Giftet Inc.
Journal of Geolocation, Geo-information, and Geo-intelligence
Volume 2024, article ID 2024071602, 19 pages
https://doi.org/10.18610/JG3.2024.071602

Research Article

On the Computation of the Probability Density Function of the Exponential Generalized Beta Family of Distributions

To my Prof., John Orr, a phenomenal teacher, scholar, researcher, friend, department head, the best Ph.D. Dissertation academic advisor who I have ever known in my life.[i]

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Received May 8, 2024; Revised May 19, 2024-June 17, 2024, Accepted July 16, 2024; Published November 1, 2024.

Editor-in-Chief/Senior Scientific Editor: Ilir F. Progri

The computation of the probability density function[ii] (pdf[iii]) of the exponential generalized Beta[iv] (EGB) family of distributions is discussed in this paper in three main parts. The EGB pdf is discussed first. Second, the non-generalized[v] pdfs derived from the EBG family of distributions such exponential Weibull (EW), Gompertz, Gumbel, logistic, and normal (or Gaussian) are considered next. Third, the generalized pdfs derived from the EBG family of distributions such as Generalized Gumbel (GGu) of Type 1 and Generalized Logistics (GLo) of Type 1 through 4 pdfs are presented afterwards. A careful study which includes investigation, validation, and verification of all the closed form expressions (CFE) of the pdfs is presented in the paper. Numerical, theoretical results of the specific and generalized pdfs derived from the EGB family of distributions showing the connection with EGB pdfs are presented in the paper.
Index Terms—probability density function, exponential generalized Beta, EBG family of distributions, exponential Weibull, Gompertz, Gumbel, logistic, normal (or Gaussian), Generalized Gumbel, Generalized Logistics.

1 Introduction

The generalized Beta distribution (GBD)[vi] model was first considered by Mauldon (1959, [1]), Mathai, Saxena (1966, [2]), introduced by Rao, Garg (1969, [3]), Dubey (1970, [4]), Mielke Jr, Johnson (1974, [7]), independently of them from Jakuszenkow (1974, [8]), and Falls (1974, [9] from a NASA report in 1973 which was referred to as the generalized beta probability density function (1) and (2)).

Johnson, Kotz (1970, [5])[vii] contains an outstanding literature search of the genesis of the continuous univariate distributions with an outstanding discussion of all the mathematical models of extreme value, logistics, Laplace (or double exponential), Beta, uniform (or rectangular), F-, t-, non-central (χ^2, t-, F-), lifetime distributions in which a number of them are special cases of the new distributions to come. I believe that this publication inspired many authors and researchers in the analysis, modeling, and early computation of the special cases of new distribution models to come.

Officer (1972, [6]) points out that the distribution of the stock return was shown to be non-normal, "fat-tailed" relative to the normal distribution. Officer (1972, [6]) was also among the first or if not the first to test for the condition of the stability of the stock returns and found out that the stock return distribution is not stable. Officer (1972, [6]) concludes his paper suggesting that a class of "fat-tailed" distributions with finite second moments will be found to give a better approximation of the distribution of stock returns[viii].

It is not clear how aware Mielke Jr, Johnson (1974, [7]), independently of them from Jakuszenkow (1974, [8]), and Falls (1974, [9]) were of the work of their predecessors when they introduced the GBD model. I am certain that Mielke Jr, Johnson (1974, [7]) reference of the work performed by Johnson, Kotz (1970, [5]) is not an accident.

Prentice (1975, [10]) is another author who considered extreme value tests, generalized logistics distributions, life testing, log gamma distributions, logistic tests, model discrimination, score tests for normality, tests for exponentiality; i.e., Prentice (1975, [10]) considered the special cases of the Exponential Generalized Beta (EGB[ix]) distribution (EGBD).

Bartels (1977, [11]) lays an important groundwork for the Central Limit Theorem[x] (CLT) can also lead to other nonnormal stable limits in a number of cases familiar to econometricians which in fact are special cases of the EGBD.

Kalbfleisch, Prentice (1980, [12]) published in their first book a discussion on the *generalized F-distribution* (as a special case of the GB2) which received praise from J. Kline in the book review (1982, [13]).

The EGBD, which is an example of an asymmetrical[xi], non-stable[xii], and non-normal distribution, became a prominent distribution starting with Venter(1984, [14]), McDonald (1984, [15]), Patil (1984, [16]) whose primarily purpose was to model many distributions of the income (see Bookstaber, McDonald(1987, [17]), McDonald (1988, [18]), Cummins et al. (1990, [19]), McDonald[xiii](1991, [20])).

McDonald (1988, [18]) mentioned the obvious correlation between that the many new statistical models such as generalized Beta[xiv] of the first and second types (GB1 and GB2) developed the estimation and analysis of many new statistical models and the use of computers and computer computational software like MATLAB.

The first official mention of the EGB2D started with McDonald (1991, [21])) where he recognized the benefits of the EGBD as not only having "fat tails" but also its ability to model symmetry or a symmetry of the distribution of the stock returns. However, the EGB2D model proposed by McDonald (1991, [21] (4))) is only a special case of the EGBD that was discussed in Progri (2016, [45]), which is discussed in greater detail in Sect. 2.

In McDonald, Xu (1992, [22])) the generalization of the Beta of the first and second kind is proposed which are referred to as the EGB1 and EGB2 which are a special case of the EGB[xv].

In BarNiv, McDonald (1992, [23]) the emphasis is the study and synthesis of methodological and empirical models. Hansen, McDonald (1992, [24]) make use of artificial intelligence and generalized qualitative-response models as a means of the study and synthesis of methodological and empirical models.

McDonald, Nelson (1992, [25]) point out the departure from normality in the beta estimation[xvi] in the market model due to the effect of the estimation of the skewness[xvii] and leptokurtosis[xviii]. The empirical investigation conducted by McDonald, Nelson (1992, [25]), which is the first application of this procedure in regression models, reveals that both skewness and kurtosis[xix] can affect β estimates.

Mcdonald, White (1993, [26]) were the first to compare the relative performance of some adaptive robust estimators (partially adaptive and adaptive procedures) with some common nonadaptive robust estimators. According to Mcdonald, White (1993, [26]) the adaptive and partially adaptive estimators perform very well relative to the other estimation procedures considered, and preliminary results suggest that in some important cases they can perform much better than ordinary least squares (OLS) with fifty to eighty percent reductions in standard errors.

Mcdonald, Xu (1994, [27]) discuss some forecasting applications of partially adaptive estimators of Autoregressive Integrated Moving Average (ARIMA[xx]) models.

