the inverse of (21) we obtain the following

$$\frac{1}{\infty} < \frac{1}{(1+e^{-x_1})^d} \leq \frac{1}{(1+e^{-x_2})^d} < 1 \tag{22}$$

which is equivalent to the desired result

$$0 < F_{\mathrm{GL1}}(x;d) \leq F_{\mathrm{GL1}}(x;d) < 1 \tag{23}$$

Equations (19) through (22) complete the proof of (23).

We could have used a different approach to prove the monotonic nature of the GL1 [47] cdf [38] by taking the derivative of (12) or (13) and showing that the derivative is equal to the GL1 [47] pdf [37] Progri (2024, [7] (36)) always greater or equal to zero as shown in Progri (2024, [7] (37)).

In summary on the GL1 [47] cdf [38] I have shown the CFE (12), (13), bounded inequalities (14), (15) the limits in (16) through (18), and the monotonic nature in (19) through (23), and I have shown the connection with Progri (2024, [7] (36), (37)) and Progri (2024, [8] (8), (9)).

This concludes the specific discussion on the GL1 [47] cdf [38]. Next, we provide an explicit description of the Generalized Logistics of Type 2 cdf.

3.3 The Generalized Logistics of Type 2 CDF

The Generalized Logistic distribution of Type II was introduced by Gupta and Kundu (2010, [34]). This distribution is a generalization of the standard logistic distribution and is defined as a member of the proportional reversed hazard family with the logistic distribution as its baseline.

The GL of type 2 (or GL2) [47] cdf [38] can be obtained from the EGB3 cdf [38] for a particular set of parameters such as *a* equal to one, *b* equal to zero, *d* equal to one, *p* equal to *d*, and *q* equal to one as follows:

$$\begin{aligned}
F_{\mathrm{GL2}}(x;d) &= F_{\mathrm{EGB}}(x; a=1, b=0, d=1, p=d, q=1) \\
&= \int_{-\infty}^{x} \frac{de^u}{(1+e^u)^{d+1}} du \\
&= \int_{-\infty}^{x} \frac{d}{(1+e^u)^{d+1}} de^u; u \to |_{-\infty}^{x} \\
&= \int_{0}^{e^x} \frac{d}{(1+z)^{d+1}} d(z \equiv e^u); z \to |_{0}^{e^x} \\
&= d \int_{0}^{e^x} \frac{1}{(1+z)^{d+1}} dz \\
&= d \int_{1}^{e^x+1} w^{-d-1} d(w \equiv z+1); w \to |_{1}^{e^x+1} \\
&= -w^{-d}|_{1}^{e^x+1}; d > 0 \\
&= 1 - (1+e^x)^{-d}; d > 0 \\
&= 1 - \frac{e^{-dx}}{(1+e^{-x})^d}; d > 0
\end{aligned} \tag{24}$$

The GL2 [47] cdf [38] is an analytical CFE (24) that involves the computation of only the exponential function for both positive and negative exponents. For a precise analysis on the computation of the exponential function the reader may refer to Progri (2022, [6]).

Bounded Property or Inequality: Is GL2 [47] cdf [38] a valid cdf [38]? First, it is clear that

$$F_{\mathrm{GL2}}(x;d) = 1 - (1+e^x)^{-d} \geq 0, -\infty < x < \infty, d > 0 \tag{25}$$

Ditto comments on Progri (2024, [8] (3) and (47)).

$$F_{\mathrm{GL2}}(x;d) = 1 - (1+e^x)^{-d} \leq 1, -\infty < x < \infty, d > 0 \tag{26}$$

Ditto comments on Progri (2024, [8] (4) and (47)).

Calculating Limits: Second, if we take the limit when $x \to \pm\infty$

$$\begin{aligned}
\lim_{x\to-\infty} F_{\mathrm{GL2}}(x;d) &= \lim_{x\to-\infty} [1 - (1+e^x)^{-d}] \\
&= 1 - \lim_{x\to-\infty} [(1+e^x)^{-d}] \\
&= 1 - \left(1 + \lim_{x\to-\infty} e^x\right)^{-d} \\
&= 1 - (1+0)^{-d} \\
&= 1 - 1 \\
&= 0
\end{aligned} \tag{27}$$

Ditto comments on Progri (2024, [8] (5) and (41)).

Next, we examine the following limit when $x \to \infty$ then we have

$$\begin{aligned}
\lim_{x\to+\infty} F_{\mathrm{GL2}}(x;d) &= \lim_{x\to+\infty} [1 - (1+e^x)^{-d}] \\
&= 1 - \lim_{x\to+\infty} (1+e^x)^{-d} \\
&= 1 - \left(1 + \lim_{x\to+\infty} e^x\right)^{-d} \\
&= 1 - (1+\infty)^{-d} \\
&= 1 - \frac{1}{\infty} \\
&= 1 - 0
\end{aligned}$$

$$= 1 \quad (28)$$

Ditto comments on Progri (2024, [8] (6) and (42)).

The GL2 [47] cdf [38] is also right continuous meaning

$$\lim_{x \to x^+} F_{\mathrm{GL2}}(x; d) = \lim_{x \to x^+} 1 - (1 + e^x)^{-d}, d > 0$$

$$= 1 - (1 + e^x)^{-d}, d > 0 \quad (29)$$

Ditto comments on Progri (2024, [8] (7) and (43)).

Non-Decreasing Monotonicity: Third, we need to prove that $F_{\mathrm{GL2}}(x; d)$ is a *monotonically non-decreasing* function. For that we assume that for any Progri (2024, [8] (8)) we have Progri (2024, [8] (9)) for c equal to one

$$0 \le e^{x_1} \le e^{x_2} < \infty \quad (30)$$

Adding one on both sides of (30) produces

$$1 \le 1 + e^{x_1} \le 1 + e^{x_2} < \infty \quad (31)$$

Raising into power of d (31) yields

$$1 \le (1 + e^{x_1})^d \le (1 + e^{x_2})^d < \infty \quad (32)$$

Taking the inverse of (32) we obtain the following:

$$\frac{1}{\infty} < (1 + e^{x_2})^{-d} \le (1 + e^{x_1})^{-d} \le 1 \quad (33)$$

Multiplying both sides of (33) with minus one yields:

$$-1 < -(1 + e^{x_1})^{-d} \le -(1 + e^{x_2})^{-d} \le 0 \quad (34)$$

Finally, adding one on both sides of (34) produces:

$$1 - 1 < 1 - (1 + e^{x_1})^{-d} \le 1 - (1 + e^{x_2})^{-d} \le 1 + 0 \quad (35)$$

Which is equivalent to the desired answer:

$$0 < 1 - (1 + e^{x_1})^{-d} \le 1 - (1 + e^{x_2})^{-d} \le 1 \quad (36)$$

Equations (30) through (36) complete the proof of the monotonicity of $F_{\mathrm{GL2}}(x; d)$.

We could have used a different approach to prove the monotonic nature of the GL2 [47] cdf [38] by taking the derivative of (24) and showing that the derivative is equal to the GL2 [47] pdf [37] Progri (2024, [7] (40)) always greater or equal to zero as shown in Progri (2024, [7] (41)).

In summary on the GL2 [47] cdf [38] I have shown the CFE (24), bounded inequalities (25) and (26), the limits in (27) through (29), and the monotonic nature in (30) through (36), and I have shown the connection with Progri (2024, [7] (40), (41)) and Progri (2024, [8] (8), (9)).

This concludes the detailed discussion on the GL2 [47] cdf [38]. Next, follows a specific description of the Generalized Logistic distribution of Type 3 cdf.

3.4 The Generalized Logistics of Type 3 CDF

The Generalized Logistic distribution of Type III[xix] (or GL3) was not invented by a single person but was proposed by Balakrishnan and Leung (1988, [11]). They introduced the concept of generalized logistic distributions, including Type I, II, and III. Johnson et al. (1995, [12]) later summarized several generalizations of the logistic distribution, including Type III.

For example, for a equal to one, b equal to zero, d equal to d, p equal to d, and q equal to one then EGB [39] cdf [38] becomes GL of type 3 (or GL3) [47] cdf [38] as follows:

$$F_{\mathrm{Gl3}}(x; d) = F_{\mathrm{EGB}}(x; a = 1, b = 0, d, p = d, q = 1)$$

$$= \frac{\int_0^{x' \equiv e^x} v^{d-1}(1+e^v)^{-2d} dv}{\mathrm{B}(d,d)}$$

$$= \frac{\mathrm{B}\left(d,d; z \equiv \frac{x'}{1+x'}\right)}{\mathrm{B}(d,d)}$$

$$= \mathrm{I}_z(d,d)$$

$$= \frac{z^d[F_1(d,z) \equiv {}_2F_1[d,1-d;d+1;z]]}{d\mathrm{B}(d,d)} \quad (37)$$

The GL3 [47] cdf [38] is an analytical CFE (37) that involves the computation of the power function, exponential function for both positive and negative exponents, hypergeometric function (see Progri (2019, [4])), and Beta function. For an accurate analysis on the computation of the exponential function the reader may refer to Progri (2022, [6]).

Bounded Property or Inequality: Is GL3 [47] cdf [38] a valid cdf [38]? First, it is clear that

$$F_{\mathrm{GL3}}(x; d) = \frac{z^d F_1(d,z)}{d\mathrm{B}(d,d)} \ge 0, \left| \begin{matrix} -\infty < x < \infty \\ d > 0 \end{matrix} \right., \quad (38)$$

Ditto comments on Progri (2024, [8] (3) and (47)).

$$F_{\mathrm{GL3}}(x; d) = \frac{z^d F_1(d,z)}{d\mathrm{B}(d,d)} \le 1, \left| \begin{matrix} -\infty < x < \infty \\ d > 0 \end{matrix} \right. \quad (39)$$

Ditto comments on Progri (2024, [8] (4) and (47)).

Calculating Limits: Second, if we take the limit when $x \to \pm\infty$

$$\lim_{x \to -\infty} F_{\mathrm{GL3}}(x; d) = \lim_{x \to -\infty} \left[\frac{z^d F_1(d,z)}{d\mathrm{B}(d,d)}\right]$$

$$= \frac{\lim_{x \to -\infty} [z^d F_1(d,z)]}{d\mathrm{B}(d,d)}$$

$$= \frac{\lim_{x \to -\infty} [z^d F_1(d,z)]}{d\mathrm{B}(d,d)}$$

$$= \frac{0^d F_1(d,0)}{d\mathrm{B}(d,d)}$$

$$= \frac{0 \times 1}{d\mathrm{B}(d,d)}$$

$$= 0 \quad (40)$$

Ditto comments on Progri (2024, [8] (5) and (41)).

Next, we examine the following limit when $x \to \infty$ then we have

$$\lim_{x\to+\infty} F_{\mathrm{GL3}}(x;d) = \lim_{x\to+\infty}\left[\frac{z^d F_1(d,z)}{d\mathrm{B}(d,d)}\right]$$

$$= \frac{\lim_{x\to+\infty}[z^d F_1(d,z)]}{d\mathrm{B}(d,d)}$$

$$= \frac{1^d F_1(d,1)}{d\mathrm{B}(d,d)}$$

$$= \frac{\frac{\Gamma(d+1)\Gamma(d+1-d-1+d)}{\Gamma(d+1-d)\Gamma(d+1-1+d)}}{d\mathrm{B}(d,d)}$$

$$= \frac{\frac{\Gamma(d+1)\Gamma(d)}{\Gamma(1)\Gamma(2d)}}{d\mathrm{B}(d,d)}$$

$$= \frac{\frac{\Gamma(d+1)\Gamma(d)}{\Gamma(2d)}}{d\frac{\Gamma(d)\Gamma(d)}{\Gamma(2d)}}$$

$$= 1 \tag{41}$$

Ditto comments on Progri (2024, [8] (6) and (42)).

The GL3 [47] cdf [38] is also right continuous meaning

$$\lim_{x\to x^+} F_{\mathrm{GL3}}(x;d) = \lim_{z\to z^+}\frac{z^d[F_1(d,z)]}{d\mathrm{B}(d,d)},\ d>0$$

$$= \frac{z^d[F_1(d,z)]}{d\mathrm{B}(d,d)},\ d>0 \tag{42}$$

Ditto comments on Progri (2024, [8] (7) and (43)).

