Giftet Inc.
Journal of Geolocation, Geo-information, and Geo-intelligence
Volume 2024, article ID 2024071603, 15 pages
https://doi.org/10.18610/JG3.2024.071603

Research Article

On the Computation of the Cumulative Probability Distribution Function of the Exponential Generalized Beta Nongeneralized Family of Distributions

To my Prof. Dennise Nicoletti, a phenomenal teacher, scholar, researcher, the best female professor who I have ever known in my life.[i]

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Received May 8, 2024; Revised May 19, 2024-June 17, 2024, Accepted July 16, 2024; Published November 1, 2024.

Editor-in-Chief/Senior Scientific Editor: Ilir F. Progri

This is a continuation of the research on the computation of the cumulative probability distribution function (cdf) of the exponential generalized Beta (EGB) family of distributions. In 2016, I published for the first time a rather difficult journal paper on the 5-paramater EGB[ii] (or EGB3) probability density function (pdf), cdf, mean, and variance. However, years later I felt that there was so much more information in that journal paper that needed to be explained in great detail. This is the second paper that aims to do just that. The genesis and the fundamental properties of the cdf are described in thorough examination. This paper focuses on nongeneralized EGB cdfs such as the Exponential Weibull (EW), Gompertz, Gumbel, Logistic, and Normal (or Gaussian) cdf. The limit calculation and the non-decreasing monotonicity tests are described in minute details. We perform *analytical* and *conceptual* assessments to gain a comprehensive understanding of the EGB3 cdf family of nongeneralized distributions, by exploring both the *what* (*analytical*) and the *why* (*conceptual*) behind it, leading to profound understanding of the power and flexibility that comes from the analysis, modeling, and simulation of the EGB3 cdf.

Index Terms—Exponential generalized Beta, probability density function, cumulative distribution function, Exponential Weibull, Gompertz, Gumbel, Logistic, Normal.

1 Introduction

Progri (2016, [1]) was the first paper I published on the exponential generalized Beta (EGB3) distribution, also known as the EGB of type 3 (EGB3[iii]). However, in Progri (2016, [1]) I introduced the material, as I often do, very fast; meaning that

I skipped a lot of helpful or important information. There was a brief description of the EGB probability density function (pdf), cumulative probability distribution function (cdf), and the statistics parameters such as the moment generating function, mean and variance.

I cannot argue with the number of equations I created, nor can I argue with the computation of the equations. I can; however, argue with the connection that that EGB pdf and cdf from Progri (2016, [1]) have with other distribution models and how they were derived and used in technical literature and who the main contributors were and what was significant about their contributions.

In Progri (2022, [2]) I performed a study of the computation of the exponential function for both the positive and negative exponents. The wisdom, knowledge, and understanding that was produced from Progri (2022, [2]) will be employed throughout Progri (2024, [3]) and in this journal paper.

For this reason I aimed to enhance what were the issues with the lack of mention of the main contributors who created these models in the first place in Progri (2016, [1]) with a series of journal papers. The first journal paper in Progri (2024, [3]) I aimed at improving the understanding of the EGB3 pdf and of the EGB family of pdfs and added a rather unique introduction highlighting in great depth the main contributors and their most significant contributions as it relates to the fundamental nongeneralized cdfs of the EGB family of distributions.

This journal paper aims at ameliorating the content related to the EGB cdf and of the EGB family of cdfs. However, before we dive into a deeper discussion on the main theme of the journal paper I will continue with the introduction of the main contributors and their contributions that were left off in Progri (2024, [3]).

There is a section in the paper dedicated to genesis and the fundamental properties of the cdf. The connection of the fundamental theorem of calculus, the derivative properties, the fundamental inequalities, and the non-decreasing monotonicity nature of the cdf are given in the paper.

The genesis of the fundamental theorem of calculus are with James Gregory in 1600s, Isaac Barrow 1670, Isaac Newton in 1680s and Gottfried Leibniz in 1686.

The original contribution of the Exponential Weibull (EW) distribution coming from Mudholkar et al. (1995, [4]) in included in the paper.

The Gompertz distribution was developed by Benjamin Gompertz, an English mathematician and actuary. He introduced the distribution in 1825, outlining its use in describing mortality rates and actuarial tables. Gompertz's work showed that mortality rates tend to increase exponentially with age, a concept central to the Gompertz distribution.

The Gumbel distribution was developed by Emil Julius Gumbel and published in his book Statistics of Extremes in 1958. Gumbel was a German mathematician and political writer who made significant contributions to the field of extreme value theory.

The logistic distribution was first introduced by Pierre-François Verhulst in 1838 [10], [18]. Verhulst developed it to model population growth and later named it *logistic* in his 1845 publication Nassar, Elmasry (2012, [10]). The function itself was also independently rediscovered and applied in various fields, including autocatalytic chemical reactions.

This paper is organized as follows: in Sect. 2 The Genesis and Fundamental Properties of cdf are deliberated in comprehensive narration. Non-generalized cdfs derived from the EGB3 cdf are detailed in Sect. 3. Analytical, conceptual results are deliberated in Sect. 4. Conclusion is provided in Sect. 5, Acknowledgment in Sect. 6, along with a list of references in Sect. 7.

2 The Genesis and Fundamental Properties of CDF

The relationship between the pdf [11] and the cdf [12] was established through the *Fundamental Theorem of Calculus*. Specifically, the cdf of a continuous random variable is found by integrating the pdf from negative infinity to a given value, x as follows:

$$\begin{aligned} F(x) &= \Pr(X \geq x) \\ &= \int_{-\infty}^{x} f(t)dt \end{aligned} \tag{1}$$

Fundamental Theorem of Calculus: Equation (1) shows the fundamental relationship between the cdf, $F(x)$, and the pdf, $f(t)$, via the *Fundamental Theorem of Calculus*.

The *Fundamental Theorem of Calculus* was developed through the work of several mathematicians. James Gregory first formulated and proved a rudimentary form or a restricted version in the 1600s.

Isaac Barrow significantly contributed to the fundamental theorem of calculus, proving it in a more general form than previous versions around 1670. He achieved this by linking the problems of finding areas (quadratures) and tangents.

Isaac Barrow, a professor at Cambridge University, made a significant improvement by proving a more general version of the theorem. He did this in his lectures from 1664-1666, which were later published as Lectiones Geometricae in 1670.

Barrow's proof used a geometric approach, connecting the problems of finding areas under curves (quadratures) and determining tangents to curves. This established the relationship between *differentiation and integration*, a *key insight of the fundamental theorem*.

Isaac Newton, Barrow's student, further developed the theorem and the surrounding mathematical theory. According to some sources, Newton also independently discovered the theorem around the same time, recognizing its power in solving integration problems.

Isaac Newton did not *improve* the fundamental theorem of calculus in the sense of refining a pre-existing theorem. Instead, he and Gottfried Wilhelm Leibniz independently developed and published it in the late 17th century. While Newton began working on calculus in the 1660s, his ideas were not widely published until the 1680s and 1690s.

Newton's work on fluxions (a method of infinitesimal calculus) was completed in 1671 and later published in 1736. His first independent treatise on calculus, "De Analysi," was written in 1669 and published in 1711.

Gottfried Leibniz began working on calculus in 1674 and published his work in 1684, independently of Newton. He also developed the notation and theoretical framework for calculus that we still use today.

A controversy arose over who discovered calculus first, but it is now widely accepted that both Newton and Leibniz independently developed the calculus. Newton was credited with the first discovery, while Leibniz was credited with the first published work.

Differential Calculus: The fundamental theorem of calculus, in its modern form, connects *differential calculus* and *integral calculus*, showing that differentiation and integration are inverse operations. It states that if there exists a function $F(x)$ that represents the area under the curve $f(x)$ from 0 to x, then the derivative of $F(x)$ is $f(x)$. Taking the derivative of (1) shows the link between the *differential calculus* of pdf, $f(x)$, as derivative of the *integral calculus* cdf, $F(x)$.

$$f(x) = \frac{dF(x)}{dx} \quad (2)$$

The invention of the cdf did not have a single inventor, but its development and formalization involved several key figures. Abraham de Moivre approximated the cdf of the normal distribution in 1733.