Mcdonald, Xu (1995, [28]) introduced a 5-parameter GBD which (1) nests the GB and GG distributions; (2) leads to an EGBD; (3) includes more than thirty distributions as limiting or special cases known as the EGB family of distributions; (4) includes generalized forms of logistics, exponential, Gompertz, Gumbell and the normal distributions; (5) provides the basis for partially adaptive estimation of econometric models with possibly skewed and leptokurtic error distributions.

The book chapter by Mcdonald (1996, [29]) includes fourteen unique distributions for financial models that includes normal, lognormal, and other diversions from normality distributions.

When I produced my Ph.D. dissertation in Progri (2003, [30]) I studied only four probability distribution models: Normal (or Gaussian), Logo-normal, Rayleigh, and Rician. In fact I proposed a theoretical approach on the unified multipath distribution model Progri (2003, [30], pp. 179-184), which was later revied in Progri et al. (2004, [31]), Progri (2006, [32]), and Progri et al. (2007, [33]).

Ten to fifteen years later when I first begun to develop the initial manuscript of the second pioneer publication, I decided to do a much deeper analysis on the mathematical statistical distribution models as indicated in Progri et al (2016, [43], [44]), Progri (2016, [45]), Progri et a. (2017, [46]), and Progri (2019, [47]).

When I first developed mathematical derivations of the EGB in Progri (2016, [45]) I had very little knowledge of the contributions that were made in area of the EGB and its connection to the EGB family of distributions.

From 2016 until 2024 I felt compelled to revisit this work and perform a comprehensive literature review on the genesis of EGB, its family of distributions, and highlight the contributions made from so many individuals because I felt that this was lacking in all of my previous publications which are referenced.

The introduction in Progri (2024, [49]) includes a continuation of the discussion on the application of EGBD and the family of distributions from 1996 until 2006.

This paper is organized as follows: in Sect. 2 the computation of the EGBD pdf is presented. Non-generalized PDFs derived from the EGBD PDF are discussed in Sect. 3. Section 4 contains a detailed discussion on Generalized PDFs derived from EGBD PDFs. Qualitative, Quantitative Outcome are presented in Sect. 5; Conclusions are provided in Sect. 6, Acknowledgment in Sect. 7, along with a list of references in Sect. 8. The derivations of EW PDF as a special case of EGB PDF are presented in Appen. A.

This concludes with the introduction Sect. of the paper. Next, we start with the computation of the EGBD pdf.

2 The EGBD PDF

In this section the computation of the EGBD pdf is performed. This Sect. contains one main SubSect.: the derivation EGBD pdf from the GBD pdf is discussed in Sect. 2.1.

2.1 The Derivation of EGBD PDF from GBD PDF

Letting $X \sim f_{GB}(x; a, c, d, p, q)$ be the 5-paramater GBD and the random variable $Y = \log X$[1], then

$$f_Y(y) = f_X(x)\frac{dx}{dy} = f_X(x)x \tag{1}$$

Furthermore, using the identity

$$\left(\frac{x}{a}\right)^c = e^{c[\log(x)-\log(a)]} = e^{\frac{y-b\prime}{a\prime}} \tag{2}$$

where

$$b' = \log(a) \tag{3}$$

$$a' = \frac{1}{c} \tag{4}$$

With re-parameterization, EGB [50] is distributed with the following pdf [36]:

$$f_{\mathrm{EGB}}(x; a, b, d, p, q) = \begin{vmatrix} f(x), & -\infty < x < b' \\ 0, & b' \le x < \infty \end{vmatrix} \tag{5}$$

where

$$f(x) = \frac{e^{d\frac{x-b}{a}}\left[1-(1-q)e^{\frac{x-b}{a}}\right]^{p-1}}{|a|\mathrm{B}(d,p)\left(1+qe^{\frac{x-b}{a}}\right)^{d+p}}$$

$$= \frac{x'^{d}[1-(1-q)x']^{p-1}}{|a|\mathrm{B}(d,p)[1+qx']^{d+p}} \tag{6}$$

$$b' = b + |a|\log\left(\frac{1}{1-q}\right) \tag{7}$$

$$x' = e^{\frac{x-b}{a}} \tag{8}$$

The EGB was proposed for the first time by McDonald (1991, [21] (4)); however, in the CFE of EGB proposed by McDonald (1991, [21] (4)) is only a special case of the EGB proposed by Mcdonald, Xu (1995, [28]) and Progri (2016, [45]) for the parameter q equal to one.

[1] $\log(X) \equiv \ln(X)$ this is the natural logarithm.

It should be noted that $f_{\text{EGB}}(x; a, b, d, p, q)$ is continuous for x when $p \neq 1$; else, for $p = 1$, $f_{EGB}(x; a, b, d, p, q)$ is not continuous for all x. For example, for $b \neq 1$

$$f_{\text{EGB}}(x \to b'^{-}; a, b, d, p = 1, q) = \frac{b'^{d}}{|a|\mathrm{B}(d,p)\left(1+\frac{q}{1-q}\right)^{d+p}} \neq 0 \quad (9)$$

$$f_{\text{EGB}}\left(x \to b'^{+}; a, b, d, p = 1, q\right) = 0 \quad (10)$$

The EGB is a 5-paramater distribution model. It is a mathematical model that is supposed to generalize a wide range of distribution models, known as EGB family of distributions, as discovered by Mcdonald, Xu (1995, [28]).

3 Non-generalized PDFs Derived from the EGBD PDF

There is a wide range of distribution models that can be generalized from the 5-paramater model of the EGBD. As an example, Bookstaber, McDonald (1987, [17]) Fig. 1 provides a block diagram of the GB2 family tree[xxi]. In this section we consider the EW PDF in Sect. 3.1; The Gompertz PDF in Sect. 3.2; The Gumbel PDF in Sect. 3.3; The Logistic PDF in Sect. 3.4; and the Norma PDF in Sect. 3.5.