Non-Decreasing Monotonicity: Third, we need to prove that $F_{\mathrm{GL3}}(x;d)$ is a *monotonically non-decreasing* function. For that we need to prove that the derivative of the numerator of $F_{\mathrm{GL3}}(x;d)$ is always positive:

$$\frac{\partial\{z^d F_1(d,z)\}}{\partial x} = dz^{d-1}\frac{\partial z}{\partial x}F_1(d,z) + z^d\frac{\partial F_1(d,z)}{\partial x} \tag{43}$$

Next, we take the derivative of the hypergeometric function Progri (2016, [1], [2]; 2019, [3], [5])

$$\frac{\partial F_1(d,z)}{\partial x} = \frac{d(1-d)}{d+1}\left\{F_2(d,z) \equiv {}_2F_1\left[\begin{matrix}d+1,2-d; \\ d+2;\end{matrix} z\right]\right\}\frac{\partial z}{\partial x} \tag{44}$$

Next, substituting (44) into (43) produces

$$\frac{\partial\{z^d F_1(d,z)\}}{\partial x} = dz^{d-1}\frac{\partial z}{\partial x}\left[F_1(d,z) + \frac{(1-d)z}{d+1}F_2(d,z)\right] \tag{45}$$

We have to compute the partial derivative $\partial z/\partial x$ as follows:

$$\frac{\partial z}{\partial x} = \frac{e^x(1+e^x)-e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} = \frac{z}{1+e^x} \tag{46}$$

Next, substituting (46) into (45) produces:

$$\frac{\partial\{z^d F_1(d,z)\}}{\partial x} = \frac{dz^d}{1+e^x}\left[F_1(d,z) + \frac{(1-d)z}{d+1}F_2(d,z)\right] \tag{47}$$

Next, it remains to compute the derivative of $F_{\mathrm{Gl3}}(x;d)$ as follows:

$$\frac{\partial F_{\mathrm{Gl3}}(x;d)}{\partial x} = \frac{\partial}{\partial x}\left[\frac{z^d F_1(d,z)}{d\mathrm{B}(d,d)}\right]$$

$$= \frac{1}{d\mathrm{B}(d,d)}\frac{\partial[z^d F_1(d,z)]}{\partial x}$$

$$= \frac{1}{d\mathrm{B}(d,d)}\frac{dz^d}{(1+e^x)}\left[F_1(d,z) + \frac{(1-d)z}{d+1}F_2(d,z)\right]$$

$$= \frac{z^d}{\mathrm{B}(d,d)}\frac{F_1(d,z)+\frac{(1-d)z}{d+1}F_2(d,z)}{(1+e^x)} \tag{48}$$

On the other hand from Progri (2024, [7] (44)) we should obtain the following:

$$\frac{\partial F_{\mathrm{Gl3}}(x;d)}{\partial x} = \frac{e^{dx}}{\mathrm{B}(d,d)(1+e^x)^{2d}}$$

$$= \frac{z^d}{\mathrm{B}(d,d)(1+e^x)^d} \geq 0 \tag{49}$$

Since (48) and (49) must be identical that leaves the following:

$$\frac{F_1(d,z)+\frac{(1-d)z}{d+1}F_2(d,z)}{(1+e^x)} = \frac{1}{(1+e^x)^d}$$

$$= (1+e^x)^{-d} \tag{50}$$

Next, we obtain an original, novel, or innovative CFE that perhaps is not published anywhere else by applying the techniques that were developed in Progri (2016, [1], [2]; 2019, [3], [5]) as follows:

$$F_1(d,z) + \frac{(1-d)z}{d+1}F_2(d,z) = \frac{(1+e^x)}{(1+e^x)^d}$$

$$= \left(\frac{1}{1+e^x}\right)^{d-1}$$

$$= \left(\frac{1+e^x-e^x}{1+e^x}\right)^{d-1}$$

$$= \left(1-\frac{e^x}{1+e^x}\right)^{d-1}$$

$$= (1-z)^{d-1} \tag{51}$$

Finally, the last proof we need to show is that if we start from the left side of (51) we will indeed arrive at the right side of (51). This proof is shown in a short journal letter Progri (2024, [9]) since this proof is really outside the scope of this journal paper.

In summary on the GL3 [47] cdf [38] I have shown the CFE (37), bounded inequalities (38) and (39), the limits in (40) through (42), and the monotonic nature in (43) through (49), an original expression of the binomial expansion (50) and (51), and I have shown the connection with Progri (2024, [7] (40), (41)) and Progri (2024, [8] (8), (9)).

This concludes with the intricate discussion on the GL3 cdf. Next, we elaborate on the intricate discussion of the Generalized Logistics of Type 4 cdf.

3.5 The Generalized Logistics of Type 4 CDF

The Type IV Generalized Logistic (GLIV or GL4) distribution was proposed by Richard Prentice (1976, [10]), Kalbfleisch, Prentice (2002, [28]) as indicted by Nassar, Elmasry (2012, [36]). Prentice (1976, [10]) introduced GL4 as an extended distribution to model binary response data under the usual symmetric logistic distribution. The GL4 distribution is also known as the beta generalized logistic distribution.

For example, for the set of parameters *a* equal to one, *b* equal to zero, and *q* equal to one, EGB3 [39] cdf [38] becomes GL4 [47] cdf [38] as follows:

$$F_{\mathrm{GI4}}(x;d,p) = F_{\mathrm{EGB}}(x;a=1,b=0,d,p,q=1)$$
$$= \frac{\int_0^{x'\equiv e^{-x}} \frac{v^{d-1}}{(1+e^v)^{d+p}}dv}{\mathrm{B}(d,p)};\, d,p>0$$
$$= \frac{\mathrm{B}\left(d,p;z\equiv\frac{x'}{1+x'}\right)}{\mathrm{B}(d,p)}$$
$$= \mathrm{I}_z(d,p)$$
$$= \frac{z^d\{F_1(d,p,z)\equiv {}_2F_1[d,1-p;1+d;z]\}}{d\mathrm{B}(d,p)} \tag{52}$$

The GL4 [47] cdf [38] is an analytical CFE (52) that involves the computation of the power function, exponential function for both positive and negative exponents, hypergeometric function (see Progri (2019, [4])), and Beta function. For an accurate analysis on the computation of the exponential function the reader may refer to Progri (2022, [6]).

Bounded Property or Inequality: Is GL4 [47] cdf [38] a valid cdf [38]? First, it is clear that

$$F_{\mathrm{GL4}}(x;d,p) = \frac{z^d F_1(d,p,z)}{d\mathrm{B}(d,p)} \geq 0, \begin{vmatrix} -\infty < x < \infty \\ d,p>0 \end{vmatrix}, \tag{53}$$

Ditto comments on Progri (2024, [8] (3) and (47)).

$$F_{\mathrm{GL4}}(x;d,p) = \frac{z^d F_1(d,p,z)}{d\mathrm{B}(d,p)} \leq 1, \begin{vmatrix} -\infty < x < \infty \\ d,p>0 \end{vmatrix} \tag{54}$$

Ditto comments on Progri (2024, [8] (4) and (47)).

Calculating Limits: Second, if we take the limit when $x \to \pm\infty$ of $F_{\mathrm{GL4}}(x;d,p)$ we obtain

$$\lim_{x\to-\infty} F_{\mathrm{GL4}}(x;d,p) = \lim_{x\to-\infty}\left[\frac{z^d F_1(d,p,z)}{d\mathrm{B}(d,p)}\right]$$
$$= \frac{\lim_{x\to-\infty}[z^d F_1(d,p,,z)]}{d\mathrm{B}(d,p)}$$
$$= \frac{\lim_{x\to-\infty}[z^d F_1(d,p,z)]}{d\mathrm{B}(d,p)}$$
$$= \frac{0^d F_1(d,p,0)}{d\mathrm{B}(d,p)}$$
$$= \frac{0\times 1}{d\mathrm{B}(d,p)}$$
$$= 0 \tag{55}$$

Ditto comments on Progri (2024, [8] (5) and (41)).

Next, we examine the following limit when $x\to\infty$ then we have

$$\lim_{x\to+\infty} F_{\mathrm{GL4}}(x;d,p) = \lim_{x\to+\infty}\left[\frac{z^d F_1(d,p,z)}{d\mathrm{B}(d,p)}\right]$$
$$= \frac{\lim_{x\to+\infty}[z^d F_1(d,p,z)]}{d\mathrm{B}(d,p)}$$
$$= \frac{1^p F_1(d,p,1)}{d\mathrm{B}(d,p)}$$
$$= \frac{\frac{\Gamma(d+1)\Gamma(d+1-d-1+p)}{\Gamma(d+1-d)\Gamma(p+1-1+d)}}{d\mathrm{B}(d,p)}$$
$$= \frac{\frac{\Gamma(d+1)\Gamma(p)}{\Gamma(1)\Gamma(p+d)}}{d\mathrm{B}(d,p)}$$
$$= \frac{\frac{d\Gamma(d)\Gamma(p)}{\Gamma(1)\Gamma(p+d)}}{d\frac{\Gamma(d)\Gamma(p)}{\Gamma(d+p)}}$$
$$= 1 \tag{56}$$

Ditto comments on Progri (2024, [8] (6) and (42)).

The GL4 [47] cdf [38] is also right continuous meaning

$$\lim_{x\to x^+} F_{\mathrm{GL4}}(x;d,p) = \lim_{z\to z^+}\frac{z^d[F_1(d,p,z)]}{d\mathrm{B}(d,p)},\, d>0$$
$$= \frac{z^d[F_1(d,p,z)]}{d\mathrm{B}(d,p)},\, d>0 \tag{57}$$

Ditto comments on Progri (2024, [8] (7) and (43)).

Non-Decreasing Monotonicity: Third, we need to prove that $F_{\mathrm{GL4}}(x;d,p)$ is a *monotonically non-decreasing* function. For that we need to prove that the derivative of the numerator of $F_{\mathrm{GL4}}(x;d,p)$ is always positive:

$$\frac{\partial\{z^d F_1(d,p,z)\}}{\partial x} = dz^{d-1}\frac{\partial z}{\partial x}F_1(d,p,z) + z^d\frac{\partial F_1(d,p,z)}{\partial x} \tag{58}$$

Next, we take the derivative of the hypergeometric function

$$\frac{\partial F_1(d,p,z)}{\partial x} = \frac{d(1-p)}{d+1}\left\{F_2(d,p,z) \equiv {}_2F_1\begin{bmatrix} d+1,2-p; \\ d+2; \end{bmatrix} z\right\}\frac{\partial z}{\partial x} \tag{59}$$

Next, substituting (59) into (58) produces

$$\frac{\partial\{z^p F_1(d,p,z)\}}{\partial x} = dz^{d-1}\frac{\partial z}{\partial x}\left[F_1(d,p,z) + \frac{(1-p)z}{d+1}F_2(d,p,z)\right] \tag{60}$$

Next, substituting (46) into (60) produces:

$$\frac{\partial\{z^d F_1(d,p,z)\}}{\partial x} = \frac{dz^d}{1+e^x}\left[F_1(d,p,z) + \frac{(1-p)z}{d+1}F_2(d,p,z)\right] \tag{61}$$

Next, it remains to compute the derivative of $F_{\mathrm{GI4}}(x;d,p)$ Progri (2016, [1], [2]; 2019, [3], [5]) as follows:

$$\frac{\partial F_{\mathrm{GI4}}(x;d,p)}{\partial x} = \frac{\partial}{\partial x}\left[\frac{z^d F_1(d,p,z)}{d\mathrm{B}(d,p)}\right]$$

$$= \frac{1}{d\mathrm{B}(d,p)} \frac{\partial[z^d F_1(d,p,z)]}{\partial x}$$

$$= \frac{1}{d\mathrm{B}(d,p)} \frac{dz^d}{(1+e^x)} \left[F_1(d,p,z) + \frac{(1-p)z}{d+1} F_2(d,p,z)\right]$$

$$= \frac{z^d}{\mathrm{B}(d,p)} \frac{F_1(d,p,z)+\frac{(1-p)z}{d+1}F_2(d,p,z)}{(1+e^x)} \quad (62)$$

On the other hand from Progri (2024, [7] (48)) we should obtain the following:

$$\frac{\partial F_{\mathrm{GL4}}(x;d,p)}{\partial x} = \frac{e^{dx}}{\mathrm{B}(d,p)(1+e^x)^{d+p}}$$

$$= \frac{z^d}{\mathrm{B}(d,p)(1+e^x)^p} \geq 0 \quad (63)$$

Since (62) and (63) must be identical that leaves the following:

$$\frac{F_1(d,p,z)+\frac{(1-p)z}{d+1}F_2(d,p,z)}{(1+e^x)} = \frac{1}{(1+e^x)^p}$$

$$= (1+e^x)^{-p} \quad (64)$$

Next, we obtain an original, novel, or innovative CFE that perhaps is not published anywhere else by applying the techniques that were developed in Progri (2016, [1], [2]; 2019, [3], [5]) as follows:

$$F_1(d,p,z) + \frac{(1-p)z}{d+1} F_2(d,p,z) = \frac{(1+e^x)}{(1+e^x)^p}$$

$$= \left(\frac{1}{1+e^x}\right)^{p-1}$$

$$= \left(\frac{1+e^x-e^x}{1+e^x}\right)^{p-1}$$

$$= \left(1 - \frac{e^x}{1+e^x}\right)^{p-1}$$

$$= (1-z)^{p-1} \quad (65)$$

Finally, the last proof we need to show is that if we start from the left side of (65) we will indeed arrive at the right side of (65). This proof of (65) is shown in a short journal letter Progri (2024, [9]) since this proof is really outside the scope of this journal paper.

Equation (65) is a generalized CFE of (51). It is no surprise that when d equal to p, (65) becomes identical to (51).

In summary on the GL4 [47] cdf [38] I have shown the CFE (52), bounded inequalities (53) and (54), the limits in (55) through (57), and the monotonic nature in (58) through (63), an original expression of the binomial expansion (64) and (65), and I have shown the connection with Progri (2024, [7] (40), (41)) and Progri (2024, [8] (8), (9)).