Pafnuty Chebyshev's publication titled "Mémoire sur les valeurs moyennes" (Memoir on Average Values), published in 1846, discusses the cumulative distribution function. In this paper, Chebyshev introduced what is now known as Chebyshev's inequality, which provides a bound on the probability of a random variable deviating from its mean (or expected value) by a certain amount. This inequality is closely related to the cdf, as it helps determine the probability of observing a value within a specific range based on the distribution's overall shape.

Fundamental Inequalities: There are two fundamental inequalities of the cdf, $F(x)$.

First, the cdf function is always greater than or equal to zero meaning, the probability of the event has to be a non-negative number, in other words, the event has to be greater than minus infinity; meaning no event can occur before minus infinity

$$F(x) \geq 0 = \Pr(X \leq x > -\infty) \quad (3)$$

Second, the cdf function is always less than or equal to one, meaning the probability of the event has to be a number smaller than or equal to one; i.e., the event has to be less than infinity; i.e. all events have occurred before infinity.

$$F(x) \leq 1 = \Pr(X \leq x < \infty) \quad (4)$$

Equations (3) and (4) are known as the two fundamental inequalities of the cdf.

Calculating Limits: The two fundamental inequalities of the cdf are based on the calculation of the limits. As x approaches negative infinity, the probability of X being less than or equal to x also approaches zero, so $F(x)$ approaches 0 because only the empty space $\emptyset$ exists before minus infinity; i.e.,

$$\lim_{x \to -\infty} F(x) = 0 = \Pr(\emptyset \leq x = -\infty) \quad (5)$$

As x approaches positive infinity, the probability of X being less than or equal to x approaches one, so $F(x)$ approaches one, meaning the entire observation space has been accounted for when the main variable x approaches plus infinity; hence, the probability must be equal to one as shown below:

$$\lim_{x \to +\infty} F(x) = 1 = \Pr(S \leq x = +\infty) \quad (6)$$

These limits are fundamental to the concept of probability and the cdf and are not the result of a calculation by any specific individual. They are consequences of the definition of probability and the properties of continuous probability distributions.

The right-continuity of the cdf is a fundamental property inherent to its definition and not explicitly formulated by a single individual. It arises naturally from the properties of probabilities and how the cdf is defined, specifically its limit as x approaches a given point from the right. The right-continuity ensures that the cdf is a non-decreasing function and allows for the accurate calculation of probabilities.

For a cdf to be right-continuous at point x, the limit as you approach x from values greater than x (a+) must equal the function's value at a ($F(a)$).

$$\lim_{x\to a_+} F(x) = F(a) = \Pr(X \leq x = a) \tag{7}$$

Equation (7) essentially says that as x gets closer and closer to a from values larger than a, the probability $F(x)$ approaches the probability $F(a)$.

In 1933, Andrey Kolmogorov's publication "Sulla determinazione empirica di una legge di distribuzione" in Giornale dell\'Istituto Italiano degli Attuari discusses the cdf. His paper "On the empirical determination of a distribution" introduced the concept of the empirical distribution function and explored how well it approximates the true distribution when the true distribution is continuous.

Non-Decreasing Monotonicity: It is difficult to pinpoint one specific individual as the first to recognize the monotonicity of the cdf. The concept is deeply rooted in probability theory and is inherent to how cdfs are defined.

The concept was not discovered by one person but rather evolved as probability theory developed. It is a necessary property that arises from the way probabilities are combined and understood.

While no single individual is credited with discovering monotonicity, mathematicians like Paul Lévy, who worked on characteristic functions and distributions, are important figures in the development of probability theory, including the study of cdfs.

The cdf function $F(x)$ is monotonically *non-decreasing* meaning, for any two quantities, x_1 less than or equal to x_2 the corresponding functions $F(x_1)$ will be less than or equal to $F(x_2)$ as shown below:

$$-\infty < x_1 \leq x_2 < \infty \tag{8}$$

$$0 \leq F(x_1) \leq F(x_2) \leq 1 \tag{9}$$

CDFs are defined as non-decreasing functions, meaning they always increase or stay the same as the input value increases. This is a direct consequence of the definition of the cdf as a cumulative probability.

This concludes the brilliant description of the genesis and fundamental properties of cdf which I am certain cannot be found in this level of detain from any of the probability theory textbooks.

3 Nongeneralized CDFs Derived from the EGB3 CDF

In Progri (2024, [3]) the connection of the nongeneralized pdfs from the EGB3 pdf family of distribution is shown. In this section we consider the EW cdf in Sect. 3.1; The Gompertz cdf in Sect. 3.2; The Gumbel cdf in Sect. 3.3; The Logistic cdf in Sect. 3.4; and the Norma cdf in Sect. 3.5.

3.1 The Exponential Weibull CDF

The Weibull distribution, a foundation for the EW distribution, was invented by Swedish engineer and mathematician Waloddi Weibull in 1937. The EW distribution, a modification of the original, was developed later by researchers like Mudholkar and Srivastava (1995, [4]).

Weibull's original work focused on the Weibull distribution, a probability distribution used to model failure rates and other phenomena.

EW is a generalized form of Weibull distribution, where the Weibull cdf function is exponentiated. This addition of a second shape parameter provides greater flexibility in modeling various types of data.

Mudholkar and Srivastava (1995, [4]) are credited with introducing the exponentiated Weibull distribution, a key modification of the original Weibull distribution.

The EW [14] cdf [12], also a special case of EGB [13] cdf, is given by

$$\begin{aligned} F_{\mathrm{EW}}(x; b, c) &= \int_{-\infty}^{x} f_{\mathrm{EW}}(t; b, c) dt, b > 0, c > 0 \\ &= 1 - e^{-be^{cx}}, b > 0, c > 0 \end{aligned} \tag{10}$$

The EW [14] cdf [12] is an analytical CFE that involves the computation of only the exponential function for both positive and negative exponents. For a comprehensive discussion on the computation of the exponential functions the reader may refer to Progri (2022, [2]).

Calculating Limits[iv]: Let us examine what happens when $x \to -\infty$ then we have

$$\begin{aligned} F_{\mathrm{EW}}(x \to -\infty; b, c) &= \lim_{x\to-\infty} \left(1 - e^{-be^{cx}}\right) \\ &= 1 - \lim_{x\to-\infty} e^{-be^{cx}} \end{aligned}$$

$$= 1 - e^{\lim_{x\to-\infty} -be^{cx}}$$
$$= 1 - e^{-b \lim_{x\to-\infty} e^{cx}}$$
$$= 1 - e^{-b\times 0}$$
$$= 1 - 1$$
$$= 0 \quad (11)$$

The technique that was used for calculating the first limit is the direct substitution. As x approaches negative infinity, the F_{EW} cdf approaches zero (no events occur before negative infinity).

Next, we examine the following limit when $x \to \infty$ then we have

$$F_{\text{EW}}(x \to \infty; b, c) = \lim_{x\to\infty}\left(1 - e^{-be^{cx}}\right)$$
$$= 1 - \lim_{x\to\infty} e^{-be^{cx}}$$
$$= 1 - e^{\lim_{x\to\infty} -be^{cx}}$$
$$= 1 - e^{-b \lim_{x\to\infty} e^{cx}}$$
$$= 1 - e^{-b\times\infty}$$
$$= 1 - 0$$
$$= 1 \quad (12)$$

The technique that was used for calculating the second limit is the direct substitution. As x approaches positive infinity, the F_{EW} cdf approaches one (all events are less than or equal to positive infinity).

The EW [14] cdf [12] is also right continuous meaning

$$\lim_{x\to x^+} F_{\text{EW}}(x; b, c) = 1 - \lim_{x\to x^+} e^{-be^{cx}}, b > 0, c > 0$$
$$= 1 - e^{-be^{cx}}, b > 0, c > 0 \quad (13)$$

Although. for most continuous cdfs this property of the cdfs is often understood.