3.1 The Exponential Weibull PDF

The EW [51] pdf is a special case of EGB [50] pdf whose closed form expression (CFE) pdf is given by

$$f_{\text{EW}}(x; b, c) = |f(x), \quad -\infty < x < \infty$$

$$= |ce^{cx}be^{-be^{cx}}, \quad -\infty < x < \infty \quad (11)$$

In Appen. A, the EGB pdf is reduced in an expression very close to EW [51] pdf. Starting with the CFE (55) in Append A, if we set

$$c = 1/a \quad (12)$$

$$d = 1 \quad (13)$$

and make the substitution

$$b = e^{-cb} \quad (14)$$

we obtain the exact CFE of the EW pdf as follows

$$f(x) = \frac{e^{d\frac{x-b}{a}}e^{-e^{\frac{x-b}{a}}}}{|a|\Gamma(d)}$$

$$= ce^{cx}e^{-cb}e^{-e^{-cb}e^{cx}}$$

$$= ce^{cx}be^{-be^{cx}} \quad (15)$$

Is EW pdf a valid pdf? First, it is clear that

$$f(x) \geq 0, -\infty < x \leq \infty \quad (16)$$

Second, if we take the limit when $x \to \pm\infty$[xxii]

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} ce^{cx}be^{-be^{cx}}$$

$$= bc \lim_{x\to\pm\infty} e^{cx-be^{cx}}$$

$$= bce^{\lim_{x\to\pm\infty} cx-be^{cx}}$$

$$= bc \times 0$$

$$= 0 \quad (17)$$

Third, we need to prove the following:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} ce^{cx}be^{-be^{cx}}dx$$

$$= 1 \quad (18)$$

This proof is given later on in Progri (2024, [49]). This concludes, or ends, the brief discussion that the first special case of EGB pdf which was initially derived in Append. A, and then for a set of parameters it was reduced to the exact CFE of the EW pdf as given in (11) and (15).

3.2 The Gompertz PDF

The EGB [50] pdf [36] includes generalizations of the Gompertz [52] pdf.

Because the EW [51] pdf is valid for all the values of $-\infty < x < \infty$; hence, we need $F_{\text{Gom}}(x = 0; b, c) = 0$; hence, Gompertz [52] pdf [36] is given by

$$f_{\text{Gom}}(x; a, b) = \begin{vmatrix} 0, & -\infty < x \leq 0 \\ ae^{ax}be^{b}\exp(-be^{ax}), & 0 < x < \infty \end{vmatrix}$$

$$= \begin{vmatrix} 0, & -\infty < x \leq 0 \\ e^{b}f_{\text{EW}}(x; a, b), & 0 < x < \infty; \end{vmatrix} \quad (19)$$

Is Gompertz [52] pdf [36] a valid pdf? First, it is clear that

$$f_{\text{Gom}}(x; a, b) \geq 0, -\infty < x < \infty \quad (20)$$

Second, if we take the limit when $x \to \infty$

$$\lim_{x\to\infty} f_{\text{Gom}}(x; a, b) = \lim_{x\to\infty} e^{b}f_{\text{EW}}(x; a, b)$$

$$= e^{b} \lim_{x\to\infty} f_{\text{EW}}(x; a, b)$$

$$= e^{b} \times 0$$

$$= 0 \quad (21)$$

Third, we need to prove the following:

$$\int_{-\infty}^{\infty} f_{\text{Gom}}(x; a, b)dx = \int_{0}^{\infty} e^{b}f_{\text{EW}}(x; a, b)dx$$

$$= 1 \quad (22)$$

This proof is given later on in Progri (2024, [49]). This concludes, or ends, the brief discussion that the Gompertz [52] pdf is a generalization of the EGB pdf due to its great relations with the EW pdf as given in (11) and (15).

3.3 The Gumbel PDF

If we take EW and set $a = 0$ corresponds to the same Wei for $a = 1$

$$f_{\mathrm{EW}}(x; a, c) = ce^{cx}e^{-e^{cx}}; c > 0$$
$$= ce^{cx}e^{-e^{cx}} \quad (23)$$

If we wanted this to account for negative exponent by substituting $x \to -x$ we obtain the Gumbel (type-I generalized extreme value distribution [56]) [53] pdf [36] as follows:

$$f_{\mathrm{Gum}}(x; a) = ae^{-ax}e^{-e^{-ax}}$$
$$= ae^{-ax-e^{-ax}} \quad (24)$$

Is Gumbel (type-I generalized extreme value distribution [56]) [53] pdf [36] a valid pdf? First, it is clear that

$$f_{\mathrm{Gum}}(x; a) = ae^{-ax-e^{-ax}} \geq 0, -\infty < x < \infty, a > 0 \quad (25)$$

Second, if we take the limit when $x \to \pm\infty$

$$\lim_{x\to\pm\infty} f_{\mathrm{Gum}}(x; a) = \lim_{x\to\pm\infty} ae^{-ax-e^{-ax}}$$
$$= a \lim_{x\to\infty} e^{-ax-e^{-ax}}$$
$$= a \times 0$$
$$= 0 \quad (26)$$

Third, we need to prove the following:

$$\int_{-\infty}^{\infty} f_{\mathrm{Gum}}(x; a)dx = \int_{-\infty}^{\infty} ae^{-ax-e^{-ax}}dx$$
$$= 1 \quad (27)$$

This proof is given later on in Progri (2024, [49]).

3.4 The Logistic PDF

For example, for $a = -a$, $d = 1$, $p = 1$, and $q = 1$ EGB [50] pdf becomes logistic [57] pdf

$$f_{\mathrm{Log}}(x; a, b) = f_{\mathrm{EGB}}(x; a = -a, b, d = 1, p = 1, q = 1)$$
$$= \frac{x' \equiv e^{-\frac{x-b}{a}}}{|a|(1+x')^2}$$
$$= \frac{1}{4c}\operatorname{sech}\left(\frac{x-b}{2a}\right) \quad (28)$$

Is logistic [57] pdf [36] a valid pdf? First, it is evident that

$$f_{\mathrm{Log}}(x; a, b) = \frac{x' \equiv e^{-\frac{x-b}{a}}}{|a|(1+x')^2} \geq 0, -\infty < x < \infty, a, b > 0 \quad (29)$$

Second, if we take the limit when $x \to \pm\infty$

$$\lim_{x\to\pm\infty} f_{\mathrm{Log}}(x; a, b) = \lim_{x\to\pm\infty} \frac{x' \equiv e^{-\frac{x-b}{a}}}{|a|(1+x')^2}$$
$$= \frac{1}{|a|} \lim_{x\to\pm\infty} \frac{x' \equiv e^{-\frac{x-b}{a}}}{(1+x')^2}$$
$$= \frac{1}{|a|} \times 0$$
$$= 0 \quad (30)$$

Third, we need to prove the following:

$$\int_{-\infty}^{\infty} f_{\mathrm{Log}}(x; a, b)dx = \int_{-\infty}^{\infty} \frac{x' \equiv e^{-\frac{x-b}{a}}}{|a|(1+x')^2}dx$$
$$= 1 \quad (31)$$

This proof is given later on in Progri (2024, [49]).

3.5 The Normal PDF

For example, the normal pdf is a special case of the generalized gamma distribution pdf which is a special case of the GBD pdf. The connection between the normal pdf with the ECB pdf will be shown in a separate journal paper.