This concludes the intricate discussion on the generalized logistics of type 4 cdf. Next, we start a specific investigation into the analytical, conceptual results.

4 Analytical, Conceptual Results

The analytical, computational research results work builds on the solid foundation that was developed in Progri (2024, [7], [8]). Therefore, in this section significant improvements are shown in Sect. 4.1, description of MATLAB simulation functions in Sect. 4.2, examples test setup in Sect. 4.3, and finally graphical results in Sect. 4.4.

4.1 Significant Improvement

Significant improvements required to perform the analytical, computational research results work are already illustrated in fine elements in both Progri (2024, [7], [8]).

Some refinements of the computation of the hypergeometric function and of the additional MATLAB simulation functions are provided in the following section.

4.2 Description of MATLAB Simulation Functions

In order to obtain the MATLAB simulations of this journal paper we prepared a rather detailed MATLAB simulation test setup that is illustrated in minute characteristics in Progri (2024, [7], [8]).

The MATLAB simulation consists of six additional functions (not included in in Progri (2024, [7], [8])) which are explained in order of from slow to fast, then from sharpest to least sharp which is explained in greater detail in Sect. 4.3. The function:

1. exgenbeta3cdf(a,b,d,p,q,x-b_0) is six parameter function for computing the EGB3 cdf based on Progri (2016, [1]) and Appen. A Sect. 8.
2. genggucdf(a,b,x-b_0) is the Generalized Gumbel of type I cdf as indicated in Sect. 3.1.
3. genlog3cdf(d,x-b_0) is the Generalized Logistic of type III cdf as indicated in Sect. 3.4.
4. genlog1cdf(d,x-b_0) computed the Generalized Logistic of type I cdf as indicated in Sect. 3.2.
5. genlog2cdf(d,x-b_0) performs the computation of the Generalized Logistic of type II cdf as indicated in Sect. 3.3.
6. genlog4cdf(d,p,x-b_0) is the Generalized Logistic of type IV cdf as indicated in Sect. 3.5.

The parameter b_0 is defined as the dc offset for the value of the cdf to be equal to one half. As indicated later in Sect. 4.3, it is expected that b_0; so as the value of the cdf will be equal to half.

Of all six functions were built as part of the Giftet Indoor Geolocation Library Progri (2024, [52]).

This concludes the description of MATLAB simulation functions. Next, we consider the analytical, conceptual examples.

4.3 Examples Test Setup

The setup for the numerical examples is exactly the same as the one in Qualitative, Quantitative results in Progri (2024, [7], [8]) and Analytical, Conceptual results in Progri (2024, [8]).

We performed eight MATLAB simulation examples one for every generalized cdf plus three for the EGB3 model. There inputs for the simulation examples are already provided in Qualitative, Quantitative results in Progri (2024, [7]) and Analytical, Conceptual results in Progri (2024, [8]). The outputs are based on the computation of the MATLAB functions in Sect. 4.2. There are a total of four outputs for each simulation example:

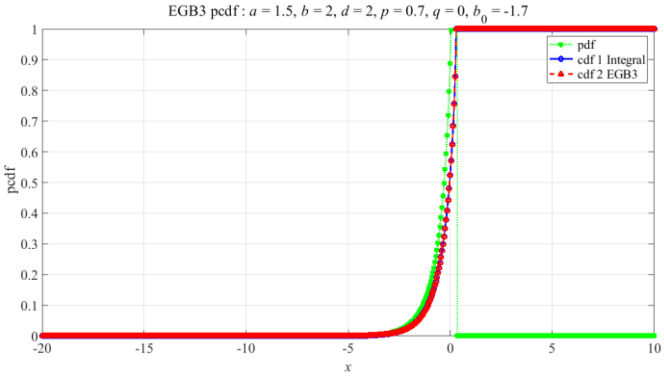
1. The first output is the direct computation of pdf based on the description of Progri (2024, [7] Sect. 5.1).
2. The second output is the computation of the MATLAB BIF integral from the integration of the pdf function Progri (2024, [7]) corresponding to the CFE of the cdf function in this journal paper.
3. The third output is the computation of the MATLAB functions based on the description in Sect. 4.2.
4. The third output if the difference of the second output with the third output.

4.4 Graphical Results

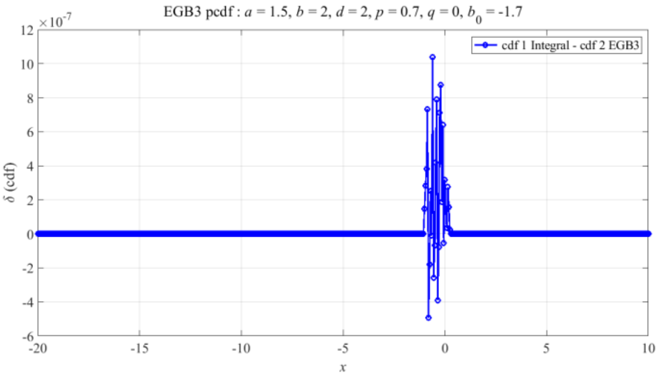
Graphical results (like MATAB plots) are used to visually represent data, making it easier to understand and analyze trends, patterns, and relationships that might be difficult to discern from raw numerical data alone. They provide a clear, concise, and often more intuitive way to convey information.

Facilitates Understanding: MATAB plots translate analytical CFE into easily digestible visual formats. They make it easier to grasp the overall shape, spread, and distribution of the data. Visuals are often more effective than text in conveying information, especially all the properties of the cdfs discussed in Sect. 3 and Appen. A Sect. 8.

Reveals Fundamental Properties: MATAB plots can highlight all the properties of the cdfs such as bounded inequalities, computation of the limits and of the limit from the right, the nondecreasing monotonicity that might not be immediately apparent in raw data. For example, a MATLAB



(a) The computation of EGB3 pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 0.7$, $q = 0$, and $b_0 = -1.7$. Cdf 1 Integral, Cdf 2 EGB3 cdf.



(b) The computation of EGB3 cdf error for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 0.7$, $q = 0$, and $b_0 = -1.7$. Cdf 1 Integral minus Cdf 2 EGB3 cdf.

Fig. 1. The computation of EGB3 pdf, cdf, and cdf error for $-20 \leq x \leq 10$.

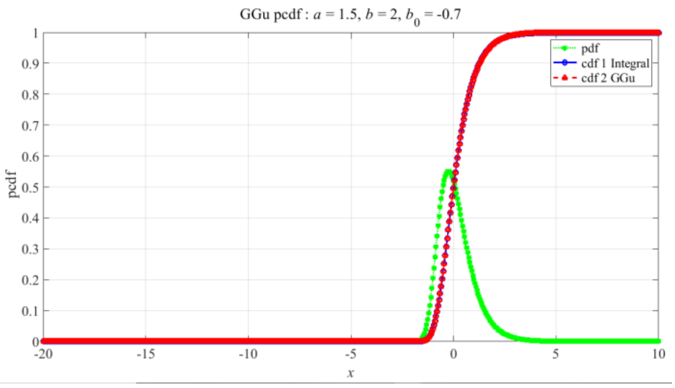
Plot can visually illustrate all the properties that we just enumerated.

Simplifies Comparisons: MATAB plots can make it easier to compare and contrast different generalized cdfs derived from the EGB3 cdf. For instance, although the shapes of different generalized cdfs derived from the EGB3 cdf might be different but their fundamental nature is expected to be the same.

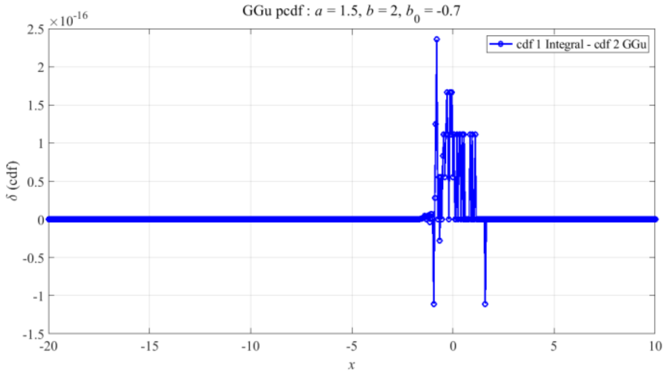
Reinforces Key Points: MATAB plots can be used to emphasize specific points or conclusions. MATAB plots can be used to visually support the fundamental analytical, conceptual findings of the cdfs, making them more persuasive and memorable.

Saves Time: MATAB plots can condense large amounts of information into a single, easily accessible plot. This allows users to quickly scan and understand the key insights without having to wade through detailed numerical data.

Improve Communication: MATAB plots can be used to communicate complex information to a wider audience,



(a) The computation of the GGu pdf and cdf for $-20 \le x \le 10$, $a = 1.5$, $b = 2$, $b_0 = -0.7$. Cdf 1 Integral, Cdf 2 GGu cdf.



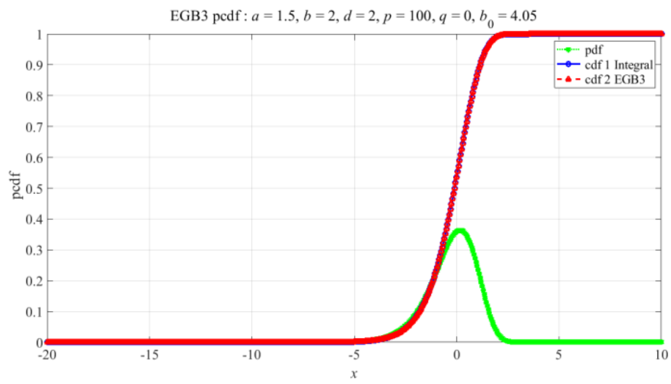
(b) The computation of the GGu cdf error for $-20 \le x \le 10$, $a = 1.5$, $b = 2$, $b_0 = -0.7$. Cdf 1 Integral minus Cdf 2 GGu cdf.

Fig. 2. The computation of GGu pdf, cdf, and cdf error for $-20 \le x \le 10$.

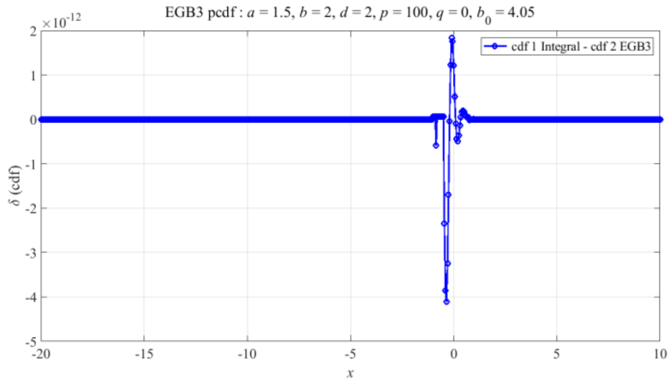
including those who may not be statistically savvy. They can help to make data more accessible and engaging, fostering better understanding and communications.

Figure 1 (a) depicts the computation of the EGB3 equal to EGB1 pdf and cdf for $-20 \le x \le 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 0.7$, $q = 0$, and $b_0 = -1.7$. Cdf 1 Integral, Cdf 2 EGB3 cdf. Is the EGB3 cdf a valid cdf? Certainly, the EGB3 cdf is always greater than or equal to zero and less than or equal to one and a nondecreasing monotonic function for all the values of the main variable from minus twenty to plus ten. The only reason why I have shown the plots of the Cdf 2 EGB3 pdf is to prove numerically that in fact the derivative of the EGB3 cdf is in fact a valid pdf. Another observation that is made about EGB3 cdf is that it produces very sharp growth in comparison to other distribution models investigated in this journal paper.

Figure 1(b) shows the error between the Cdf 1 Integral and the Cdf 2 EGB3 cdf from the exgenbeta3cdf MATLAB



(a) The computation of the EGB3 pdf and cdf for $-20 \le x \le 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 100$, $q = 0$, and $b_0 = 4.05$. Cdf 1 Integral, Cdf 2 the EGB3 cdf.

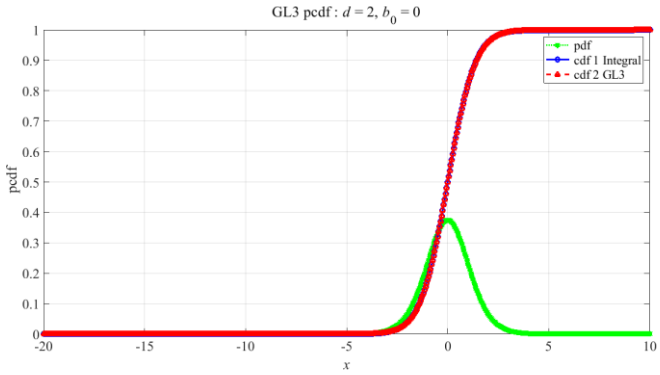


(b) The computation of the EGB3 cdf error for $-20 \le x \le 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 100$, $q = 0$, and $b_0 = 4.05$. Cdf 1 Integral minus Cdf 2 the EGB3 cdf.