Non-Decreasing Monotonicity[v]: Next, we need to prove that the function $F_{\text{EW}}(x; b, c)$ is monotonically *non-decreasing*; i.e., for any

$$-\infty < x_1 \le x_2 < \infty \quad (14)$$

We need to prove that

$$0 \le F_{\text{EW}}(x_1; b, c) \le F_{\text{EW}}(x_2; b, c) \le 1 \quad (15)$$

Let us start with a known inequality of the exponential function

$$0 \le e^{cx_1} \le e^{cx_2} < \infty, c > 0 \quad (16)$$

If we multiply both sides of (16) with $-b$ then we obtain the following

$$-\infty \le -be^{cx_2} \le -be^{cx_1} \le 0, b, c > 0 \quad (17)$$

Taking the exponential of both sides of (17) yields

$$0 \le e^{-be^{cx_2}} \le e^{-be^{cx_1}} \le 1, b, c > 0 \quad (18)$$

Multiplying both sides of (18) with -1 produces

$$-1 \le -e^{-be^{cx_1}} \le -e^{-be^{cx_2}} \le 0, b, c > 0 \quad (19)$$

Adding 1 on both sides of (19) we can obtain the desired inequality (15)

$$0 \le 1 - e^{-be^{cx_1}} \le 1 - e^{-be^{cx_2}} \le 1, b, c > 0 \quad (20)$$

We could have used another approach to show the monotonic nature of the EW [14] cdf [12] by taking the derivative of (10) and showing that the derivative is equal to the EW [14] pdf [11] Progri (2024, [3] (11)) always greater or equal to zero as shown in Progri (2024, [3] (16)).

In summary on the EW [14] cdf [12] I have shown the CFE (10), the limits in (11), (12), and (13), and the monotonic nature in (14) through (20), and I have shown the connection with Progri (2024, [3] (11), (16)).

This concludes the comprehensive discussion on the EW [14] cdf [12]. Next, we continue the description of the Gompertz cdf.

3.2 *The Gompertz CDF*

The Gompertz[vi] [15] cdf [12] is the first, the simplest, and the fundamental model of understanding human mortality. Gompertz distribution can be applied to model various engineering phenomena, like in reliability engineering and demography, where the failure rate increases with time, especially those involving growth or decay processes, such as population dynamics, tumor growth, and survival analysis.

Because the above is for all the values of $-\infty < x < \infty$; hence, we need $F_{\text{Gom}}(x = 0; b, c) = 0$; hence, Gompertz [15] cdf [12] are given by

$$F_{\text{Gom}}(x; b, c) = \int_{-\infty}^{x} f_{\text{Gom}}(t; b, c)dt$$
$$= \int_{0}^{x} f_{\text{Gom}}(t; b, c)dt$$
$$= \begin{cases} 0, & -\infty < x \le 0 \\ 1 - e^{-be^{cx}+b}, & 0 < x < \infty \end{cases}$$
$$= \begin{cases} 0, & -\infty < x \le 0 \\ 1 - e^{b}e^{-be^{cx}}, & 0 < x < \infty \end{cases} \quad (21)$$

The Gompertz [15] cdf [12] is an analytical CFE that involves only the exponential function for both positive and negative exponents. For a thorough investigation into the computation of the exponential functions the reader may refer to Progri (2022, [2]).

Calculating Limits: Let us examine what happens when $x \to 0$ then we have

$$F_{\text{Gom}}(x \to 0; b, c) = \lim_{x\to 0}\left(1 - e^{b}e^{-be^{cx}}\right)$$

$$= 1 - \lim_{x \to 0} e^{b} e^{-be^{cx}}$$
$$= 1 - e^{b} e^{\lim_{x\to 0} -be^{cx}}$$
$$= 1 - e^{b} e^{-b \lim_{x\to 0} e^{cx}}$$
$$= 1 - e^{b} e^{-b\times 1}$$
$$= 1 - 1$$
$$= 0 \quad (22)$$

The technique that was used for calculating the first limit is the direct substitution. As *x* approaches negative infinity, the F_{Gom} cdf approaches zero (no events occur before negative infinity).

Next, we examine the following limit when $x \to \infty$ then we have

$$F_{\mathrm{Gom}}(x \to \infty; b, c) = \lim_{x\to\infty}\left(1 - e^{b} e^{-be^{cx}}\right)$$
$$= 1 - \lim_{x\to\infty} e^{b} e^{-be^{cx}}$$
$$= 1 - e^{b} e^{\lim_{x\to\infty} -be^{cx}}$$
$$= 1 - e^{b} e^{-b \lim_{x\to\infty} e^{cx}}$$
$$= 1 - e^{b} e^{-b\times\infty}$$
$$= 1 - 0$$
$$= 1 \quad (23)$$

The technique that was used for calculating the second limit is the direct substitution. As *x* approaches positive infinity, the F_{Gom} cdf approaches one (all events are less than or equal to positive infinity).

The Gompertz [15] cdf [12] is also right continuous meaning

$$\lim_{x\to x^{+}} F_{\mathrm{Gom}}(x; b, c) = 1 - e^{b} \lim_{x\to x^{+}} e^{-be^{cx}}, b > 0, c > 0$$
$$= 1 - e^{b} e^{-be^{cx}}, b > 0, c > 0 \quad (24)$$

As for most continuous cdfs, property (24) of Gompertz [15] cdf [12] is often understood.

Non-Decreasing Monotonicity: Next, we need to prove that the function $F_{\mathrm{Gom}}(x; b, c)$ is *monotonically non-decreasing*; i.e., for any

$$0 \le x_1 \le x_2 < \infty \quad (25)$$

we need to prove that

$$0 \le F_{\mathrm{Gom}}(x_1; b, c) \le F_{\mathrm{Gom}}(x_2; b, c) \le 1 \quad (26)$$

Let us start with a known inequality of the exponential function

$$1 \le e^{cx_1} \le e^{cx_2} < \infty, c > 0 \quad (27)$$

If we multiply both sides of (27) with $-b$ then we obtain the following

$$-\infty < -be^{cx_2} \le -be^{cx_1} \le -b, b, c > 0 \quad (28)$$

Taking the exponential of both sides of (28) yields

$$0 \le e^{-be^{cx_2}} \le e^{-be^{cx_1}} \le e^{-b}, b, c > 0 \quad (29)$$

Multiplying both sides of (29) with $-e^{b}$ produces

$$-1 \le -e^{b} e^{-be^{cx_1}} \le -e^{b} e^{-be^{cx_2}} \le 0, b, c > 0 \quad (30)$$

Adding 1 on both sides of (30) we can obtain the desired inequality (26)

$$0 \le 1 - e^{b} e^{-be^{cx_1}} \le 1 - e^{b} e^{-be^{cx_2}} \le 1, b, c > 0 \quad (31)$$

We could have used another approach to show the monotonic nature of the Gompertz [15] cdf [12] by taking the derivative of (21) and showing that the derivative is equal to the Gompertz [15] pdf [11] Progri (2024, [3] (19)) always greater or equal to zero as shown in Progri (2024, [3] (20)).

In summary on the Gompertz [15] cdf [12] I have shown the CFE (21), the limits in (22), (23), and (24), and the monotonic nature in (25) through (31), and I have shown the connection with Progri (2024, [3] (19), (20)).

This concludes the discussion on the Gompertz [15] cdf [12]. Next, we continue the discussion of the Gumbel cdf.

3.3 The Gumbel CDF

The Gumbel distribution[vii][viii] [16] is primarily used to model the distribution of extreme values, making it useful in fields like hydrology, meteorology, and engineering for analyzing extreme events like floods, storms, and earthquakes. It is also employed in risk analysis and economics for understanding rare but significant events.

The Gumbel (type-I generalized extreme value distribution [17]) [16] cdf [12] as follows:

$$F_{\mathrm{Gum}}(x; a) = e^{-e^{-ax}}; a > 0 \quad (32)$$

The Gumbel (type-I generalized extreme value distribution [17]) [16] cdf [12] is an analytical CFE that involves the computation of only the exponential function for both positive and negative exponents. For a complete examination on the computation of the exponential functions the reader may refer to Progri (2022, [2]).

Calculating Limits: Let us examine what happens when $x \to -\infty$ then we have

$$F_{\mathrm{Gum}}(x \to -\infty; a) = \lim_{x\to-\infty} e^{-e^{-ax}}$$
$$= e^{\lim_{x\to-\infty} -e^{-ax}}$$
$$= e^{-\lim_{x\to-\infty} e^{-ax}}$$
$$= e^{-\infty}$$
$$= 0 \quad (33)$$

The technique that was used for calculating the first limit (33) is the direct substitution. As *x* approaches negative infinity, the F_{Gom} cdf approaches zero (no events occur before negative infinity).