Section 3, or the non-generalized PDFs derived from the EGBD pdf, is concluded here. Next, we begin the discussion of Sect. 4 on the generalized PDFs derived from EGBD pdf.

4 Generalized PDFs Derived from EGBD PDF

The 5-paramater EGBD includes also a number of generalized pdfs derived from the EGBD pdf such as the generalized Gumbel of type 1 PDF in Sect. 4.1; the generalized Logistics of type 1 PDF in Sect. 4.2; The generalized Logistics of type 2 PDF in Sect. 4.3; The generalized Logistics of type 3 PDF in Sect. 4.4; and The Generalized Logistics of Type 4 PDF in Sect. 4.5.

4.1 The Generalized Gumbel of Type 1 PDF

The generalized Gumbel (or type 1 Gumbel [54]) can be obtained from substituting $x \to -x$ into the EW as follows:

$$f_{\mathrm{GGu}}(x; a, b) = |ae^{-ax}be^{-be^{-ax}}, \quad -\infty < x < \infty$$
$$= |abe^{-ax-be^{-ax}}, \quad -\infty < x < \infty \quad (32)$$

Is generalized Gumbel (GGu or type I Gumbel [54]) pdf [36] a valid pdf? First, it is clear that for $a, b > 0$

$$f_{\mathrm{GGu}}(x; a, b) = abe^{-ax-be^{-ax}} \geq 0, -\infty < x < \infty \quad (33)$$

Second, if we take the limit when $x \to \pm\infty$

$$\lim_{x\to\pm\infty} f_{\mathrm{GGu}}(x; a, b) = \lim_{x\to\pm\infty} abe^{-ax-be^{-ax}}$$
$$= ab \lim_{x\to\pm\infty} e^{-ax-be^{-ax}}$$
$$= a \times 0$$
$$= 0 \quad (34)$$

Third, we need to prove the following:

$$\int_{-\infty}^{\infty} f_{\mathrm{GGu}}(x; a, b)dx = \int_{-\infty}^{\infty} abe^{-ax-be^{-ax}}dx$$

$$= 1 \tag{35}$$

This proof of (35) is given later on in Progri (2024, [49]).

This concludes the brief discussion on the Gumbel [54] and GGu cdfs.

4.2 The Generalized Logistics of Type 1 PDF

Next, for example, $a = 1$, $b = 0$, $d = d$, $p = 1$ and $q = 1$ EGB [50] becomes GL of type 1 [58]

$$f_{\mathrm{GL1}}(x; d) = f_{\mathrm{EGB}}(x; a = 1, b = 0, d = d, p = 1, q = 1)$$
$$= \frac{x'}{\mathrm{B}(d,1)(1+x')^{d+1}}$$
$$= \frac{de^{-x}}{(1+e^{-x})^{d+1}}$$
$$= \frac{de^{dx}}{(1+e^{x})^{d+1}} \tag{36}$$

Is GL of type 1 [58] pdf [36] a valid pdf? First, it is clear that

$$f_{\mathrm{GL1}}(x; d) = \frac{de^{dx}}{(1+e^{x})^{d+1}} \geq 0, -\infty < x < \infty, d > 0 \tag{37}$$

Second, if we take the limit when $x \to \pm\infty$

$$\lim_{x\to\pm\infty} f_{\mathrm{GL1}}(x; d) = \lim_{x\to\pm\infty} \frac{de^{-x}}{(1+e^{-x})^{d+1}}$$
$$= d \lim_{x\to\pm\infty} \frac{e^{-x}}{(1+e^{-x})^{d+1}}$$
$$= d \times 0$$
$$= 0 \tag{38}$$

Third, we need to prove the following:

$$\int_{-\infty}^{\infty} f_{\mathrm{GL1}}(x; d)\, dx = \int_{-\infty}^{\infty} \frac{de^{dx}}{(1+e^{x})^{d+1}} dx$$
$$= 1 \tag{39}$$

The proof of (39) is given later in Progri (2024, [49]).

4.3 The Generalized Logistics of Type 2 PDF

Next, let us consider the case when we multiply the GL of type 1 [58] pdf [36] with the $e^{-(d-1)x}$ we get GL of type 2 [58] pdf [36]

$$f_{\mathrm{GL2}}(x; d) = f_{\mathrm{GL1}}(x; d) e^{-(d-1)x}$$
$$= \frac{de^{-x}e^{-(d-1)x}}{(1+e^{-x})^{d+1}}$$
$$= \frac{de^{-dx}}{(1+e^{-x})^{d+1}}$$
$$= \frac{de^{x}}{(1+e^{x})^{d+1}}; d > 0 \tag{40}$$

Is GL of type 2 [58] pdf [36] a valid pdf? First, it is clear that

$$f_{\mathrm{GL2}}(x; d) = \frac{de^{x}}{(1+e^{x})^{d+1}} \geq 0, -\infty < x < \infty, d > 0 \tag{41}$$

Second, if we take the limit when $x \to \pm\infty$

$$\lim_{x\to\pm\infty} f_{\mathrm{GL2}}(x; d) = \lim_{x\to\pm\infty} \frac{de^{x}}{(1+e^{x})^{d+1}}$$
$$= d \lim_{x\to\pm\infty} \frac{de^{x}}{(1+e^{x})^{d+1}}$$
$$= d \times 0$$
$$= 0 \tag{42}$$

Third, we need to prove the following:

$$\int_{-\infty}^{\infty} f_{\mathrm{GL2}}(x; d)\, dx = \int_{-\infty}^{\infty} \frac{de^{x}}{(1+e^{x})^{d+1}} dx$$
$$= 1 \tag{43}$$

The proof of (43) is given later in Progri (2024, [49]).

4.4 The Generalized Logistics of Type 3 PDF

Next, for example, for $a = 1$, $b = 0$, and $p = d$, $q = 1$, EGB [50] becomes GL of type 3 [58] as given below

$$f_{\mathrm{GL3}}(x; d) = f_{\mathrm{EGB}}(x; a = 1, b = 0, d, p = d, q = 1)$$
$$= \frac{x'^{d}}{\mathrm{B}(d,d)(1+x')^{2d}}$$
$$= \frac{e^{-dx}}{\mathrm{B}(d,d)(1+e^{-x})^{2d}}$$
$$= \frac{e^{dx}}{\mathrm{B}(d,d)(1+e^{x})^{2d}} \tag{44}$$

Is GL of type 3 [58] pdf [36] a valid pdf? First, it is clear that

$$f_{\mathrm{GL3}}(x; d) = \frac{e^{dx}}{\mathrm{B}(d,d)(1+e^{x})^{2d}} \geq 0, -\infty < x < \infty, d > 0 \tag{45}$$

Second, if we take the limit when $x \to \pm\infty$

$$\lim_{x\to\pm\infty} f_{\mathrm{GL3}}(x; d) = \lim_{x\to\pm\infty} \frac{e^{dx}}{\mathrm{B}(d,d)(1+e^{x})^{2d}}$$
$$= \frac{1}{\mathrm{B}(d,d)} \lim_{x\to\pm\infty} \frac{e^{dx}}{(1+e^{x})^{2d}}$$
$$= \frac{1}{\mathrm{B}(d,d)} \times 0$$
$$= 0 \tag{46}$$

Third, we need to prove the following:

$$\int_{-\infty}^{\infty} f_{\mathrm{GL3}}(x; d)\, dx = \int_{-\infty}^{\infty} \frac{e^{dx}}{\mathrm{B}(d,d)(1+e^{x})^{2d}} dx$$
$$= 1 \tag{47}$$

The proof of (47) is given later in Progri (2024, [49]).