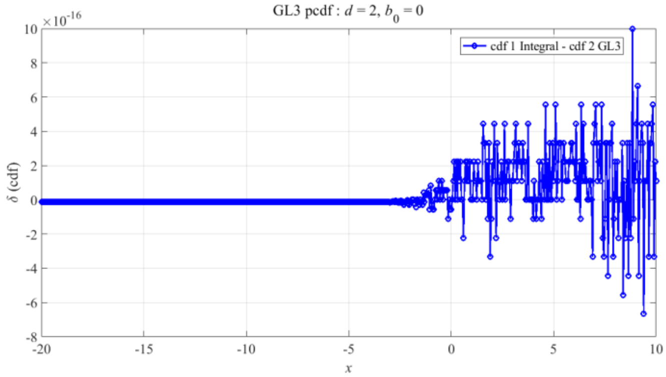
Fig. 3. The computation of the EGB3 pdf, cdf, and cdf error for $-20 \le x \le 10$.

function. As illustrated from the analytical results of Fig. 1(b) the error is in the order of 1.0372e+09 femto^{xx} or 1.0372e-06 or micro only for the very small portion of the cdf that corresponds to the growth section which is most likely due to the computation of the integral. Outside of that portion the error is practically zero.

In Fig. 2(a) it is presented the computation of the GGu, an acronym for the generalized Gumbel of type I, pdf and cdf for $-20 \le x \le 10$, $a = 1.5$, $b = 2$, $b_0 = -0.7$. Cdf 1 *Integral* MATLAB Cdf 2 GGu cdf is the mirror image of the MATLAB BIF, exponential function cdf genggucdf. Is GGu cdf a valid cdf? For all values of the main variable x from minus twenty to plus ten the GGu cdf is always greater than or equal to zero, less than or equal to one, and a nondecreasing monotonic function. If we take the derivative of the GGu cdf from minus infinity to x the derivative is never negative meaning always greater than or equal to zero. The GGu cdf is

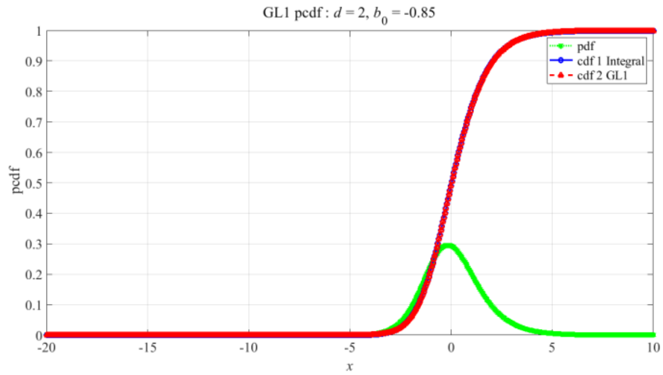


(a) The computation of GL3 pdf and cdf for $-20 \le x \le 10$, $d = 2$, $b_0 = 0$. Cdf 1 Integral, Cdf 2 GL3 cdf.

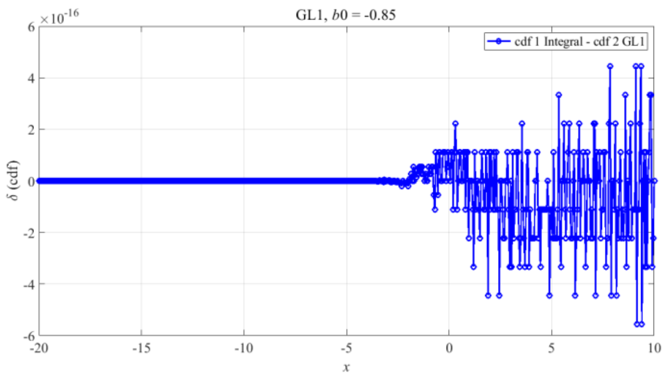


(b) The computation of GL3 cdf error for $-20 \le x \le 10$, $d = 2$, $b_0 = 0$. Cdf 1 Integral minus Cdf 2 GL3 cdf.

Fig. 4. The computation of GL3 pdf, cdf, and cdf error for $-20 \le x \le 10$.



(a) The computation of GL1 pdf and cdf for $-20 \le x \le 10$, $d = 2$, $b_0 = -0.85$. Cdf 1 Integral, Cdf 2 GL1 cdf.



(b) The computation of GL1 cdf error for $-20 \le x \le 10$, $d = 2$, $b_0 = -0.85$. Cdf 1 Integral, Cdf 2 GL1 cdf.

Fig. 5. The computation of GL1 pdf, cdf, and cdf error for $-20 \le x \le 10$.

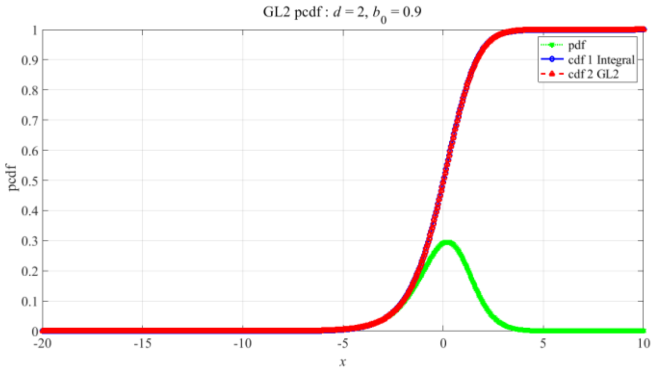
ranked as the second fastest than the growth rate.

Figure 2(b) shows the error between the Cdf 1 Integral and the Cdf 2 GGu cdf from the genggucdf MATLAB function. As illustrated from the analytical results of Fig. 2(b) the error is in the order of 0.2359 femto or 10^{-15} for the very small portion of the cdf that corresponds to the growth section. Outside of that portion the error is practically zero. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 GGu cdf from the genggucdf MATLAB function.

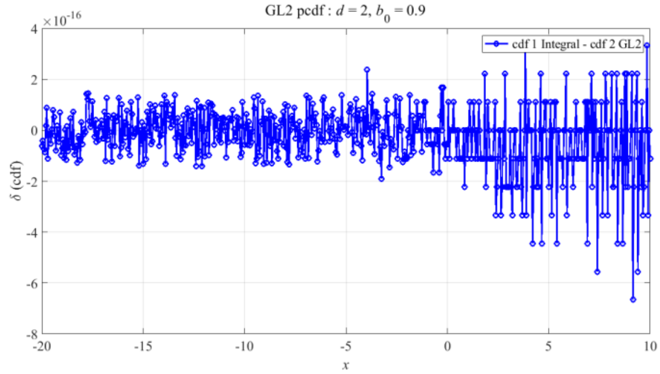
Figure 3(a) depicts the computation of the EGB3 exgenbeta3cdf equal to EGB1 pdf and cdf for $-20 \le x \le 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 100$, $q = 0$, and $b_0 = 4.05$. Cdf 1 Integral, Cdf 2 EGB3 cdf. Is the EGB3 cdf a valid cdf? Certainly, the EGB3 cdf is always greater than or equal to zero and less than or equal to one and a nondecreasing monotonic function for all the values of the main variable from minus twenty to plus ten. The only reason why I have shown the plots of the Cdf 2 EGB3 pdf is to prove numerically that in fact the derivative of the EGB3 cdf is in fact a valid pdf. We can say that the parameter p is the sharpness parameter. The higher the value of p the smoother EGB3 cdf and the smaller the value of p the sharper EGB3 cdf is. This is a small example to illustrate the flexibility that EGB3 cdf offers in modeling the various stock returns.

Figure 3(b) shows the error between the Cdf 1 Integral and the Cdf 2 EGB3 cdf from the exgenbeta3cdf MATLAB function. As illustrated from the analytical results of Fig. 3(b) the error is in the order of 1.8286e+03 femto or 1.8286e-12 or pico only for the very small portion of the cdf that corresponds to the growth section which is most likely due to the computation of the integral. Outside of that portion the error is practically zero.

Figure 4(a) presents the computation of GL3, an anacronym that describes the Generalized Logistics of Type III in Sect. 3.4, pdf and cdf for $-20 \le x \le 10$, $d = 2$, $b_0 = 0$. Cdf 1 Integral, Cdf 2 GL3 cdf. GL3 cdf main computation is performed by computing CFE (37), the limits in (38)-(40), and the monotonic nature in (49) which appears to grow monotonically from zero to one. All of these observations

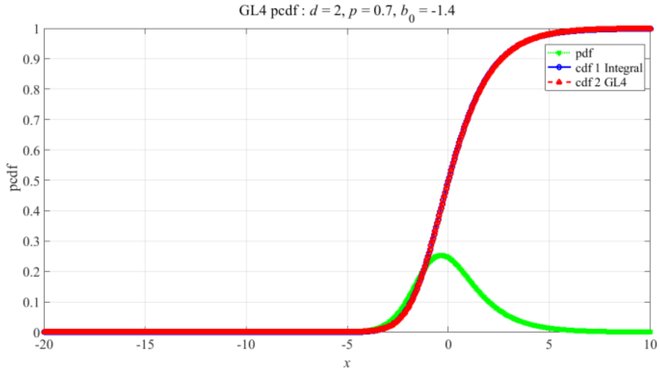


(a) The computation of GL2 pdf and cdf for $-20 \leq x \leq 10$, $d = 2$, $b_0 = 0.9$. Cdf 1 Integral, Cdf 2 GL2 cdf.

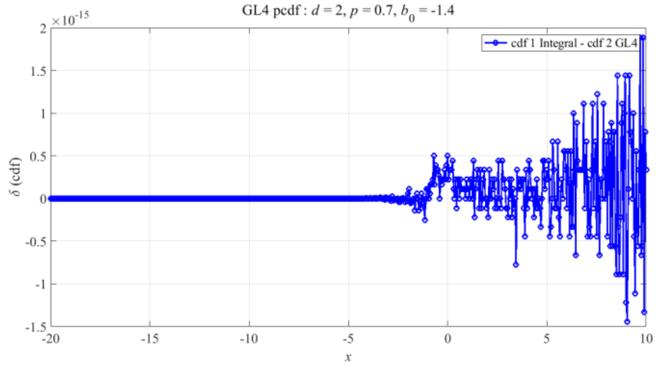


(b) The computation of GL2 cdf error for $-20 \leq x \leq 10$, $d = 2$, $b_0 = 0.9$. Cdf 1 Integral minus Cdf 2 GL2 cdf.

Fig. 6. The computation of GL2 pdf, cdf, and cdf error for $-20 \leq x \leq 10$.



(a) The computation of GL4 pdf and cdf for $-20 \leq x \leq 10$, $d = 2$, $p = 0.7$, $b_0 = -1.4$. Cdf 1 Integral, Cdf 2 GL4 cdf.



(b) The computation of GL4 cdf error for $-20 \leq x \leq 10$, $d = 2$, $p = 0.7$, $b_0 = -1.4$. Cdf 1 Integral minus Cdf 2 GL4 cdf.

Fig. 7. The computation of GL4 pdf, cdf, and cdf error for $-20 \leq x \leq 10$.

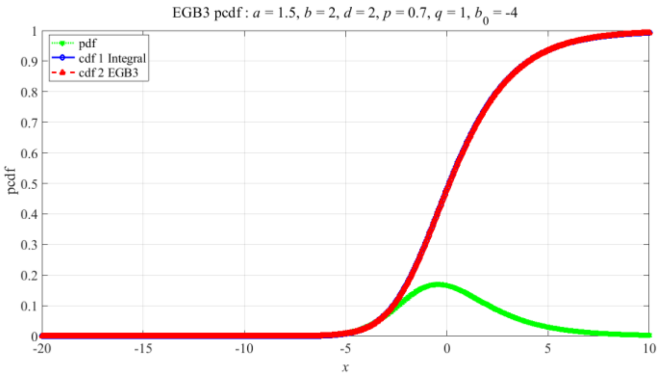
together indicate that GL3 cdf is a valid cdf model. In terms of sharpness for the values of the parameters considered, GL3 cdf is the less sharp cdf model from the EGB family of distributions.

Figure 4(b) shows the error between the Cdf 1 Integral and the Cdf 2 GL3 cdf from the genlog3cdf MATLAB function. As illustrated from the analytical results of Fig. 4(b) the error is in the order of 0.9992 femto or 10^{-15} which corresponds to the segment for which the cdf becomes positive and the main variable is greater than zero. Outside of this segment, the error is practically zero. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 GL3 cdf from the genlog3cdf MATLAB function.

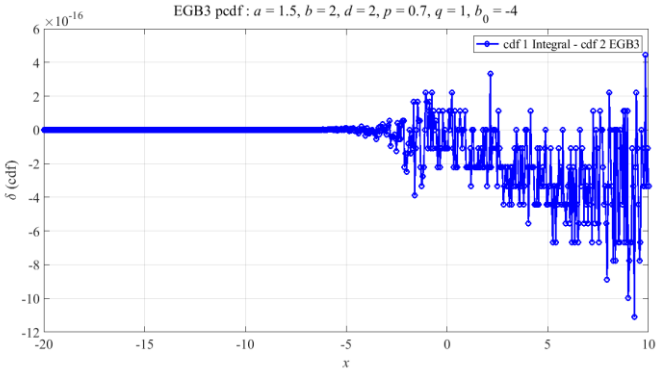
Figure 5(a) presents the computation of GL1, an anacronym that describes the Generalized Logistics of Type I in Sect. 3.2, pdf and cdf for $-20 \leq x \leq 10$, $d = 2$, $b_0 = -0.85$. Cdf 1 Integral, Cdf 2 GL1 cdf. GL1 cdf main computation is performed by computing CFE (12) or (13), the limits in (16)-(18), and the monotonic nature in (19)-(23) which appears to grow monotonically from zero to one. All of these observations together indicate that GL1 cdf is a valid cdf model. In terms of sharpness for the values of the parameters considered, GL1 cdf is the less sharp cdf than GL3 model from the EGB family of distributions.