Next, we examine the following limit when $x \to \infty$ then we have

$$
\begin{aligned}
F_{\text{Gum}}(x \to \infty; a) &= \lim_{x\to\infty} e^{-e^{-ax}} \\
&= e^{\lim_{x\to\infty} -e^{-ax}} \\
&= e^{-\lim_{x\to\infty} e^{-ax}} \\
&= e^{-0} \\
&= 1 \qquad (34)
\end{aligned}
$$

The technique that was used for calculating the second limit (34) is the direct substitution. As x approaches positive infinity, the F_{Gum} cdf approaches one (all events are less than or equal to positive infinity).

The Gumbel (type-I generalized extreme value distribution [17]) [16] cdf [12] is also right continuous meaning

$$
\begin{aligned}
\lim_{x\to x^+} F_{\text{Gum}}(x; a) &= \lim_{x\to x^+} e^{-e^{-ax}}, a > 0 \\
&= e^{-e^{-ax}}, a > 0 \qquad (35)
\end{aligned}
$$

As for most continuous cdfs, property (35) of Gumbel (type-I generalized extreme value distribution [17]) [16] cdf [12] is often understood.

Monotonicity: Next, we need to prove that the function $F_{\text{Gum}}(x; a)$ is monotonically increasing; i.e., for any (14) we need to prove that

$$0 \le F_{\text{Gum}}(x_1; a) \le F_{\text{Gum}}(x_2; a) \le 1 \qquad (36)$$

Let us start with a known inequality of the exponential function

$$\infty > e^{-ax_1} \ge e^{-ax_2} \ge 0, a > 0 \qquad (37)$$

If we multiply both sides of (37) with -1 then we obtain the following

$$-\infty < -e^{-ax_1} \le -e^{-ax_2} \le 0, a > 0 \qquad (38)$$

Taking the exponential of both sides of (38) yields the desired result (36)

$$0 < e^{-e^{-ax_1}} \le e^{-e^{-ax_2}} \le 1, a > 0 \qquad (39)$$

We could have used a different approach to prove the monotonic nature of the Gumbel (type-I generalized extreme value distribution [17]) [16] cdf [12] by taking the derivative of (32) and showing that the derivative is equal to the Gumbel (type-I generalized extreme value distribution [17]) [16] pdf [11] Progri (2024, [3] (24)) always greater or equal to zero as shown in Progri (2024, [3] (25)).

In summary on the Gumbel (type-I generalized extreme value distribution [17]) [16] cdf [12] I have shown the CFE (32), the limits in (33), (34), and (35), and the monotonic nature in (36) through (39), and I have shown the connection with Progri (2024, [3] (24), (25)).

This concludes the discussion on the Gumbel [17] and the generalized Gumbel (or type I Gumbel [17]) cdf [12]. Next, we continue the discussion of the Logistic cdf.

3.4 The Logistic CDFs

Verhulst's Work (1838): Verhulst introduced the logistic function[ix] as a model for population growth, extending the exponential growth model to include a carrying capacity.

The Logistic [18] cdf [12] is obtained from the EGB3 in Progri (2016, [1]) cdf by setting the parameters as a equal to minus a, b equal to b, d equal to one, p equal to one, and q equal to one as follows:

$$
\begin{aligned}
F_{\text{Log}}(x; a, b) &= F_{\text{EGB}}(x; a = -a, b, d = 1, p = 1, q = 1) \\
&= \frac{\int_0^{x' \equiv e^{-\frac{x-b}{a}}} (1+e^v)^{-2} dv}{\mathrm{B}(1,1)} \\
&= \mathrm{I}_z(1,1) \\
&= \frac{\mathrm{B}(1,1;z)}{\mathrm{B}(1,1)} \\
&= (1 + x')^{-1} \\
&= \left(1 + e^{-\frac{x-b}{a}}\right)^{-1} \qquad (40)
\end{aligned}
$$

The Logistic [18] cdf [12] is an analytical CFE that involves the computation of only the exponential function for both positive and negative exponents. For an elaborate deliberation on the computation of the exponential functions the reader may refer to Progri (2022, [2]).

Calculating Limits: Let us examine what happens when $x \to -\infty$ then we have

$$
\begin{aligned}
F_{\text{Log}}(x \to -\infty; a, b) &= \lim_{x\to-\infty} \frac{1}{1+e^{-\frac{x-b}{a}}} \\
&= \frac{1}{\lim_{x\to-\infty} 1+e^{-\frac{x-b}{a}}} \\
&= \frac{1}{1+\lim_{x\to-\infty} e^{-\frac{x-b}{a}}} \\
&= \frac{1}{1+e^{\infty}} \\
&= \frac{1}{1+\infty} \\
&= \frac{1}{\infty} \\
&= 0 \qquad (41)
\end{aligned}
$$

The technique that was used for calculating the first limit (41) is the direct substitution. As x approaches negative infinity, the F_{Log} cdf approaches zero (no events occur before negative infinity).Next, we examine the following limit when $x \to +\infty$ then we have

$$\begin{aligned} F_{\text{Log}}(x \to +\infty; a, b) &= \lim_{x\to+\infty} \frac{1}{1+e^{-\frac{x-b}{a}}} \\ &= \frac{1}{\lim_{x\to+\infty} 1+e^{-\frac{x-b}{a}}} \\ &= \frac{1}{1+\lim_{x\to+\infty} e^{-\frac{x-b}{a}}} \\ &= \frac{1}{1+e^{-\infty}} \\ &= \frac{1}{1+0} \\ &= \frac{1}{1} \\ &= 1 \end{aligned} \tag{42}$$

The technique that was used for calculating the second limit (42) is the direct substitution. As x approaches positive infinity, the F_{Log} cdf approaches one (all events are less than or equal to positive infinity).

The Logistic [18] cdf [12] is also right continuous meaning

$$\begin{aligned} \lim_{x\to x^+} F_{\text{Log}}(x; a, b) &= \lim_{x\to x^+} \left(1 + e^{-\frac{x-b}{a}}\right)^{-1}, a > 0, b > 0 \\ &= \left(1 + e^{-\frac{x-b}{a}}\right)^{-1}, a > 0, b > 0 \end{aligned} \tag{43}$$

As for most continuous cdfs, property (43) of Logistic [18] cdf [12] is often understood.

Non-Decreasing Monotonicity: Next, we need to prove that the function $F_{\text{Log}}(x; a, b)$ is *monotonically non-decreasing*; i.e., for any (14) we need to prove that

$$0 \le F_{\text{Log}}(x_1; a, b) \le F_{\text{Log}}(x_2; a, b) \le 1 \tag{44}$$

Let us start with a known inequality of the exponential function

$$\infty > e^{-a^{-1}x_1+a^{-1}b} \ge e^{-ax_2+a^{-1}b} \ge 0, a > 0, b > 0 \tag{45}$$

If we add one on both sides of (45) then we obtain the following

$$\infty > 1 + e^{-a^{-1}x_1+a^{-1}b} \ge 1 + e^{-ax_2+a^{-1}b} \ge 0, \left| \begin{matrix} a > 0 \\ b > 0 \end{matrix} \right. \tag{46}$$

Taking the inverse of both sides of (46) yields the desired result (44)

$$0 < \frac{1}{1+e^{-a^{-1}x_1+a^{-1}b}} \le \frac{1}{+e^{-ax_2+a^{-1}b}} \le 1, \left| \begin{matrix} a > 0 \\ b > 0 \end{matrix} \right. \tag{47}$$

We could have used a different approach to prove the monotonic nature of the Logistic [18] cdf [12] by taking the derivative of (40) and showing that the derivative is equal to the Logistic [18] pdf [11] Progri (2024, [3] (28)) always greater or equal to zero as shown in Progri (2024, [3] (29)).

In summary on the Logistic [18] cdf [12] I have shown the CFE (40), the limits in (41), (42), and (43), and the monotonic nature in (44) through (47), and I have shown the connection with Progri (2024, [3] (28), (29)).

This concludes the complete discussion on the Logistic [18] cdf [12]. Next, we continue the thorough description of the Normal cdf [12].

3.5 The Normal CDF Description

The normal cdf and the EGB cdf are related through the concept of a generalized distribution. The EGB distribution, which includes the generalized exponential distribution (GED) as a special case, can be seen as a generalization of the normal distribution. Specifically, when certain parameters of the EGB distribution are set to specific values, the EGB cdf converges to the normal cdf.

GED: The GED or EGB1 is a flexible family of distributions that includes the normal distribution as a special case. It is characterized by a parameter that controls the shape of the distribution, allowing it to be more or less skewed than a normal distribution.