4.5 The Generalized Logistics of Type 4 PDF

For example, for $a = 1$, $b = 0$, and, $q = 1$ EGB [50] becomes GL of type 4 [58];

$$f_{\mathrm{GL4}}(x; d, p) = f_{\mathrm{EGB}}(x; a = 1, b = 0, d, p, q = 1)$$

$$= \frac{x'^{d}}{\mathrm{B}(d,p)(1+x')^{d+p}}$$

$$= \frac{e^{-px}}{\mathrm{B}(d,p)(1+e^{-x})^{d+p}}$$

$$= \frac{e^{dx}}{\mathrm{B}(p,d)(1+e^{x})^{d+p}} \tag{48}$$

Is GL of type 4 [58] pdf [36] a valid pdf? First, it is clear that

$$f_{\mathrm{GL4}}(x; d, p) = \frac{e^{dx}}{\mathrm{B}(p,d)(1+e^{x})^{d+p}} \geq 0, \begin{array}{l} -\infty < x < \infty \\ d, p > 0 \end{array} \tag{49}$$

Second, if we take the limit when $x \to \pm\infty$

$$\lim_{x\to\pm\infty} f_{\mathrm{GL4}}(x; d, p) = \lim_{x\to\pm\infty} \frac{e^{dx}}{\mathrm{B}(p,d)(1+e^{x})^{d+p}}$$

$$= \frac{1}{\mathrm{B}(p,d)} \lim_{x\to\pm\infty} \frac{e^{dx}}{(1+e^{x})^{d+p}}$$

$$= \frac{1}{\mathrm{B}(d,d)} \times 0$$

$$= 0 \tag{50}$$

Third, we need to prove the following:

$$\int_{-\infty}^{\infty} f_{\mathrm{GL4}}(x; d, p)\, dx = \int_{-\infty}^{\infty} \frac{e^{dx}}{\mathrm{B}(p,d)(1+e^{x})^{d+p}}\, dx$$

$$= 1 \tag{51}$$

The proof of (51) is given later in Progri (2024, [49]).

This concludes the discussion on the generalized PDFs derived from EGBD PDF. Next, we start a brief investigation into the numerical, theoretical results.

5 Qualitative, Quantitative Outcome

The qualitative and quantitative outcome consists of the description of the MATLAB simulations presented in SubSects. 4.1 and of the Qualitative, Quantitative Examples in SubSects. 4.2.

5.1 Description of MATLAB Simulation Functions

In order to obtain the MATLAB simulations of this journal paper we prepared a rather detailed MATLAB simulation test setup.

The MATLAB simulation consists of thirteen functions which are explained in order of from slow to fast growth then from sharpest to least sharp which is explained in greater detail in Sect. 5.2. The function:

1. lognpdf(-x+b_0,a,b) is the mirror image of the MATLAB BIF, Logo-normal pdf.
2. exppdf(-x+b_0,a) is the mirror image of the MATLAB BIF, exponential function pdf.
3. exgenbeta3pdf(a,b,d,p,q,x-b_0) is six parameter function for computing the EGB of the third kind based on Progri(2016, [45]).
4. gomppdf(a,b,x-b_0) computes of the Gompertz pdf as depicted in Sect. 3.2.
5. gumbpdf(a,x-b_0) performs the computation of the Gumbel pdf as indicated in Sect. 3.3.
6. genggupdf(a,b,x-b_0) is the Generalized Gumbel of type I as indicated in Sect. 4.1.
7. normpdf(x,b-b_0,a/sqrt(2)) is the Nornal or Gaussian MATLAB BIF pdf.
8. genlog3pdf(d,x-b_0) is the Generalized Logistic of type III as indicated in Sect. 4.4.
9. genlog1pdf(d,x-b_0) computed the Generalized Logistic of type I as indicated in Sect. 4.2.
10. genlog2pdf(d,x-b_0) performs the computation of the Generalized Logistic of type II as indicated in Sect. 4.3.
11. genlog4pdf(d,p,x-b_0) is the Generalized Logistic of type IV as indicated in Sect. 4.5.
12. ewpdf(a,b,x-b_0) computes of the Exponential Weibull pdf which was explained in great detail in Sect. 3.1.
13. logpdf(a,b,x-b_0) performs the computation of the Logistic pdf as shown in Sect. 3.4.

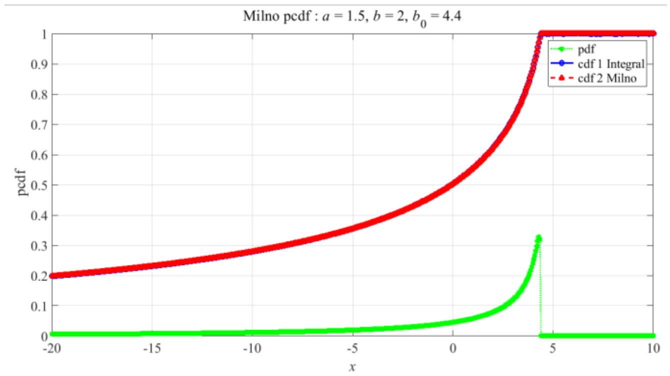
The parameter b_0 is defined as the dc offset for the value of the CDF to be equal to one half. As indicated later in Sect. 5.2, it is expected that b_0 so as the value of the CDF will be equal to half.

Of all thirteen functions, ten of them were built as part of the Giftet Indoor Geolocation Library Progri (2025, [60]).