Figure 5(b) shows the error between the Cdf 1 Integral and the Cdf 2 GL1 cdf from the genlog1cdf MATLAB function. As illustrated from the analytical results of Fig. 5(b) the error is in the order of 0.4441 femto or 10^{-15} which corresponds to the segment for which the cdf becomes positive and the main variable is greater than zero. Outside of this segment, the error is practically zero. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 GL1 cdf from the genlog1cdf MATLAB function.

Figure 6(a) presents the computation of GL2, an anacronym that describes the Generalized Logistics of Type II in Sect. 3.3, pdf and cdf for $-20 \leq x \leq 10$, $d = 2$, $b_0 = -0.85$. Cdf 1 Integral, Cdf 2 GL2 cdf. GL2 cdf main computation is performed by computing CFE (24), the limits in (25)-(29), and the monotonic nature in (30)-(36) which appears to grow monotonically from zero to one. All of these observations together indicate that GL2 cdf is a valid cdf model. In terms of sharpness for the values of the parameters considered, GL2



(a) The computation of the EGB3 pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 0.7$, $q = 1$, and $b_0 = -4.0$. Cdf 1 Integral, Cdf 2 the EGB3 cdf.



(b) The computation of the EGB3 cdf error for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 0.7$, $q = 1$, and $b_0 = -4.0$. Cdf 1 Integral minus Cdf 2 the EGB3 cdf.

Fig. 8. The computation of EGB3 pdf, cdf, and cdf error for

cdf is the less sharp cdf than GL1 model from the EGB family of distributions.

Figure 6(b) shows the error between the Cdf 1 Integral and the Cdf 2 GL2 cdf from the genlog2cdf MATLAB function. As illustrated from the analytical results of Fig. 6(b) the error is in the order of 0.3331 femto or 10^{-15} which corresponds to entire simulation segment. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 GL2 cdf from the genlog2cdf MATLAB function.

Figure 7(a) presents the computation of GL4, an anacronym that describes the Generalized Logistics of Type IV in Sect. 3.5, pdf and cdf for $-20 \leq x \leq 10$, $d = 2$, $b_0 = 0.8$. Cdf 1 Integral, Cdf 2 GL4 cdf. GL4 cdf main computation is performed by computing CFE (52), the limits in (53)-(37), and the monotonic nature in (58)-(63) which appears to grow monotonically from zero to one. All of these observations together indicate that GL4 cdf is a valid cdf model. In terms of sharpness for the values of the parameters considered, GL4 cdf is the least sharp cdf from the EGB family of distributions.

Figure 7(b) shows the error between the Cdf 1 Integral and the Cdf 2 GL4 cdf from the genlog4cdf MATLAB function. As illustrated from the analytical results of Fig. 7(b) the error is in the order of 1.8874 femto or 10^{-15} which corresponds to entire simulation segment. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 GL4 cdf from the genlog4cdf MATLAB function.

Figure 8(a) depicts the computation of the EGB3 exgenbeta3cdf equal to EGB2 pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 0.7$, $q = 1$, and $b_0 = -4$. Cdf 1 Integral, Cdf 2 EGB3 cdf. Is the EGB3 cdf a valid cdf? Certainly, the EGB3 cdf is always greater than or equal to zero and less than or equal to one and a nondecreasing monotonic function for all the values of the main variable from minus twenty to plus ten. The only reason why I have shown the plots of the Cdf 2 EGB3 pdf is to prove numerically that in fact the derivative of the EGB3 cdf is in fact a valid pdf. We can say that the parameter p is the sharpness parameter. The higher the value of p the smoother EGB3 cdf and the smaller the value of p the sharper EGB3 cdf is. This is a small example to illustrate the flexibility that EGB3 cdf offers in modeling the various stock returns.

Figure 8(b) shows the error between the Cdf 1 Integral and the Cdf 2 EGB3 cdf is identical to the EGB2 cdf from the exgenbeta3cdf MATLAB function. As illustrated from the analytical results of Fig. 8(b) the error is in the order of 0.4441 femto only for the almost one-half portion of the cdf that corresponds to the growth section. Outside of that portion the error is practically zero.

This concluded the discussion on examples and numerical, theoretical results.

5 Conclusions

In conclusion, this is the third iconic journal paper that was written to fill in the void that was created from the lack pf content in Progri (2016, [1]).

Let us enumerate the accomplishment so far from the three iconic journal papers in Progri (2024, [7]). First, the material in Progri (2024, [7]) became so big that there was no way to publish the entire content in just one journal paper. So, the original paper in Progri (2024, [7]) was split into two journal papers: Progri (2024, [7]) for the EGB3 pdf family of distributions and Progri (2024, [8]) corresponding to the EGB3 cdf family of distributions.

I believe that the content in Progri (2024, [7]) as it relates to the EGB3 pdf family of distributions is very solid; meaning, it addresses in both qualitative, and quantitative, in analytical and conceptual, all the fundamental principles and has enough MATLAB simulation examples for validation.

Similarly, the content in Progri (2024, [8]) became enormous; therefore, I decided to split Progri (2024, [8]) into two journal papers: Progri (2024, [8]) corresponding to the EGB3 nongeneralized cdf family of distributions and this journal paper corresponding to the EGB3 generalized cdf family of distributions.

In all three journal-papers the historical context and content is beyond remarkable. While all these contributions and contributors might be known in our communities of mathematics, physics, science, and engineering they are put in a seemingly unique and original way to facilitate and broaden the understanding of the reader. Take for example the unique jewel in Appen. C, the proof provided in Appen. C adds to the knowledge, originality, and uniqueness of this research. The proof provided in Appen. C would have been more appropriate for the content in Progri (2024, [7]) because it involves the CFE of the EGB2 pdf not its cdf. Other content in this journal paper also would have been more appropriate for the content in Progri (2024, [7]).

All three journal papers contain a total of two-hundred and fifteen equations divided as follows: Fifty-nine in Progri (2024, [7]), forty-seven in Progri (2024, [8]), and one-hundred and nine in this journal paper. This journal-paper contains more than fifty percent of the total number of equations.

In number of pages: nineteen pages in Progri (2024, [7]), fifteen pages in Progri (2024, [8]), and twenty-four pages in this journal-paper. All three journal papers come close to fifty-eight pages.

In number of MATLAB plots: fourteen plots in Progri (2024, [7]), fourteen plots in Progri (2024, [8]), and sixteen plots in this journal paper for a total of forty-four plots. In almost all of them accuracy that is obtained from the analytical and conceptual results exceeds the maximum precision offered by MATLAB which is in excess of sixteen digits numerical precision.

I believe with these statistics alone elevate these three journal papers as the most comprehensive study into the description of all the fundamental properties of pdf and cdf of the EGB3, including special cases, nongeneralized, and generalized family of distributions; i.e., they are another step closer to perfection.

EGB3 is a very powerful model that offers infinite ways of modeling pdfs and cdfs that range from as sharp as exponential or as it relates to the nongeneralized family of EGB3 distributions to normal and to the very flat and smooth as the generalized logistic distributions.

6 Acknowledgement

This work was supported by Giftet Inc. executive office.

I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [53] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications*. *Vol. I* (Not yet available in print) [52].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [53] reminds me of my grandfather who I loved very much.

Finally, I am also dedicating this journal paper to our most beloved President Ronald Reagan, who reminds me of my grandfather, [54] who spoke from the heart, who spoke the truth, who was a staunch supporter of personal freedom, the greatest leader of the free world, and perhaps the greatest critic of the wasteful government control and spending[xxi].

"*Government is not the answer, government is the problem*."—Ronald Reagan

I would like to express my deepest gratitude to my mother Lumturi Progri, my sister Adriana Dine, and Ms. Elizabeth Demir for their support, loyally, and dedication to me during some of the most difficult times in my life as a boy, man, and scholar.

7 References

[1] I.F. Progri, "Exponential generalized Beta distribution," *J. Geol. Geoinfo. Geointel.,* vol. 2016, article ID 2016071603, 18 pg., Nov. 2016. DOI: http://doi.org/10.18610/JG3.2016.071603

[2] I. Progri, "Hypergeometric function partial derivatives," *J. Geol. Geoinfo. Geointel.*, vol. 2016, article ID

2016071604, 21 pg., Nov. 2016. DOI: https://doi.org/10.18610/JG3.2016.071604

[3] I.F. Progri, "The 4-parameter ionospheric channel model: part 1—theory and simulation," in *Proc. ION-ITM*, Reston, VA, pp. 713-750, Jan. 29-Feb. 1, 2018, https://doi.org/10.33012/2018.15590

[4] I.F. Progri, "Efficient computation of the special cases of the generalized Bessel function distributions," *J. Geol. Geoinfo. Geointel.,* vol. 2019, article ID 2019071601, 31 pg., Nov. 2019. DOI: https://doi.org/10.18610/JG3.2019.071601

[5] I.F. Progri, "The 4-parameter ionospheric channel model: part 1—theory and simulation," *J. Geol. Geoinfo. Geointel.,* vol. 2019, article ID 2019071603, 34 pg., Nov. 2019. DOI: https://doi.org/10.18610/JG3.2019.071603

[6] I.F. Progri, "A study of the computation of the exponential function," *J. Geol. Geoinfo. Geointel.,* vol. 2022, article ID 2022071602, 10 pg., Nov. 2022. DOI: https://doi.org/10.18610/JG3.2022.071602

[7] I.F. Progri, "On the computation of the probability density function of the exponential generalized beta family of distributions," vol. 2024, article ID 2024071602, 19 pg., Nov. 2024. DOI: https://doi.org/10.18610/JG3.2024.071602

[8] I.F. Progri, "On the computation of the cumulative probability distribution function of the exponential generalized Beta nongeneralized family of distributions," vol. 2024, article ID 2024071603, 15 pg., Nov. 2024. DOI: https://doi.org/10.18610/JG3.2024.071603

[9] I.F. Progri, "Reduction formulae of a binomial expansion via a 2F1 hypergeometric function and its derivative," vol. 2024, article ID 2024071605, 10 pg., Nov. 2024. DOI: https://doi.org/10.18610/JG3.2024.071605

[10] R.L. Prentice, "A generalization of the probit and logit methods for dose response curves," *Biometrics*, vol. 32, no. 4, pp. 761-768, Dec. 1976.

[11] N. Balakrishnan, M.Y. Leung, "Order statistics from the type I generalized logistic distribution," *Commun. in Stat.-Simul. & Comput.*, vol 17, no. 1, pp. 25-50, 1988. DOI: https://doi.org/10.1080/03610918808812648

[12] N.L. Johnson, S. Kotz, N. Balakrishnan, *Continuous Univariate Distributions*. Vol. 2, Hoboken, NJ: John Wiley & Sons, 1995. ISBN: 980-0-417-58-494-0

[13] J.B. McDonald, Y.J. Xu, "A generalization of the beta distribution with applications," *J. Econ.*, vol. 66, no. 1-2, pp. 133-152, Oct. 1995. DOI: https://doi.org/10.1016/0304-4076(94)01612-4

[14] J.B., McDonald, Y.J. Xu, "A comparison of semi-parametric and partially adaptive estimators of the censored regression model with possibly skewed and leptokurtic error distributions," *Economics Letters*, vol. 51, no. 2, pp. 153-159, May 1996. DOI: https://doi.org/10.1016/0165-1765(96)00816-6

[15] J.B. McDonald, "An application and comparison of some flexible parametric and semi-parametric qualitative response models," *Economics Letters*, vol. 53, no. 2, pp. 145-152, Nov. 1996. DOI: https://doi.org/10.1016/S0165-1765(96)00907-X

[16] K.-L. Wang, C. Fawson, C.B. Barrett, J.B. McDonald "A more general approach to modeling exchange rate volatility," *Economics Research Institute Study Paper* 4, pp. 1-47, Mar. 1997.

[17] K.-L. Wang, "Essays on time series analysis of exchange rate distributions," *Ph.D. Dissertation*, Utah State University, 1997.

[18] K.L. Wang, C. Fawson, C.B. Barrett, J.B. McDonald, "A flexible parametric GARCH model with an application to exchange rates," Economic research institute study paper ERI# 97-04, 39 pg., Apr. 1998.

[19] Y. Hwang, "Long memory and asymmetry in conditional variance models," *Ph.D. Dissertation*, Michigan State University; 1998.

[20] H., Ebens, "Realized stock volatility," *Dep. Econ.,* Johns Hopkins University, 28 pg., Mar. 1999.