GBD: The generalized beta distribution (GBD) is a broader family that includes the GED. It is defined by a shape parameter that controls the shape of the distribution, and it can be used to model a wide range of probability distributions.

EGB Distribution and Convergence: The EGB distribution can be seen as a generalization of both the normal and GEDs. When specific parameters of the EGB distribution are set to certain values, the EGB CDF converges to the normal CDF. This means that the EGB distribution can be used to approximate the normal distribution, and the EGB CDF can be used to calculate probabilities for the normal distribution.

In essence, the EGB distribution offers a way to generalize the normal distribution, allowing for more flexible modeling of data that may not perfectly fit the assumptions of the normal distribution. By controlling the shape parameters of the EGB distribution, one can approximate or even exactly replicate the normal CDF.

Because the normal cdf is such a widely used distribution, Progri (2003, [5]), Progri et al. (2004, [6]), Progri (2006, [7]), Progri et al. (2007, [8]), and Progri (2010, [9]) contain a description of the normal cdf and its relation to other distribution models.

The details discussion of the convergence of the EGB cdf to the normal (or Gaussian) cdf along with other properties of the normal pdf and other statistics will be provided in a separate journal paper.

4 Analytical, Conceptual Results

Analytical results are the findings obtained from carefully examining and analyzing the nongeneralized distributions from Sect. 3. They provide insights, discoveries, and information that can be used to make informed decisions and improve processes. These results can be presented in various formats, including reports, charts, graphs, and dashboards. Section 4.1 includes a brief discussion on significant improvements from conceptual results. Section 4.2 a description of the MATLAB simulation functions is provided. Section 4.3 includes analytical, conceptual examples.

4.1 Significant Improvement from Conceptual Results

Process Improvement: By identifying areas for improvement, analytical results can be used to optimize processes, increase efficiency, and improve overall performance.

Description of the MATLAB simulation functions contains the description of the new MATLAB functions that were built as part of the computation of the analytical and conceptual results.

4.2 Description of MATLAB Simulation Functions

In order to obtain the MATLAB simulations of this journal paper we prepared a rather detailed MATLAB simulation test setup.

The MATLAB simulation consists of seven functions which are explained in order of from slow to fast, then from sharpest to least sharp which is explained in greater detail in Sect. 4.3. The function:

1. logncdf($-x+b_0,a,b$) is the mirror image of the MATLAB BIF, Logo-normal cdf.
2. expcdf($-x+b_0,a$) is the mirror image of the MATLAB BIF, exponential function cdf.
3. gompcdf($a,b,x-b_0$) computes of the Gompertz cdf as depicted in Sect. 3.2.
4. gumbcdf($a,x-b_0$) performs the computation of the Gumbel cdf as indicated in Sect. 3.3.
5. normcdf($x,b-b_0,a$/sqrt(2)) is the Nornal or Gaussian MATLAB BIF cdf.
6. ewcdf($a,b,x-b_0$) computes of the Exponential Weibull cdf which was explained in great detail in Sect. 3.1.
7. logcdf($a,b,x-b_0$) performs the computation of the Logistic cdf as shown in Sect. 3.4.

The parameter b_0 is defined as the dc offset for the value of the cdf to be equal to one half. As indicated later in Sect. 4.3, it is expected that b_0; so as the value of the cdf will be equal to half.

Of all seven functions, four of them were built as part of the Giftet Indoor Geolocation Library Progri (2024, [20]).

This concludes the description of MATLAB simulation functions. Next, we consider the analytical, conceptual examples.

4.3 Analytical, Conceptual Examples

The MATLAB simulation setup for the analytical, conceptual examples is exactly the same as the one in Qualitative, Quantitative Sect. Progri (2024, [3]).

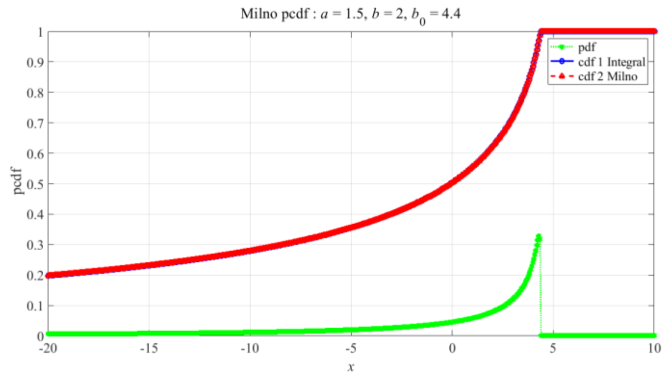
Data Analysis: Analytical results are the outcomes of analyzing data, which involves examining data sets to identify patterns, trends, and relationships as depicted in the Qualitative, Quantitative Sect. Progri (2024, [3]).

Insights and Discoveries: These results provide valuable insights and discoveries that can be used to understand mongrelized distributions CFE, make predictions based on computation of the CFEs and based on MATLAB built-in-functions such as the *integral*, and compute the error based on a simple difference.

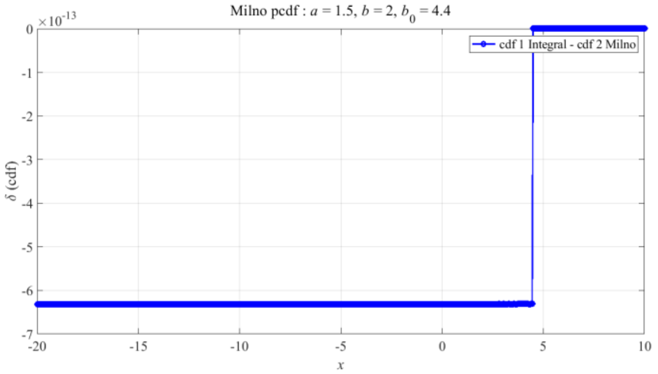
We ran the simulations on a Dell computer Intel(R) Core(TM) i5-2400 CPU @ 3.10 GHz using MATLAB 2024a [23] using a vector of $-20 \le x \le 10$ with a sample increment of $\Delta x = 0.05$ that leads to 601 points.

Figure 1 (a) depicts the computation of the Milno, is an acronym for the mirror image of the logo-normal, pdf and cdf for $-20 \le x \le 10$, $a = 1.5$, $b = 2$, $b_0 = 4.4$. Cdf 1 Integral, Cdf 2 Milno cdf. Is the Milno cdf a valid cdf? Certainly, the Milno cdf is always greater than or equal to zero and less than or equal to one and a nondecreasing monotonic function for all the values of the main variable from minus twenty to plus ten. The only reason why I have shown the plots of the Cdf 2 Milno pdf is to prove numerically that in fact the derivative of the Milno cdf is in fact a valid pdf. Another observation that is made about Milno cdf is that it produces the fattest tale of any of the cdf discussed in the EGB family of distributions. However, it exhibits very slow growth in comparison to other distribution models investigated in this journal paper.

Figure 1(b) shows the error between the Cdf 1 Integral and the Cdf 2 Milno cdf from the logncdf MATLA BIF function. As illustrated from the analytical results of Fig. 1(b) the error



(a) The computation of Milno pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $b_0 = 4.4$. Cdf 1 Integral, Cdf 2 Milno cdf.



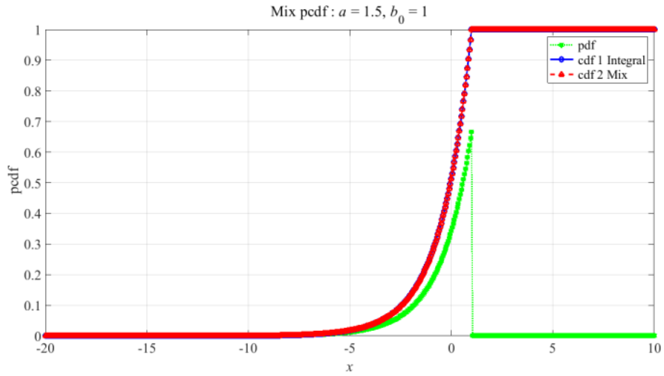
(b) The computation of Milno cdf error for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $b_0 = 4.4$. Cdf 1 Integral minus Cdf 2 Milno cdf.

Fig. 1. The computation of Milno pdf and cdf and the error for $-20 \leq x \leq 10$.