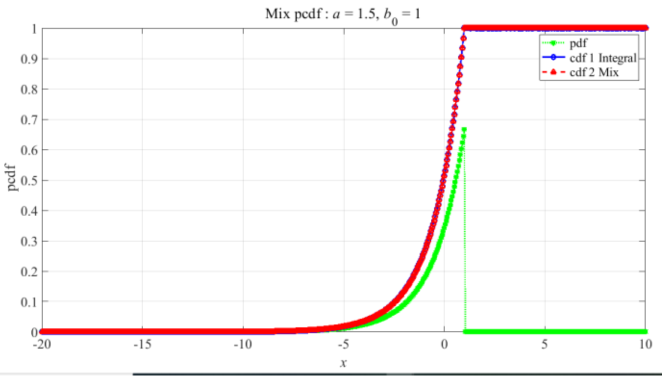
This concludes the description of MATLAB simulation functions. Next, we consider the qualitative and quantitative examples.

5.2 Qualitative, Quantitative Examples

The MATLAB simulation setup for the qualitative, quantitative examples is exactly the same as the one in Numerical Examples Sect. Progri (2016, [45]).

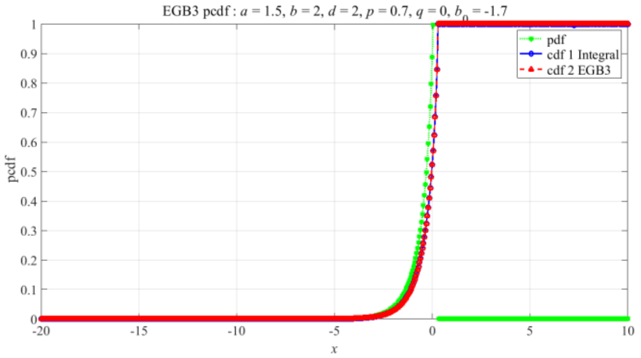


(a) The computation of Milno pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $b_0 = 4.4$. Cdf 1 Integral, Cdf 2 Milno cdf.

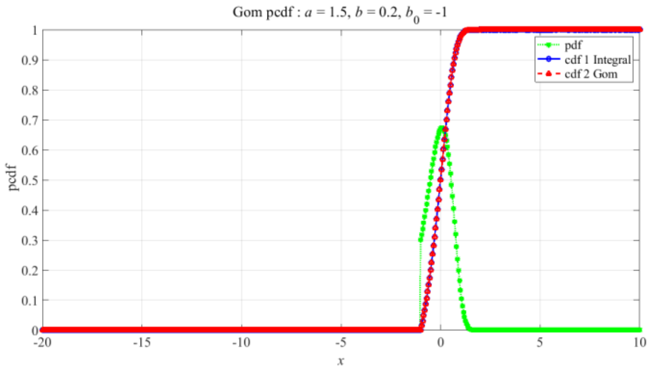


(b) The computation of MIX pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b_0 = 1$. Cdf 1 Integral, Cdf 2 MIX cdf.

Fig. 1. The computation of EGB3 and MIX pdf and cdf for $-20 \leq x \leq 10$.



(a) The computation of EGB3 pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 0.7$, $q = 0$, and $b_0 = -1.7$. Cdf 1 Integral, Cdf 2 EGB3 cdf.



(b) The computation of Gom pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 0.2$, and $b_0 = -1$. Cdf 1 Integral, Cdf 2 Gom cdf

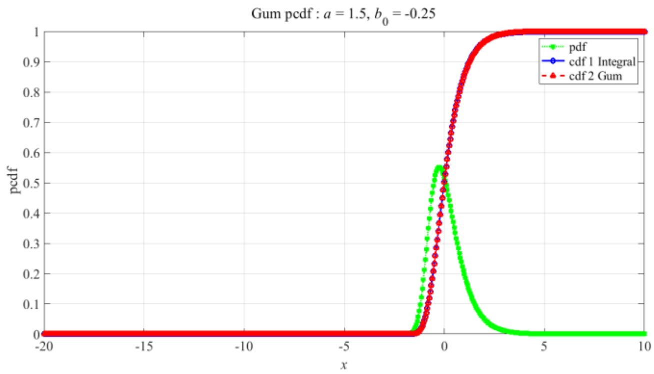
Fig. 2. The computation of EGB3 and Gom pdf and cdf for $-20 \leq x \leq 10$.

"We ran the simulations on a Dell computer Intel(R) Core(TM) i5-2400 CPU @ 3.10 GHz using MATLAB (2024b, [61]) using a vector of $-20 \leq x \leq 10$ with a sample increment of $\Delta x = 0.05$ that leads to 601 points."—Taken and modified from Progri (2024, [48]).

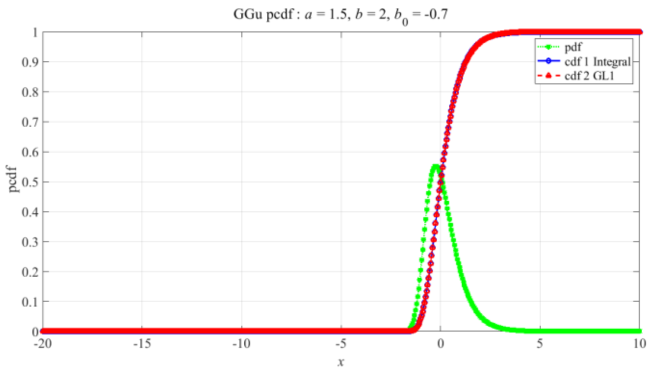
Figure 1 (a) the computation of the Milno, is an acronym for the mirror image of the logo-normal, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $b_0 = 4.4$. Cdf 1 Integral, Cdf 2 Milno cdf. Is the Milno pdf a valid pdf? Certainly, the Milno pdf is always greater than or equal to zero for all the values of the main variable from minus twenty to plus ten. The only reason why I have shown the plots of the Cdf 1 Integral, Cdf 2 Milno cdf is to prove numerically that in fact the integral of the Milno pdf is in fact a valid pdf. Detailed discussion of the Cdf 1 Integral, Cdf 2 Milno cdf are given in Progri (2024, [49]). Another observation that is made about Milno pdf is that it produces the fattest tale of any of the pdf discussed in the EGB family of distributions. However, it exhibits very slow growth in comparison to other distribution models investigated in this journal paper.

In Fig. 1(b) shows the computation of the MIX, an acronym for the mirror image of the exponential function, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b_0 = 1$. Cdf 1 *Integral* MATLAB BIF, Cdf 2 MIX cdf is the mirror image of the MATLAB BIF, exponential function cdf expcdf(-x+b_0,a). Is in fact the MIX pdf a valid pdf? For all values of the main variable x from minus twenty to plus ten the MIX pdf is always greater than or equal to zero. If we take the integral of the MIX pdf from minus infinity to x the integral is monotonic increasing from zero to one. The MIX pdf is ranged the second as the second pdf that produces the second fattest tale of any of the EGB family of distributions. The growth that results from the MIX pdf is faster than the growth rate that is produced by Milno pdf.