[21] R. Barniv, J.B. McDonald, "Review of categorical models for classification issues in accounting and finance," *Review of Quantitative Finance & Accounting*, vol. 13, pp. 39-62, July 1999. DOI: https://doi.org/10.1023/A:1008348403869

[22] J.V. Hansen, J.B. McDonald, R.D. Nelson, "Time series prediction with Genetic-Algorithm designed neural networks: An empirical comparison with modern statistical models," *Comp. Intell.*, vol. 15, no. 3, pp. 171-184, Aug. 1999. DOI: https://doi.org/10.1111/0824-7935.00090

[23] E.J. Miravete, "Quantity discounts for taste–varying consumers," *CARESS Working Paper* 99-11, 63 pg., Sep. 1999.

[24] J.B. Mcdonald, J.V. Hansen, "An application and comparison of some flexible parametric and

semiparametric qualitative-response models with heteroskedasticity," *Inter. J. Sys. Sci.*, vol. 31, no. 1, pp. 27-33, 2000. DOI: https://doi.org/10.1080/002077200291424

[25] E.P. Goss, G.S. Vozikis, "Prediction of insolvency of life insurers through neural networks," in *Proc. ECIS 2000. 17.*, pp. 850-857. 2000. https://aisel.aisnet.org/ecis2000/17

[26] M.J. Fischer, "The folded EGB2 distribution and its application to financial return data," No. 32/2000. Diskussionspapier, 24 pg., Sep. 2000. Friedrich-Alexander-Universität ErlangenNürnburg, Lehrstuhl für Statistik und Ökonometrie, Nürnberg

[27] M.J. Fischer, "The Esscher-EGB2 option pricing model," Diskussionspapier, No. 31. 24 pg., Sep. 2000, Friedrich-Alexander-Universität Erlangen-Nürnburg, Lehrstuhl für Statistik und Ökonometrie, Nürnberg

[28] J.D. Kalbfleisch, R.L. Prentice. *The Statistical Analysis of Failure Time Data*. Hoboken, NJ: John Wiley & Sons, 2002. ISBN: 0-471-36357-X

[29] I.F., Progri, "An assessment of indoor geolocation systems," *Ph.D. Dissertation*, Worcester Polytech. Inst., Massachusetts, 408 pg., May 2003.

[30] I.F. Progri, W.R. Michalson, D. Cyganski, "An OFDM/FDMA indoor geolocation system," *Navigation,* vol. 51, no. 2, pp. 133-142, summer 2004, DOI: https://doi.org/10.1002/j.2161-4296.2004.tb00346.x

[31] I.F. Progri, "A unified geolocation channel model−−part II (multipath distribution)," in *Proc. WTS-2006*, Pomona, CA, pp. 1-8, Apr. 2006, DOI; https://doi.org/10.1109/WTS.2006.334542

[32] I.F. Progri, W.R. Michalson, J. Wang, M.C. Bromberg, "Theoretical data on support of a unified indoor geolocation channel model," in *Proc. ION-NTM 2007*, San Diego, CA, pp. 577-584, Jan. 2007.

[33] I.F. Progri, "Wireless-enabled GPS indoor geolocation system," in *Proc. IEEE/ION-PLANS 2010*, Palm Spring, CA, pp. 526-538, May 2010, DOI: https://doi.org/10.1109/PLANS.2010.5507256

[34] R.D. Gupta, D. Kundu, "Generalized logistic distributions," *J. Appl. Stat. Sci.*, vol. 18, no. 1, pp. 51-, 2010.

[35] I.S. Gradshteyn, I.M. Ryzhik, (A. Jeffrey, D. Zwillinger, editors) *Table of Integrals, Series, and Products*, 7th ed., Burlington, MA: Academic Press, 1171 pg., 2007.

[36] M.M. Nassar, A. Elmasry, "A study of generalized logistic distributions," *J. Egyptian Mathem. Soc.*, vol. 20, no. 2, pp. 126-133, 2012. DOI: https://doi.org/10.1016/j.joems.2012.08.011

[37] Anon., "Probability density function," *Wikipedia*, the free encyclopedia, Jul. 2023, http://en.wikipedia.org/wiki/Probability_density_function

[38] Anon., "Cumulative distribution function," *Wikipedia*, the free encyclopedia, Jul. 2023, http://en.wikipedia.org/wiki/Cumulative_distribution_function

[39] Anon., "Generalized beta distribution," *Wikipedia*, the free encyclopedia, June 2023, http://en.wikipedia.org/wiki/Generalized_beta_distribution

[40] M.Z. Raqab, D. Kundu, "Burr type X distribution: revisited," *J. Prob. Stat. Sci.*, vol. 4, no. 2, pp. 179-193, 2006.

[41] Anon., "Gompertz distribution," *Wikipedia*, the free encyclopedia, July 2023, http://en.wikipedia.org/wiki/Gompertz_distribution

[42] Anon., "Gumbel distribution," *Wikipedia*, the free encyclopedia, July 2023, http://en.wikipedia.org/wiki/Gumbel_distribution

[43] Anon., "Type-1 Gumbel distribution," *Wikipedia*, the free encyclopedia, July 2023, http://en.wikipedia.org/wiki/Type-1_Gumbel_distribution

[44] Anon., "Type-2 Gumbel distribution," *Wikipedia*, the free encyclopedia, July 2023, http://en.wikipedia.org/wiki/Type-2_Gumbel_distribution

[45] Anon., "Generalized extreme value distribution," *Wikipedia*, the free encyclopedia, July 2023, https://en.wikipedia.org/wiki/Generalized_extreme_value_distribution

[46] Anon., "Logistic distribution," *Wikipedia*, the free encyclopedia, July 2023, http://en.wikipedia.org/wiki/Logistic_distribution

[47] Anon., "Generalized logistic distribution," *Wikipedia*, the free encyclopedia, July 2023, http://en.wikipedia.org/wiki/Generalized_logistic_distribution

[48] Anon., "Distribution (mathematics)," *Wikipedia*, the free encyclopedia, July 2023, https://en.wikipedia.org/wiki/Distribution_(mathematics)

[49] A.M. Mathai, S.B., Provost, "On the distribution of order statistics from generalized logistic samples," *METRON - Inter. J. Statis.*, vol. LXII, no. 1, pp. 63-71, 2004.

[50] I.F. Progri, "The computation of a 2F2 hypergeometric function," *J. Geol. Geoinfo. Geointel.,* vol. 2020, article ID 2020071601, 12 pg., Nov. 2020. DOI: https://doi.org/10.18610/JG3.2020.071601

[51] I.F. Progri, "Confluent hypergeometric function irregular singularities," vol. 2021, article ID 2021071601, 12 pg., Nov. 2021. DOI: https://doi.org/10.18610/JG3.2021.071601

[52] I. Progri, *Indoor Geolocation Systems—Theory and Applications*. *Vol. I*, 1st ed., Worcester, MA: Giftet Inc., ~800 pp., ~ 2024 (not yet available in print).

[53] Anon., "Jack Nicklaus," *Wikipedia, the free encyclopedia*, June 2024. https://en.wikipedia.org/wiki/Jack_Nicklaus

[54] Anon., "Ronald Reagan," *Wikipedia, the free encyclopedia*, June 2024. https://en.wikipedia.org/wiki/Ronald_Reagan

[55] Anon, "MATLAB 2023b," *The MathWorks, Inc.*, Natick, MA, Copyright © 1994-2024. The MathWorks, Inc., https://mathworks.com/products/matlab.html

8 Appendix A: Derivation of the EGB3 CDF

The EGB [39] cdf [38] can be computed via the integral of the EGB [39] pdf [37] as follows

$$F_{\mathrm{EGB}}(x;a,b,d,p,q)=\begin{vmatrix}F(x), & -\infty<x\le b' \\ 1, & b'<x<\infty\end{vmatrix} \tag{66}$$

where the

$$\begin{aligned}F(x)&=\int_{-\infty}^{x}f(x)du\\ &=\int_{-\infty}^{x}\frac{e^{d\frac{u-b}{a}}\left[1-(1-q)e^{\frac{u-b}{a}}\right]^{p-1}}{|a|\mathrm{B}(d,p)\left[1+qe^{\frac{u-b}{a}}\right]^{d+p}}du\\ &=\frac{\int_{-\infty}^{x}\frac{e^{d\frac{u-b}{a}}\left[1-(1-q)e^{\frac{u-b}{a}}\right]^{p-1}}{\left[1+qe^{\frac{u-b}{a}}\right]^{d+p}}du}{|a|\mathrm{B}(d,p)}\end{aligned} \tag{67}$$

In order to reduce this expression into the familiar expression shown in Progri (2016, [1]) we make two substitutions. In the first substitution we get rid of the exponential function from both the numerator and the denominator of the integral as follows:

$$v=e^{\frac{u-b}{a}} \tag{68}$$

$$dv=e^{\frac{u-b}{a}}\frac{1}{a}du \tag{69}$$

$$u\rightarrow|_{-\infty}^{x}\Rightarrow v\rightarrow|_{0}^{x'\equiv e^{\frac{x-b}{a}}} \tag{70}$$

Next, after we substitute (68)-(70) into (67) we obtain the following:

$$F(x')=\frac{\int_0^{x'}\frac{v^{d-1}[1-(1-q)v]^{p-1}}{(1+qv)^{d+p}}dv}{\mathrm{B}(d,p)} \tag{71}$$

One important observation we need to make is that while the main variable x goes from minus infinity to plus infinity, the substituted variable x' goes from zero to plus infinity. It is important to keep track of the range of values of the main variable. In the second substitution we get rid of the denominator from the integral as follows:

$$v'=\frac{v}{1+qv} \tag{72}$$

$$dv'=\frac{(1+qv)-qv}{(1+qv)^2}dv=\frac{dv}{(1+qv)^2}\Rightarrow dv=(1+qv)^2dv' \tag{73}$$

$$v\rightarrow|_0^{x'}\Rightarrow v'\rightarrow|_0^{z} \tag{74}$$

$$z=\frac{x'\equiv e^{\frac{x-b}{a}}}{1+qx'} \tag{75}$$

Another important observation we need to make is that while the first substituted variable x' goes from zero to plus infinity, the second substituted variable z goes from zero to one. After the substitution of (72), (74), and (75) into (71) we obtain the following

$$\begin{aligned}F(x')&=\frac{\int_0^{x'}\frac{v^{d-1}[1-(1-q)v]^{p-1}}{(1+qv)^{d+p}}dv}{\mathrm{B}(d,p)}\\ &=\frac{\int_0^{x'}\frac{v^{-p-1}[1-(1-q)v]^{p-1}}{\frac{(1+qv)^{d+p}}{v^{d+p}}}dv}{\mathrm{B}(d,p)}\\ &=\frac{\int_0^{x'}v'^{d+p}v^{-p-1}(1+qv-v)^{p-1}dv}{\mathrm{B}(d,p)}\\ &=\frac{\int_0^{x'}v'^{d+p}v^{-2}\left(v'^{-1}-1\right)^{p-1}dv}{\mathrm{B}(d,p)}\end{aligned} \tag{76}$$

After the substitution of (73) into (76), yields

$$\begin{aligned}F(z)&=\frac{\int_0^{z}v'^{d+p}v^{-2}\left(v'^{-1}-1\right)^{p-1}(1+qv)^2dv'}{\mathrm{B}(d,p)}\\ &=\frac{\int_0^{z}\frac{v'^{d+p}}{v'^{p-1}}(1-v')^{p-1}\left(\frac{1+qv}{v}\right)^2dv'}{\mathrm{B}(d,p)}\\ &=\frac{\int_0^{z}v'^{d+1}v'^{-2}(1-v')^{p-1}dv'}{\mathrm{B}(d,p)}\\ &=\frac{\int_0^{z}v'^{d-1}(1-v')^{p-1}dv'}{\mathrm{B}(d,p)}\end{aligned} \tag{77}$$

The integral in (77) does not contain any expressions in the denominator; moreover, the integral is the well-known expression of the incomplete Beta function Gradshteyn et al. (2007, [35] pg. 910 ex. 8.39), which reduces the EGB cdf as follows:

$$F(x) = \frac{\mathrm{B}(d,p;z)}{\mathrm{B}(d,p)}$$
$$= \mathrm{I}_z(d,p) \tag{78}$$

Finally, substitute (78) into (66) yields the well-known CFE of the EGB [39] cdf [38] as follows:

$$F_{\mathrm{EGB}}(x;a,b,d,p,q) = \begin{vmatrix} \mathrm{I}_z(d,p), & -\infty < x \le b' \\ 1, & b' < x < \infty \end{vmatrix} \tag{79}$$

Equation (72) is known as the regularized incomplete Beta function Gradshteyn et al. (2007, [35] 910, 8.39 8.391 8.392) which has the following properties:

$$\mathrm{I}_0(d,p) = 0 \tag{80}$$
$$\mathrm{I}_1(d,p) = 1 \tag{81}$$
$$\mathrm{I}_z(d,1) = z^d \tag{82}$$
$$\mathrm{I}_z(1,p) = 1-(1-z)^p \tag{83}$$
$$\mathrm{I}_z(d,p) = \mathrm{I}_z(d,p-1) + \frac{z^d(1-z)^{p-1}}{(p-1)B(d,p-1)} \tag{84}$$
$$\mathrm{I}_z(d,p) = \mathrm{I}_z(d-1,p-1) - \frac{z^{d-1}(1-z)^p}{(d-1)B(d-1,p)} \tag{85}$$
$$\mathrm{I}_z(d,p) = \frac{z^d}{d}\left[F_1(d,p,z) \equiv {}_2F_1(d,1-p;d+1;z)\right]$$
$$= \frac{z^d}{d}\sum_{k=0}^{\infty}\frac{(d)_k(1-p)_k}{(d+1)_k}z^k \tag{86}$$

where ${}_2F_1(a,b;a+1;z)$ is defined as the Gauss Hypergeometric function and $(a)_k$ is defined as the as the rising factorial.