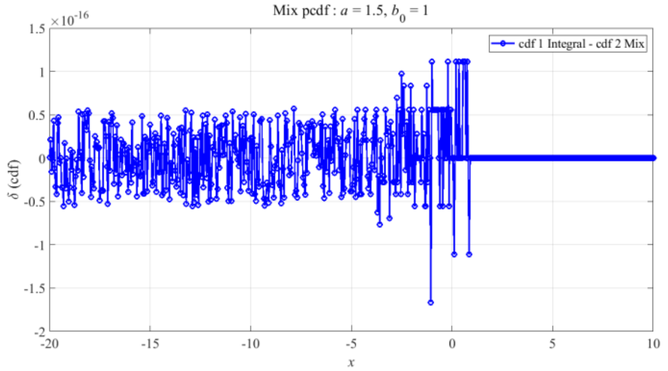
is in the order of 1e-13. The bias that is shown in Fig. 1(b) is most likely due to the computation of the integral.

In Fig. 2(a) it is presented the computation of the MIX, an acronym for the mirror image of the exponential function, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b_0 = 1$. Cdf 1 *Integral* MATLAB BIF, Cdf 2 MIX cdf is the mirror image of the MATLAB BIF, exponential function cdf expcdf($-x+b_0,a$). Is in fact the MIX cdf a valid cdf? For all values of the main variable x from minus twenty to plus ten the MIX cdf is always greater than or equal to zero, less than or equal to one, and a nondecreasing monotonic function. If we take the derivative of the MIX cdf from minus infinity to x the derivative is never negative meaning always greater than or equal to zero. The MIX cdf is ranged as the second cdf that produces the second fattest tale of any of the EGB family of distributions. The growth that results from the MIX cdf is faster than the growth rate that is produced by Milno cdf.

Figure 2(b) shows the error between the Cdf 1 Integral and the Cdf 2 MIX cdf from the expcdf MATLA BIF function. As illustrated from the analytical results of Fig. 2(b) the error is



(a) The computation of MIX pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b_0 = 1$. Cdf 1 Integral, Cdf 2 MIX cdf.

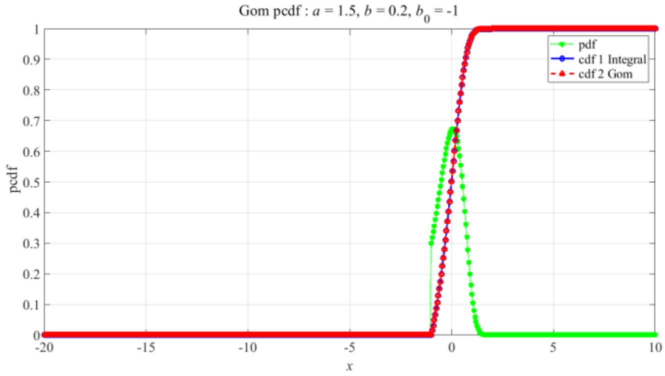


(b) The computation of MIX cdf error for $-20 \leq x \leq 10$, $a = 1.5$, $b_0 = 1$. Cdf 1 Integral minus Cdf 2 MIX cdf.

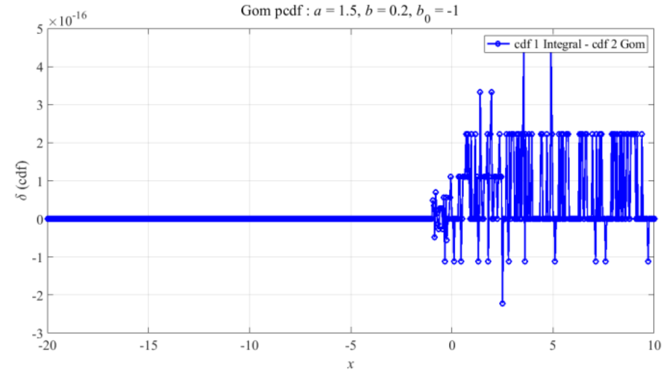
Fig. 2. The computation of MIX pdf and cdf and its error for $-20 \leq x \leq 10$.

on the order of 0.222 femto^x or 10^{-15}. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 MIX cdf from the expcdf MATLAB BIF function.

Figure 3(a) depicts the computation of the Gom, in an acronym for Gompertz, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 0.2$, and $b_0 = -1$. Cdf 1 Integral, Cdf 2 the Gom cdf. The computation of Gom cdf was performed employing (21), the limits in (22), (23), and (24), and the monotonic nature in (25) through (31) in Sect. 3.2. The first question that can be asked about the computation of Gom cdf is about the validity of the Gom cdf for all the values of the main variable x from minus twenty to plus ten. As depicted in Fig. 3(a), Gom cdf is always greater than or equal to zero in accordance with (22) and less than or equal to one in in accordance with (23) Sect. 3.2. If we were to take the derivative with respect to x the value of the derivative is always greater than or equal to zero and the Gom cdf grows monotonically from zero to one in complete agreement with (25)-(31) in Sect. 3.2. Because Gom cdf is such a sharp cdf we took $b = 0.2$ which is ten times smaller than any of the values of b used for the computation

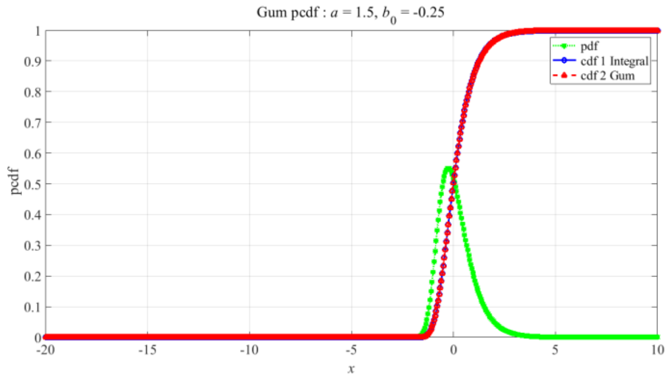


(a) The computation of Gom pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 0.2$, and $b_0 = -1$. Cdf 1 Integral, Cdf 2 Gom cdf

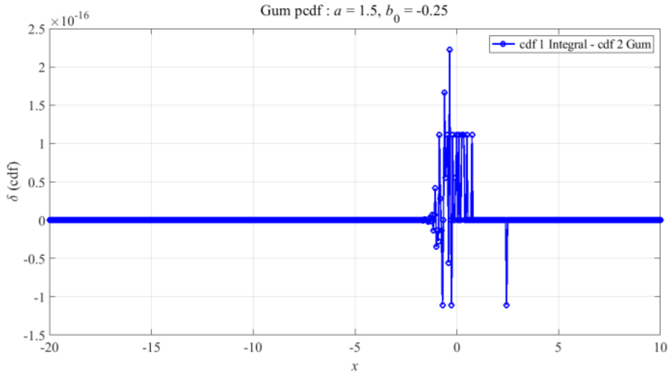


(b) The computation of the Gom cdf error for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $b_0 = -0.7$. Cdf 1 Integral minus Cdf 2 Gom cdf

Fig. 3. The computation of Gom pdf and cdf and its cdf error for $-20 \leq x \leq 10$



(a) The computation of Gum pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$ and $b_0 = -0.25$. Cdf 1 Integral, Cdf 2 Gum cdf



(b) The computation of the Gum cdf error for $-20 \leq x \leq 10$, $a = 1.5$ and $b_0 = -0.25$. Cdf 1 Integral minus Cdf 2 the Gum cdf

Fig. 4. The computation of the Gum pdf and cdf and its cdf error for $-20 \leq x \leq 10$.

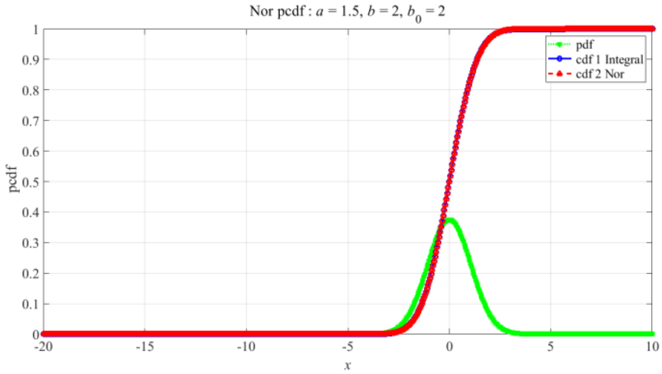
of the other EGB family of distributions. Gom cdf can be employed to describe phenomenon that exhibit very steep growth. In contrast to EGB3 cdf which uses six parameters Gom cdf only uses half of them equal to three. In that context Gom cdf would be a faster model that should provide reasonable accuracy.