Figure 2 (a) presents the computation of the EGB3 pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 0.7$, $q = 0$, and $b_0 = -1.7$. Cdf 1 represents the computation via



(a) The computation of Gum pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$ and $b_0 = -0.25$. Cdf 1 Integral, Cdf 2 Gum cdf

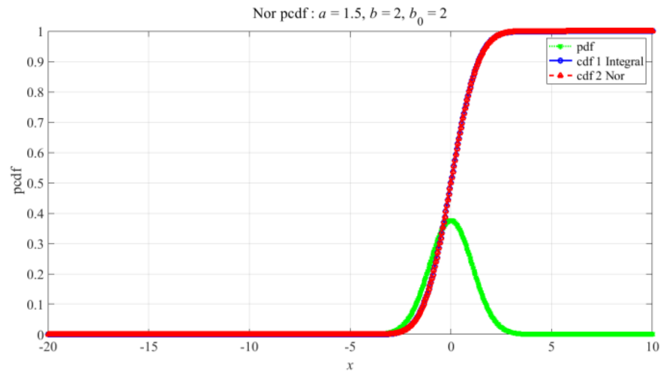


(b) The computation of the GGu pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $b_0 = -0.7$. Cdf 1 Integral, Cdf 2 GGu cdf

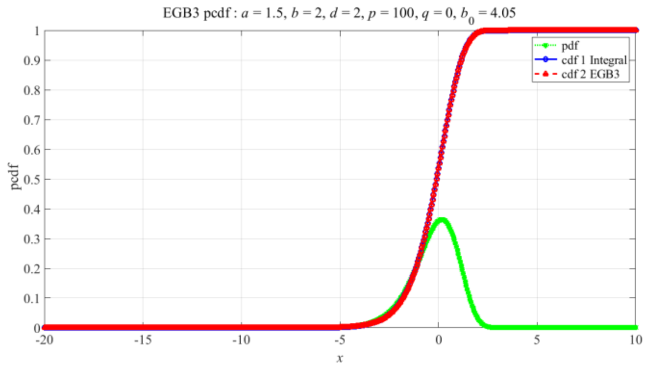
Fig. 3. The computation of Gum and GGu pdf and cdf for $-20 \leq x \leq 10$

Integral MATLAB BIF, Cdf 2 EGB3 cdf via exgenbeta3cdf(a,b,d,p,q,x-b_0) Giftet function. Again the same question is asked. Is EGB3 pdf a valid pdf? The simulation result of EGB3 pdf which are in accordance with Progri (2016, [45] (1)-(6)) indicate that EGB3 pdf is always greater than equal to zero for all values of main variable x from minus twenty to plus ten. Moreover, if we take the integra, of EGB3 pdf from minus infinity to x this integral increases monotonically from zero to one. EGB3 pdf is ranked the third fattest pdf model. We notice that EGB3 pdf for values of the six parameters given has perhaps the steepest growth than any of the EGB family of the pdf models considered in this journal paper.

Figure 2(b) displays the computation of the Gom, in an acronym for Gompertz, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 0.2$, and $b_0 = -1$. Cdf 1 Integral, Cdf 2 the Gom cdf. The computation of Gom pdf was performed employing (19)-(22) in Sect. 3.2. The first question that can be asked about the computation of Gom pdf is about the validity of the Gom pdf for all the values of the main variable x from minus twenty to



(a) The computation of the Nor pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, and $b_0 = 2$. Cdf 1 Integral, Cdf 2 the Nor cdf

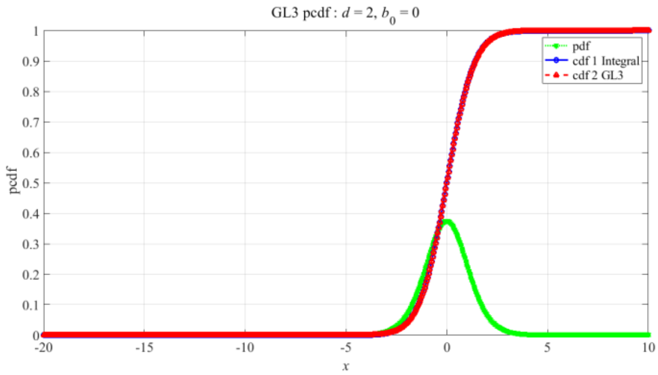


(b) The computation of the EGB3 pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 100$, $q = 0$, and $b_0 = 4.05$. Cdf 1 Integral, Cdf 2 the EGB3 cdf

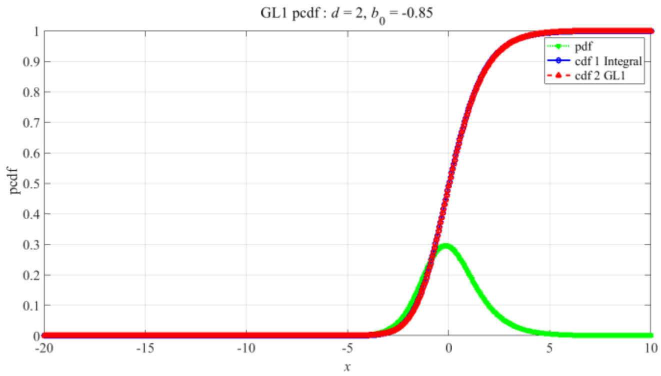
Fig. 4. The computation of the Nor and EGB3 pdf and cdf for $-20 \leq x \leq 10$.

plus ten. As depicted in Fig. 2(a), Gom pdf is always greater than or equal to zero in accordance with (20), (21) in Sect. 3.2. If we were to take the integral from minus infinity to x the value of the integral grows monotonically from zero to one in complete agreement with (22) in Sect. 3.2. Because Gom pdf is such impulsive pdf we took $b = 0.2$ which is ten times smaller than any of the values of b used for the computation of the other EGB family of distributions. Gom pdf can be employed to describe phenomenon that exhibit very steep growth. In contrast to EGB3 pdf which uses six parameters Gom pdf only uses half of them equal to three. In that context Gom pdf would be a faster model that should provide reasonable accuracy.

Figure 3(a) depicts the computation of Gum, an acronym that describes the Gumbel pdf as shown in Sect. 3.3, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$ and $b_0 = -0.25$. Cdf 1 Integral, Cdf 2 Gum cdf. Does the computation of Gum pdf show that Gum pdf is a valid pdf in accordance with (24) through (27) in Sect. 3.3?