The regularized incomplete beta function, $\mathrm{I}_z(d,p)$, is a monotonic function with respect to z because its derivative with respect to z is always positive (or zero) for z in the interval, and the parameters d and p are positive.

The derivative of $\mathrm{I}_z(d,p)$ with respect to z, by applying the techniques that were developed in Progri (2016, [1], [2]; 2019, [3], [5]), is

$$\frac{\partial \mathrm{I}_z(d,p)}{\partial z} = \frac{z^d}{\mathrm{B}(d,p)}\frac{F_1(d,p,z)+\frac{(1-p)z}{d+1}F_2(d,p,z)}{a(1+qx')}$$
$$= \frac{z^{d-1}}{\mathrm{B}(d,p)}\left[F_1(d,p,z)+\frac{(1-p)z}{d+1}F_2(d,p,z)\right]\frac{\partial z}{\partial x} \tag{87}$$
$$\frac{\partial z}{\partial x} = \frac{\frac{\partial x'}{\partial x}(1+qx')-x'q\frac{\partial x'}{\partial x}}{(1+qx')^2} = \frac{\frac{\partial x'}{\partial x}\equiv\frac{1}{a}x'}{(1+qx')^2} = \frac{z}{|a|(1+qx')} \tag{88}$$

where $F_2(d,p,z)$ is given by (59) with z however given by (75).

Non-Decreasing Monotonicity: For z in the interval from zero to one, the term

$$\frac{\partial \mathrm{I}_z(d,p)}{\partial z} = f(x) \sim z^{d-1}(1-z)^{p-1} \tag{89}$$
$$f(x) = \frac{z^d[1-(1-q)x']^{p-1}}{|a|\mathrm{B}(d,p)(1+qx')^p}$$
$$= \frac{z^d(1+qx'-x')^{p-1}}{|a|\mathrm{B}(d,p)(1+qx')^p}$$
$$= \frac{z^d(1-z)^{p-1}}{|a|\mathrm{B}(d,p)(1+qx')}$$
$$= \frac{z^{d-1}(1-z)^{p-1}}{\mathrm{B}(d,p)}\frac{\partial z}{\partial x} \ge 0 \tag{90}$$

Therefore, the since (90) is identical to (89) produces

$$z^{d-1}(1-z)^{p-1} \ge 0 \tag{91}$$

is always non-negative, meaning the derivative is always either positive or zero. This implies that the incomplete beta function, $\mathrm{I}_z(d,p)$, is *monotonically nom-decreasing* function with respect to z.

Furthermore, equating both sides of (80) with (83) produces the following identity:

$$F_1(d,p,z) + \frac{(1-p)z}{d+1}F_2(d,p,z) = (1-z)^{p-1} \tag{92}$$

Equation (92) is ditto comment on (51) and (65).

This concludes the precise discussion on Append. A, the derivation of the EGB3 cdf and its fundamental monotonic increasing nature from zero to one.

9 Appendix B: Special Cases of EGB CDF: EGB1 and EGB2 CDF

There are two fundamental special cases of the EGB3 cdf: the EGB1 cdf and the EGB2 cdf.

First, the EGB1 cdf which stands for the generalized exponential distribution cdf is obtained from the EGB3 cdf by setting the parameter *q* equal to zero as follows

$$F(x') = \frac{\int_0^{x'\equiv e^{\frac{x-b}{a}}} v^{d-1}(1-v)^{p-1}dv}{\mathrm{B}(d,p)}$$
$$= \frac{\mathrm{I}_{x'}(d,p)}{\mathrm{B}(d,p)} \tag{93}$$

Second, the EGB2 cdf which stands for the generalized logistics distribution cdf is obtained from the EGB3 cdf by setting the parameter *q* equal to one as follows

$$F(x) = \frac{\mathrm{B}\left(d,p;z\equiv\frac{x'}{1+x'}\right)}{\mathrm{B}(d,p)}$$
$$= \mathrm{I}_z(d,p) \tag{94}$$

The monotonic nature of EGB1 and EGB2 cdf follows as special cases of the monotonic nature of EGB.

This completes the discussion on the special cases of the EGB cdf: EGB1 and EGB2 cdf.

10 Appendix C: On the Origin of EGB2

Fischer (2000, [26]) states that EGB2 appears for the first time in work of Fisher (1921, [*]). However, Fischer (2000, [26]) left out all of the details which I am about to include in this appendix because of its historical context.

First, let us start with the definition of EGB2 given by McDonald (1991, [+]) as mentioned in Fischer (2000, [26] (2.2)) and in Progri (2024, [7] (6)) for q equal to one

$$f_{\mathrm{EGB2}}(x;a,b,d,p)=\frac{e^{d\frac{x-b}{a}}}{|a|\mathrm{B}(d,p)\left(1+e^{\frac{x-b}{a}}\equiv 1+x\prime\right)^{d+p}}$$

$$=\frac{x\prime^{d}}{|a|\mathrm{B}(d,p)(1+x\prime)^{d+p}} \tag{95}$$

Second, let us examine the formula (viii) given by Fisher (1921, [*] 18 pg.) of the *probable error of the intraclass correlations* (PEIC) distribution

$$f_{\mathrm{PEIC}}(x;a,b,d,p)=Ce^{c(x-b)}\mathrm{sech}^{d+p}(x-b) \tag{96}$$

Where

$$C=\frac{\frac{sn-3}{2}!}{2^{\frac{sn-3}{2}}\frac{(s-1)n-2}{2}!\frac{n-3}{2}!} \tag{97}$$

$$c=\frac{(s-2)n+1}{2} \tag{98}$$

$$b=\varphi \tag{99}$$

$$d+p=\frac{sn-1}{2} \tag{100}$$

$$\mathrm{sech}^{d+p}(x-b)=2^{d+p}\frac{e^{(d+p)(x-b)}}{\left[1+e^{2(x-b)}\right]^{(d+p)}} \tag{101}$$

$$e^{c(x-b)}\mathrm{sech}^{d+p}(x-b)=2^{d+p}\frac{e^{(d+c+p)(x-b)}}{\left[1+e^{2(x-b)}\right]^{(d+p)}} \tag{102}$$

We want the following relation

$$d+c+p=$$

$$=\frac{sn-1}{2}+\frac{(s-2)n+1}{2}$$

$$=\frac{2(s-1)n}{2}=\frac{(s-1)n}{2}\times\frac{1}{\frac{1}{2}} \tag{103}$$

Next, we examine the relation

$$\frac{(x-b)}{a}=2(x-b)\Rightarrow a=\frac{1}{2} \tag{104}$$

Next, we want (104) to be equal to

$$d+c+p=\frac{d}{a}=2d\Rightarrow c+p=d=\frac{(s-1)n}{2} \tag{105}$$

Next we need to find out what the value of p is from the following:

$$d+p=\frac{sn-1}{2}\Rightarrow p=\frac{sn-1}{2}-\frac{(s-1)n}{2}=\frac{n-1}{2} \tag{106}$$

Next, substituting (104) into (101) produces

$$e^{c(x-b)}\mathrm{sech}^{d+p}(x-b)=2^{d+p}\frac{e^{\frac{d}{a}(x-b)}}{\left(1+e^{\frac{x-b}{a}}\right)^{(d+p)}} \tag{107}$$

Next, substituting (98), (105), and (106), into a product of two in power *d* plus *p* with (97) yields:

$$C2^{d+p}=\frac{2^{d+p}}{2^{d+p-1}}\frac{(d+p-1)!}{(d-1)!(p-1)!}$$

$$=2\frac{\Gamma(d+p)}{\Gamma(d)\Gamma(p)}\text{ Gamma function}$$

$$=\frac{1}{a}\frac{1}{\mathrm{B}(d,p)}\text{ Beta Function} \tag{108}$$

Finally, substituting (107) and (108) into (96) yields an identity of the

$$f_{\mathrm{PEIC}}(x;a,b,d,p)=\frac{1}{a}\frac{1}{\mathrm{B}(d,p)}\frac{e^{\frac{d}{a}(x-b)}}{\left(1+e^{\frac{x-b}{a}}\right)^{(d+p)}}$$

$$=f_{\mathrm{EGB2}}(x;a,b,d,p) \tag{109}$$

Therefore, I have proved that what was stated by Fischer (2000, [26]) that EGB2 appears for the first time in work of Fisher (1921, [*]) is in fact true. I think this is a very important contribution from Fischer (2000, [26]); a little gem of very significant historical perspective.

[*] R.A. Fisher, "On the probable error of a coefficient of correlation deduced from a small sample," *Metron.*, vol. 1, pp. 3-32, 1921.

[+] J.B. McDonald, "Parametric models for partially adaptive estimation with skewed and leptokurtic residuals," *Econ. Let.*, vol. 37, no. 3, pp. 273-278, Nov. 1991. DOI: https://doi.org/10.1016/0165-1765(91)90222-7

This concludes the derivation of Appendix C: on the origin of EGB2.

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[i] I am dedicating this journal paper to my only female Prof. Denisse Nicoletti. Prof. Denisse Nicoletti was my Prof. on "Probability, Statistics, and Random Variables" during my Master of Science in Electrical Engineering at the Department of Electrical and Computer Engineering, Worcester Polytechnic Institute in fall of 2005. I remember her office was close to Prof. Alex Emanuel's office. In fall of 2005, I took three classes, one class in signals and systems, one class in probability and random variables, and another class in power systems engineering. Of all the three classes, her class on "Probability, Statistics, and Random Variables" was the most challenging at the time. I would go and ask her questions in her office about various topics. After the final exam I see her in the Electrical and Computer Engineering Department's

office. She said you got this, and she showed me an 'A' upside down like '∀' with her fingers. Anyways, I did get an 'A' in her class, and I remember how passionate she was about the subject matter. Unfortunately, she died tragically in a head-on collision in a car accident. "We all continue to grieve the loss of Denise Nicoletti, co-founder and director of Camp Reach from 1997-2002, who died in a head-on car crash on July 21, 2002, the first full day of the 2002 program." I remember going to her funeral and talking to Prof. Alex Emanual, her husband, and other guests about her. It was heart-breaking. WPI had a memorial built in front of the ECE Department to honor her legacy while at WPI. I am very grateful to having taken her class and getting to know her and having learned to fundamentals in such a difficult class on Probability, Statistics, and Random Variables. She encouraged me to become passionate about the fundamental concepts such as the probability density function and cumulative probability distribution functions, and how to compute the fundamental statistics such as the mean and variance.

[ii] The 5-paramater EGB will be referred to as EGB3 throughout the journal paper.

[iii] Ditto ii.

[iv] *Censored regression models*, like the Tobit model, are statistical methods used to estimate relationships between variables when the dependent variable is partially or completely unknown (censored) above or below a certain threshold.
What is Censoring? In censored data, you observe a value for some observations, but for others, you only know that the true value is above or below a certain limit (e.g., you know someone's income is at least $50,000, but not the exact amount).
Why Use Censored Regression Models? Ordinary least squares regression (OLS) can produce biased estimates when dealing with censored data. Censored regression models, like the Tobit model, are designed to address this issue by accounting for the non-randomness of the censored observations.
Tobit Model: The Tobit model is a common type of censored regression model used to estimate relationships when the dependent variable is censored from below or above. It's particularly useful when many observations on the dependent variable are zero or at a specific threshold.
How it Works: The Tobit model modifies the likelihood function to account for the unequal sampling probability of each observation, depending on whether the latent dependent variable falls above or below the determined threshold.
Examples: *Income Data*: If you have data on household income, and some individuals report that their income is below a certain threshold (e.g., $25,000), but you don't know the exact amount, then censored regression models can be used to estimate the relationship between income and other factors.
Survival Data: In survival analysis, if you observe the time until an event occurs for some individuals, but not for others (e.g., they are still alive at the end of the study), censored regression models can be used to model the relationship between survival time and other variables.
Other Censored Regression Models: Besides the *Tobit model*, other censored regression models include *quantile regression* and *nonparametric estimators*.