Figure 3(b) shows the error between the Cdf 1 Integral and the Cdf 2 Gom cdf from the gompcdf MATLAB function. As illustrated from the analytical results of Fig. 3(b) the error is on the order of 0.222 femto or 10^{-15}. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 Gom cdf from the gompcdf MATLAB function.

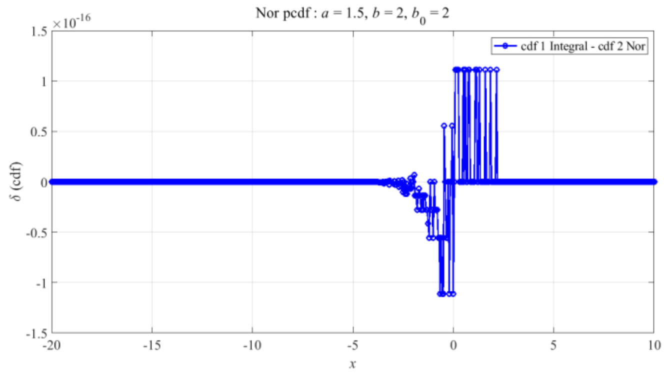
Figure 4(a) depicts the computation of Gum, an acronym that describes the Gumbel pdf as shown in Sect. 3.3 pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$ and $b_0 = -0.25$. Cdf 1 Integral, Cdf 2 Gum cdf. Does the computation of Gum cdf show that Gum cdf is a valid cdf in accordance with (32), the limits in (33), (34), and (35), and the monotonic nature in (36) through (39) in Sect. 3.3? As presented in Fig. 4(a) Gum cdf is always greater than or equal to zero, less than or equal to one, and a monotonically nondecreasing function for all values of the main variable x from minus twenty to plus ten as indicated in (36) through (39). Moreover, if we take the derivative of Gum cdf with respect to x as depicted in (32), it is always greater than or equal to zero. Gum pdf is also an impulsive pdf; however, in contrast to Gom cdf, Gum cdf is only a 2-parameter model and not as sharp as the Gom cdf.

Figure 4(b) shows the error between the Cdf 1 Integral and the Cdf 2 Gum cdf from the gumbcdf MATLA function. As illustrated from the analytical results of Fig. 4(b) the error is in the order of 0.1665 femto or 10^{-15}. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 Gum cdf from the gumbcdf MATLA function.

Figure 5(a) the computation of Nor, it is an acronym that describes the Normal (or Gaussian), pdf and cdf for $-20 \leq$

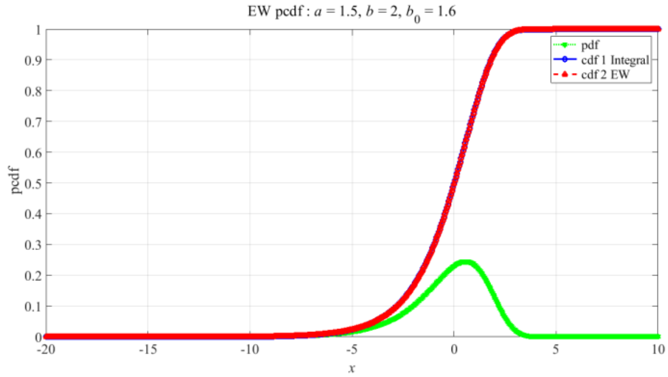


(a) The computation of the Nor pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, and $b_0 = 2$. Cdf 1 Integral, Cdf 2 the Nor cdf

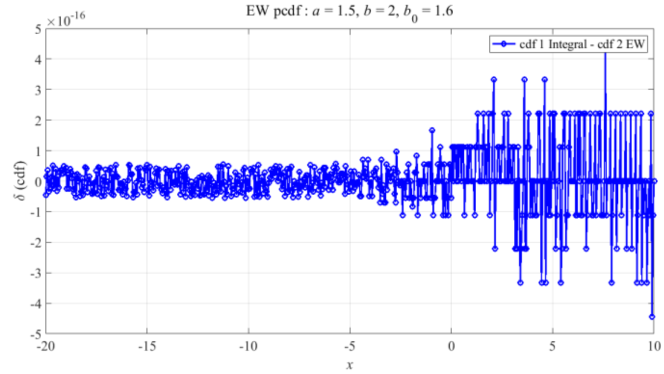


(b) The computation of Nor cdf error for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, and $b_0 = 2$. Cdf 1 Integral minus Cdf 2 Nor cdf

Fig. 5. The computation of Nor pdf and cdf and its error for $-20 \leq x \leq 10$



(a) The computation of the EW pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, and $b_0 = 1.6$. Cdf 1 Integral, Cdf 2 the EW cdf.



(b) The computation of EW cdf error for $-20 \leq x \leq 10$, $a = 1.5$, and $b_0 = 1.6$. Cdf 1 Integral minus Cdf 2 EW cdf.

Fig. 6. The computation of GL2 and GL4 pdf and cdf for $-20 \leq x \leq 10$

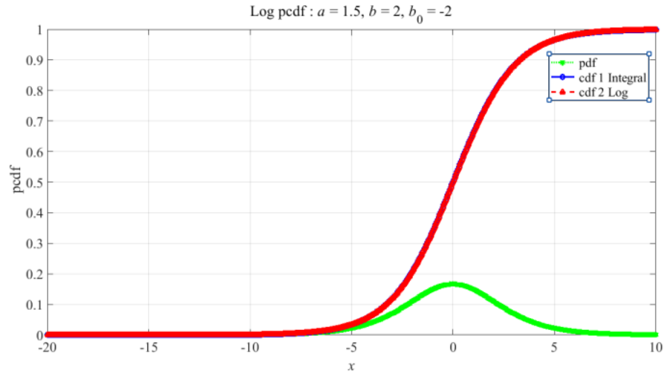
$x \leq 10$, $a = 1.5$, $b = 2$, $b_0 = 2$. Cdf 1 Integral, Cdf 2 Nor cdf. Much has been said about the computation of Nor pdf as indicated in Progri (2003, [5]), Progri et al. (2004, [6]), Progri (2006, [7]), Progri et al. (2007, [8]), and Progri (2010, [9]). However, the computation of Nor cdf in the context of EGB family of distributions is very unique. Nor cdf is perhaps one of the most studied and computed cdf models. It obeys all the laws of the cdf models. In the context of EGB family of distributions, Nor cdf is ranked right in the middle of sharpness and growth. It shows the transition from sharper cdfs to smoother cdfs.

Figure 5(b) shows the error between the Cdf 1 Integral and the Cdf 2 Nor cdf from the normcdf MATLAB BIF function. As illustrated from the analytical results of Fig. 5(b) the error is in the order of 0.1110 femto or 10^{-15}. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 Nor cdf from the normcdf MATLAB BIF function.

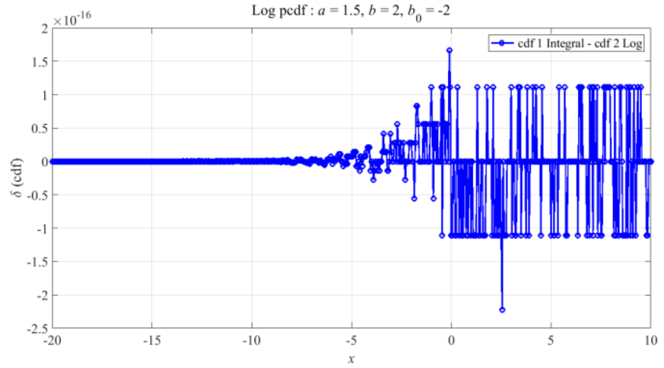
Figure 6(a) depicts the computation of EW, an anacronym that describes the Exponential Weibull in Sect. 3.1, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, and $b_0 = 1.6$. Cdf 1 Integral, Cdf 2 EW cdf. The main computation of EW cdf is performed by computing (10) which appears to indicate the computation of the limits values is accomplished via (11), (12), and (13), and the monotonic nature in (14) through (20).

The MATALB integral and EW cdf appear to grow monotonically from zero to one. And finally the computation of the derivative (10) is shown via the EW pdf which appears to be nonnegative meaning greater than or equal to zero. All of these observations together appear to indicate that EW cdf is a valid cdf model. In terms of sharpness for the values of the parameters considered, EW cdf is the second least sharp (or the second smoothest) pdf from the EGB family of distributions.

Figure 6(b) shows the error between the Cdf 1 Integral and the Cdf 2 EW cdf from the ewcdf MATLAB BIF function. As illustrated from the analytical results of Fig. 6(b) the error is in the order of 0.5551 femto or 10^{-15}. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 EW cdf from the ewcdf



(a) The computation of the Log pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$ and $b_0 = -2$. Cdf 1 Integral, Cdf 2 the Log cdf.



(b) The computation of the Log cdf error for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$ and $b_0 = -2$. Cdf 1 Integral minus Cdf 2 the Log cdf.

Fig. 7. The computation of Log pdf and cdf and its error for $-20 \leq x \leq 10$

MATLAB function.

Figure 7(a) presents the computation of Log, an anacronym that describes the Logistics in Sect. 3.4, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$ and $b_0 = -2$. Cdf 1 Integral, Cdf 2 Log cdf. Log cdf main computation is performed by computing CFE (40), the limits in (41), (42), and (43), and the monotonic nature in (44) through (47) which appears to grow monotonically from zero to one. All of these observations together indicate that Log cdf is a valid cdf model. In terms of sharpness for the values of the parameters considered, Log cdf is the least sharp cdf model from the EGB family of distributions.

Figure 7(b) shows the error between the Cdf 1 Integral and the Cdf 2 Log cdf from the logcdf MATLAB BIF function. As illustrated from the analytical results of Fig. 4(b) the error is in the order of 0.1110 femto or 10^{-15}. In this case there is no bias meaning, there is almost perfect agreement between the Cdf 1 Integral and the Cdf 2 Log cdf from the logcdf MATLAB function.

This concluded the discussion on examples and analytical, conceptual results.

5 Conclusions

In this journal paper the fundamental properties of the cdf are examined in fine detail.

Both the abstract and the introduction of the paper provide a thorough examination of the fundamental properties of the cdf, the main contributions and the main contributors. At the center piece of this journal paper is the equivalence between integration and differentiation.

The meaning of the fundamental inequalities, limits, and the monotonicity of the cdf are deliberated successfully and insightfully in Sect. 2.

For the five nongeneralized cdfs derived from the EGB3 cdf a brilliant analytical computational work is performed in Sect. 3. For the EW, Gom, Gum, Log cdfs are analyzed analytically taking into consideration all the properties discussed in Sect. 2.

The analytical, conceptual results show an agreement between the numerical integration and the analytical CEFs that were derived in Sect. 3 up to the full MATLAB sixteen digits precision or on the order of femto units.

6 Acknowledgement

This work was supported by Giftet Inc. executive office.

I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [21] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications*. *Vol. I* (Not yet available in print) [20].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [21] reminds me of my grandfather who I loved very much.

Finally, I am also dedicating this journal paper to our most beloved President Ronald Reagan, who reminds me of my grandfather, [22] who spoke from the heart, who spoke the

truth, who was a staunch supporter of personal freedom, the greatest leader of the free world, and perhaps the greatest critic of the wasteful government control and spending[xi].

"*Government is not the answer, government is the problem*."—Ronald Reagan

I would like to express my deepest gratitude to my mother Lumturi Progri, my sister Adriana Dine, and Ms. Elizabeth Demir for their support, loyally, and dedication to me during some of the most difficult times in my life as a boy, man, and scholar.

7 References

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[i] I am dedicating this journal paper to my only female Prof. Denisse Nicoletti. Prof. Denisse Nicoletti was my Prof. on "Probability, Statistics, and Random Variables" during my Master of Science in Electrical Engineering at the Department of Electrical and Computer Engineering, Worcester Polytechnic Institute in fall of 2005. I remember her office was close to Prof. Alex Emanuel's office. In fall of 2005, I took three classes, one class in signals and systems, one class in probability and random variables, and another class in power systems engineering. Of all the three classes, her class on "Probability, Statistics, and Random Variables" was the most challenging at the time. I would go and ask her questions in her office about various topics. After the final exam I see her in the Electrical and Computer Engineering Department's office. She said you got this, and she showed me an 'A' upside down like '∀' with her fingers. Anyways, I did get an 'A' in her class, and I remember how passionate she was about the subject matter. Unfortunately, she died tragically in a head-on collision in a car accident. "We all continue to grieve the loss of Denise Nicoletti, co-founder and director of Camp Reach from 1997-2002, who died in a head-on car crash on July 21, 2002, the first full day of the 2002 program." I remember going to her funeral and talking to Prof. Alex Emanual, her husband, and other guests about her. It was heart-breaking. WPI had a memorial built in front of the ECE Department to honor her legacy while at WPI. I am very grateful to having taken her class and getting to know her and having learned to fundamentals in such a difficult class on Probability, Statistics, and Random Variables. She encouraged me to become passionate about the fundamental concepts such as the probability density function and cumulative probability distribution functions, and how to compute the fundamental statistics such as the mean and variance.

[ii] The 5-paramater EGB will be referred to as EGB3 throughout the journal paper.

[iii] Ditto ii.

[iv] Calculating limits involves determining the value a function approaches as its input approaches minus or plus infinity. Techniques include direct substitution, factoring, using conjugates, and simplifying expressions. If direct substitution results in an indeterminate form (like 0/0), further manipulation, such as factoring or using conjugates, is needed.

As x approaches negative infinity, the cdf approaches zero (no events occur before negative infinity). As x approaches positive infinity, the cdf approaches one (all events are less than or equal to positive infinity).

[v] *Monotonicity*, in its broadest sense, refers to the property of something staying consistently in one direction, either increasing or decreasing, without changing its overall trend. In mathematics, it specifically describes a function that is either always increasing or always decreasing over a given interval.

F(x) cannot decrease as x increases. This reflects that the probability of an event occurring up to a certain point cannot decrease as that point is extended.

[vi] The Gompertz cdf was first derived by Benjamin Gompertz in 1825. He introduced the function in his paper "On the nature of the function expressive of the Law of human mortality, and on a new mode of determining the value of life contingencies." Gompertz's work showed that age-specific mortality rates increased exponentially over much of the adult human lifespan.

[vii] Gumbel's work on the Gumbel distribution, also known as the type-I extreme value distribution, has been widely used in various fields, including statistics, engineering, and finance, to analyze and model the distribution of extreme values.

[viii] 1. *Extreme Value Analysis*:

Hydrology and Meteorology: The Gumbel distribution is used to model and predict the probabilities of extreme events like floods, droughts, and rare storm occurrences.

Risk Analysis: It is used to understand and model the probabilities of extreme losses in financial or other risk contexts.

Engineering: It can help analyze the reliability of structures under extreme loads or conditions.

2. *Specific Applications:*

Flood Modeling: The Gumbel distribution is used to model the maximum river levels over time, helping to determine levee heights and other infrastructure decisions, as explained in this paper.

Extreme Rainfall Analysis: It can be used to analyze annual maximum rainfall intensity, helping to assess the impact of climate change on rainfall patterns and their implications.

Earthquake Risk: It can be used to model the maximum magnitudes of earthquakes over a period, which can inform the design of earthquake-resistant structures.

Life Tables and Break Strength: Gumbel probability paper is used for graphical analysis of frequency-type data, including life tables, breaking strength of materials, and more.

3. *Other Applications*:

Stock Market Analysis: Gumbel probability paper can also be used for preliminary investigation of stock market data.

Deep Learning: It can be used in deep learning models for object detection, as mentioned in a paper on underground object detection.

Multinomial Logit Model: The errors of the latent variables in the multinomial logit model can follow a Gumbel distribution.

[ix] *Verhulst's Nomenclature* (1845): Verhulst coined the term *logistic* for the function and the associated curve in his 1845 paper.

Independent Discoveries and Applications: The logistic function was later rediscovered by others, including Wilson (1925) and Winsor (1932). It was also independently used in various fields like bioassay and autocatalytic reactions.

Berkson and the Logit (1944): Joseph Berkson introduced the *logit* as an alternative to the *probit*, further solidifying the logistic function's role in statistical modeling.

[x] The femto is a unit prefix in the International System of Units (SI) that represents a factor of 10^−15, or one quadrillionth. The prefix is derived from the Danish word femten, which means *fifteen*.

[xi] Ditto Progri(2020, [19]) EN vii.