[v] Logit and probit models are both regression models used to predict a binary outcome (0 or 1), but they differ in the way they model the relationship between the predictor variables and the probability of the outcome, using either a logistic or normal distribution, respectively.
Logit Model: Employs a logistic function (also known as a sigmoid function) to model the relationship between predictor variables and the probability of the outcome. The logistic function maps any real number to a probability between 0 and 1. The logit model is often used when the outcome variable is binary, such as whether a customer will click on an ad or not. The *logit model* is also known as *logistic regression*.
Probit Model: Uses the cumulative standard normal distribution to model the relationship between predictor variables and the probability of the outcome. The cumulative standard normal distribution maps any real number to a probability between 0 and 1. The *probit model* is also used when the outcome variable is binary, such as whether a loan applicant will default or not. The *probit model* is also known as *probit regression*.
Key Differences: *Distribution Assumption*: The fundamental difference lies in the distribution assumed for the error term: logit assumes a logistic distribution, while probit assumes a normal distribution. *Link Function*: Logit uses the logit link function, while probit uses the probit link function (which is the inverse of the cumulative standard normal distribution). *Coefficient Interpretation*: The coefficients in both models are interpreted as the change in the log-odds (logit) or the change in the z-score (probit) for a one-unit change in the predictor variable.
Similarities: Both models are used to model binary outcomes. Both models can be estimated using maximum likelihood estimation. The results of the two models are often similar, especially when the sample size is large.
In summary, the choice between *logit* and *probit* models often depends on the researcher's preference and the specific context of the analysis. Both models are valuable tools for analyzing binary outcomes and understanding the relationship between predictor variables and the probability of the outcome.

[vi] *Peakedness* refers to the degree to which data values are concentrated around the mean in a distribution, often used in statistics to describe the shape of a curve, with a high degree of *peakedness* indicating a sharp, narrow distribution.
Statistical Context: In statistics, *peakedness* is often associated with the concept of kurtosis, which measures the *tailedness* and *peakedness* of a probability distribution compared to a normal distribution.
High Peakedness (*Leptokurtic*): A distribution with *high peakedness* (also known as *leptokurtic*) has a distinct peak near the mean, with data values rapidly declining as you move away from the mean, and the tails of the distribution are heavier (more data points in the tails). A distribution with a sharp, tall peak and thin tails would be considered highly peaked or *leptokurtic*.
Low Peakedness (*Platykurtic*): A distribution with low peakedness (also known as *platykurtic*) has a flatter top near the mean, with data values declining more gradually as you move away from the mean, and the tails of the distribution are lighter (fewer data points in the tails). A distribution with a flat top and thick tails would be considered less peaked or *platykurtic*.

[vii] In the context of stock returns, an ARFIMAX model is a time series model that extends the ARIMA model by incorporating fractional differencing (ARFIMA) and exogenous variables (ARFIMAX) to capture long-range dependence and non-linear relationships in stock returns, improving forecasting accuracy.
ARIMA Model: *ARIMA* (Autoregressive Integrated Moving Average) models are a common tool for analyzing and forecasting time series data, particularly financial time series like stock returns as is the case of Hansen et al. (1999, [22]).
Fractional Differencing (ARFIMA): ARIMA models assume that the data is stationary (meaning its statistical properties, like mean and variance, don't change over time). However, stock returns often exhibit long-range dependence, meaning that past returns can influence future returns even at long lags. ARFIMA addresses this by using fractional differencing, which allows for non-integer orders of integration, capturing this long-memory behavior.
Exogenous Variables (ARFIMAX): ARFIMAX models further extend ARFIMA by allowing for the inclusion of exogenous variables, which are external factors that can influence the stock returns being modeled. These exogenous variables could include macroeconomic indicators, industry-specific data, or other relevant information.
ARFIMAX models are valuable because they can: Capture long-range dependence in stock returns, which is a common characteristic of financial time series. Account for non-linear relationships between stock returns and exogenous variables. Improve the accuracy of forecasting compared to traditional ARIMA models.
ARFIMAX models are used for: volatility modeling and forecasting; predicting stock returns; risk management; and financial analysis.

[viii] Dito v.

[ix] In statistics, *heteroskedasticity* (or *heteroscedasticity*) refers to a situation where the variance of the residual errors in a regression model is not constant across all values of the independent variable(s), meaning the spread of the errors changes as the independent variable changes.
What it is: *Heteroskedasticity* violates the assumption of *homoskedasticity* (constant variance) in linear regression models.
Visual Representation: When heteroskedasticity is present, the scatterplot of the residuals (the difference between the predicted and actual values) will show a "*fanning out*" or "*cone-shaped*" pattern, indicating that the spread of the errors increases or decreases as the independent variable changes.
Consequences: Heteroskedasticity can lead to unreliable statistical inferences, such as biased standard errors, inaccurate confidence intervals, and invalid hypothesis tests.

Detection: Several statistical tests, like the Breusch-Pagan test and the White test, can be used to detect heteroskedasticity.
Examples: Imagine predicting income based on years of education. If the spread of income residuals is larger for individuals with higher incomes than for those with lower incomes, then *heteroskedasticity* may be present.
Another example is measuring the distance traveled by a rocket. Measurements may be accurate to the nearest centimeter in the first few seconds, but after a longer time, the accuracy may be poor due to increased distance, atmospheric distortion, and other factors.
Solutions: To address heteroskedasticity, you can use weighted least squares (WLS) regression, which accounts for the varying variance in the error term or use heteroskedasticity-robust standard errors and covariance matrices.
[x] In the context of insurance insolvency, discriminant analysis is a statistical model used to classify insurance companies as either solvent or insolvent based on their financial ratios and other relevant data.
What it is: *Discriminant analysis* is a statistical technique used to classify observations into two or more groups or categories. In the insurance context, it's used to predict whether an insurance company is likely to become *insolvent* (unable to meet its financial obligations) or remain *solvent*.
How it works: The model uses financial ratios and other data points as input to create a discriminant function (or a set of discriminant functions for more than two groups) that separates the groups.
Linear Discriminant Analysis (LDA): A common type of discriminant analysis is LDA, which assumes that the data are normally distributed and that the variances are equal across groups.
Applications: *Early warning system*: *Discriminant analysis* can serve as an early warning system, helping regulators and investors identify potentially vulnerable insurance companies.
Credit risk assessment: It can be used to assess the creditworthiness of insurance companies seeking funding.
Insolvency prediction: By identifying the financial characteristics that differentiate solvent and insolvent companies, the model can improve the accuracy of insolvency predictions.
Assumptions: The sample measurements are independent from each other. Distributions are normal. Co-variance of the measurements are identical across different classes.
Limitations: The model may not be accurate if the data do not follow the assumptions of LDA. The model may not provide any estimate of the associated risk of bankruptcy; it would be preferable to classify firms in more than two classes according to the level of risk.
[xi] *Flexibility and Versatility*: The EGB distribution is a generalization of the standard beta and exponential distributions, offering more flexibility in modeling various scenarios. This flexibility allows it to capture a wider range of behaviors, including skewed distributions, which are common in many engineering applications.
Bounded and Unbounded Variables: The EGB distribution can be used to model both variables that are *bounded* (like probabilities or proportions, which fall within the range of 0 to 1) and *unbounded* variables (like time to failure or income distributions).
Engineering Applications:
Reliability Engineering: EGB distributions can be used to model failure time data, where the failure rate may increase or decrease over time.
Modeling Uncertainty: The distribution can help model uncertainties in parameters or outcomes, which is crucial in engineering simulations and analysis.
System Safety: The distribution can be used to model the behavior of complex systems, where different components may fail at different rates.
Regression Analysis: The EGB distribution can be used to model the distribution of errors in regression models, especially when the errors are not normally distributed.
Advantages over other distributions:
Flexibility in Shape: Unlike the normal distribution, the EGB distribution offers more flexibility in shaping the density function, allowing it to better fit various data patterns.
Analytical Tractability: The EGB distribution is analytically tractable, making it easier to derive statistical estimations and Bayesian updates.
Modeling Skewness and Kurtosis: The EGB distribution allows for better control over skewness and kurtosis, which are important measures of the shape of a distribution.
Limitations:
Parameter Estimation: Estimating the parameters of the EGB distribution can be challenging, especially in complex cases as is the case in Progri (2016, [1]).
Computational Complexity: In some cases, the EGB distribution can lead to more computationally intensive calculations compared to simpler distributions as is the case in Progri (2016, [1]).
[xii] *Flexibility in Modeling Data Shapes*: The EGBD can represent a wide range of distributions, including symmetric, skewed, and multimodal shapes, making it suitable for various GPS-related data. Its flexible shape parameters allow for modeling non-normal data, which is common in GPS analysis.
Modeling Proportions and Probabilities: The beta distribution, and therefore the EGBD, is well-suited for modeling proportions, probabilities, or percentages, which can be relevant in GPS data analysis. For example, it can be used to model the probability of successful GPS reception or the proportion of GPS satellites in view.
Applications in GPS Analysis, Modeling, and Simulation:
GPS Signal Strength: EGBD can model the probability density of GPS signal strength at different locations, which is crucial for optimizing GPS positioning.
Error Modeling: It can be used to model the distribution of GPS positioning errors, allowing for more accurate error assessments and corrections.
Monte Carlo Simulations: The EGBD can be integrated into Monte Carlo simulations to explore the potential range of outcomes in GPS scenarios, aiding in risk assessment and decision-making.
Trajectory Analysis: It can be used to model the trajectories of moving objects tracked by GPS, providing a more flexible alternative to traditional models.
Conjugacy in Bayesian Analysis: The beta distribution, and its generalizations like EGBD, can be used as conjugate priors in Bayesian analysis, simplifying parameter estimation and updating. This is particularly useful when dealing with limited or noisy GPS data.
Special Cases and Applications: The EGBD includes several special cases, such as the logistic distribution and the Gumbell distribution, making it versatile for various modeling tasks. It has been used in various fields, including financial modeling, actuarial science, and regression analysis.
[xiii] *Flexibility*: The EGB distribution is a five-parameter generalization of the standard logistic distribution and of the generalized exponential distribution, allowing for modeling a wider range of skewed and heavy-tailed distributions than the standard logistic. This flexibility is crucial for capturing the complex variations in tropospheric distortion.
Skewness and *Leptokurtosis*: Tropospheric distortion can exhibit both positive and negative skewness, as well as heavier tails than a normal distribution. The EGB distribution, being a generalization of the logistic distribution, can accommodate these features.
Special Cases: The EGB distribution includes other distributions as special cases, providing a framework for understanding the relationships between different distributions and their potential applications in modeling tropospheric distortion.
Modeling Complex Shapes: The generalized Beta distribution allows for modeling shapes like "∪" or "∩" hazard functions, which can be useful for representing the non-monotonic behavior of tropospheric distortion at different altitudes or over time.
In summary, the EGB distribution (or its special cases) offers a powerful and flexible tool for modeling tropospheric distortion, particularly when dealing with skewed or heavy-tailed data. It provides a more general approach than standard normal distributions and logistic distributions, allowing for a better representation of the complex patterns observed in real-world tropospheric data.
[xiv] *Multipath Channels*: In wireless communication, multipath refers to the phenomenon where radio signals travel along multiple paths from the transmitter to the receiver, leading to interference and signal distortion. Modeling the behavior of these paths, including their delays and gains, is crucial for designing robust communication systems.
EGB as a Model: The EGB provides a flexible framework for modeling the distribution of these multipath parameters. Its parameters allow for capturing a range of shapes, from exponential-like to beta-like distributions, making it suitable for diverse multipath environments.
Advantages:
Flexibility: The EGB can capture both exponential and beta-like characteristics, allowing it to model a broader range of multipath scenarios than traditional distributions.
Analytical Tractability: While more complex than some simpler distributions, the EGB still allows for some analytical computations, making it practical for analysis and optimization.

Applications:
Channel Modeling: The EGB can be used to model the statistics of delay spread, power delay profile, and other channel characteristics relevant to multipath fading.
Performance Analysis: By understanding the EGB's parameters, researchers can analyze the performance of communication systems in multipath environments, predict system reliability, and optimize system design.
Receiver Design: The EGB can inform the design of receivers that can effectively mitigate the effects of multipath interference.
In summary, the EGB is a valuable tool for modeling *multipath distributions* and *distortions* due to its flexibility, analytical tractability, and ability to capture a wide range of multipath phenomena.

[xv] *Laurent Schwartz* (late 1940s): Schwartz developed a rigorous mathematical framework for distributions, making them widely applicable in analysis. His work provided a foundation for understanding generalized functions, including the Dirac delta function.
Sergei Sobolev (1930s): Sobolev's work on partial differential equations led to the initial exploration of generalized functions, specifically in the context of finding solutions that were not necessarily ordinary functions.
Historical Context: The practical use of distributions can be traced back to the use of Green's functions in the 1830s, but they were not formalized until later.
In summary, while the roots of generalized distributions can be found in earlier work, *Laurent Schwartz* is credited with developing the comprehensive and rigorous theory of distributions in the late 1940s.

[xvi] Ditto Progri (2024, [8]) vii and viii.

[xvii] The cdf values are always between zero and one, representing the full probability of the random variable being less than or equal to x.

[xviii] The cdf values are always between zero and one, representing the full probability of the random variable being less than or equal to x.

[xix] The logistic function itself was introduced by Pierre Verhulst in 1838 as a model for population growth. Ditto Progri (2024, [8]) ix.
Balakrishnan and Leung (1988, [11]) proposed three types of generalized logistic distribution, including Type III.
Johnson et al. (1995, [12]) summarized various generalizations of the logistic distribution, including the Type III version.

[xx] The femto is a unit prefix in the International System of Units (SI) that represents a factor of 10^−15, or one quadrillionth. The prefix is derived from the Danish word femten, which means *fifteen*.

[xxi] Ditto Progri(2020, [50]) EN vii.