(a) The computation of GL3 pdf and cdf for $-20 \leq x \leq 10$, $d = 2, b_0 = 0$. Cdf 1 Integral, Cdf 2 GL3 cdf

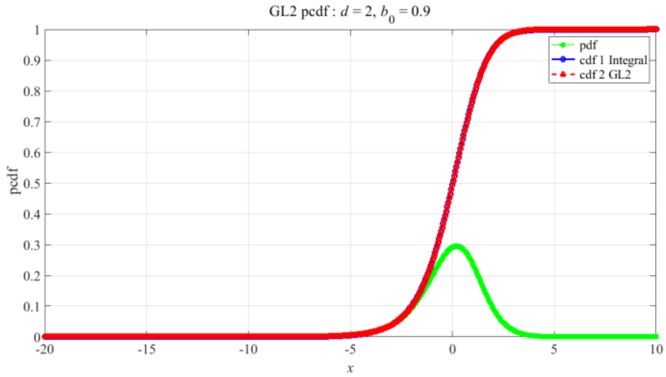


(b) The computation of GL1 pdf and cdf for $-20 \leq x \leq 10$, $d = 2, b_0 = -0.85$. Cdf 1 Integral, Cdf 2 GL1 cdf

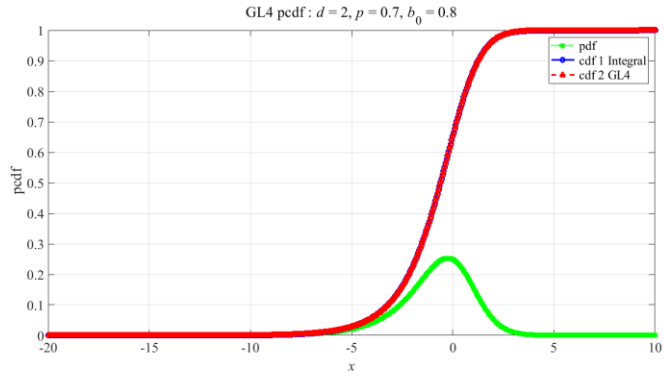
Fig. 5. The computation of GL3 and GL1 pdf and cdf for $-20 \leq x \leq 10$

As presented in Fig. 3(a) Gum pdf is always greater than or equal to zero for all values of the main variable x from minus twenty to plus ten as indicated in (25). Moreover, if we take the integral of Gum pdf from minus infinity to x as depicted in (27), the integral grows monotonically from zero to one. Gum pdf is also an impulsive pdf; however, in contrast to Gom pdf, Gum pdf is only a 2-parameter model and not as sharp as the Gom pdf.

Figure 3(b) the computation of the GGu, it is anacronym for the generalized Gumbel of type I in Sect. 4.1, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $b_0 = -0.7$. Cdf 1 Integral, Cdf 2 GGu cdf. Is the computation of the GGu pdf in accordance with (32) to (35) in Sect. 4.1a valid pdf? As illustrated in Fig. 3(b), GGu pdf is always greater than or equal to zero in accordance with (33) for all values of the main variable x from minus twenty to plus ten. The limit of GGu pdf when the main variable x goes to minus infinity is approximated with value of GGu pdf when x is equal to minus twenty in agreement with (30). And the limit of GGu pdf when x goes to plus infinity is approximated with the value of GGu pdf for the man variable x equal to ten in accordance



(a) The computation of GL2 pdf and cdf for $-20 \leq x \leq 10$, $d = 2, b_0 = 0.9$. Cdf 1 Integral, Cdf 2 GL2 cdf



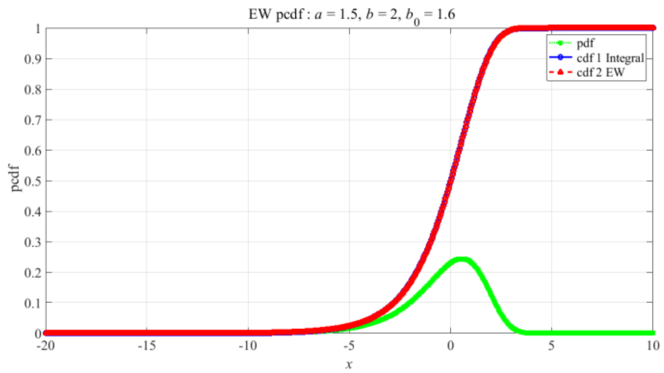
(b) The computation of GL4 pdf and cdf for $-20 \leq x \leq 10$, $d = 2, p = 0.7, b_0 = 0.8$. Cdf 1 Integral, Cdf 2 GL4 cdf.

Fig. 6. The computation of GL2 and GL4 pdf and cdf for $-20 \leq x \leq 10$

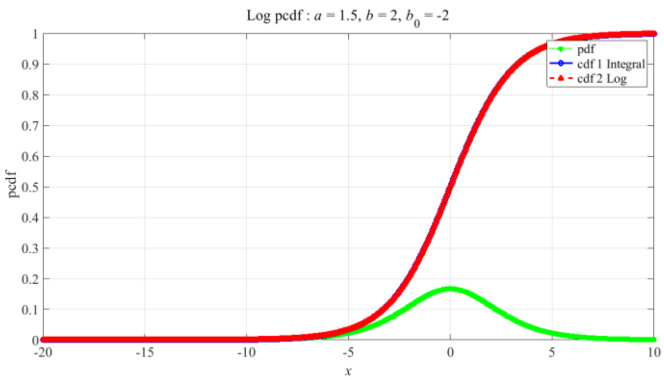
with (30). Moreover, if we take the integral of GGu pdf when the integration variable goes from minus infinity to x that integral grows monotonically from zero to one as indicated in (35). In terms of the shape of GGu pdf we can say that it has very similar shape as Gom pdf, with the same level of sharpness and it will exhibit a similar level of growth.

Figure 4(a) the computation of Nor, it is an acronym that describes the Normal (or Gaussian), pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $b_0 = 2$. Cdf 1 Integral, Cdf 2 Nor cdf. Much has been said about the computation of Nor pdf as indicated in Progri (2003, [30]), Progri et al. (2004, [31]), Progri (2006, [32]), Progri et al. (2007, [33]). However, the computation of Nor pdf in the context of EGB family of distributions is very unique. Nor pdf is perhaps one of the most studied and computed pdf models. It obeys all the laws of the pdf models. In the context of EGB family of distributions, Nor pdf is ranked right in the middle of sharpness and growth. It shows the transition from sharper pdfs to smoother pdfs.

Figure 4 (b) The computation of EGB3 pdf and cdf for



(a) The computation of the EW pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, and $b_0 = 1.6$. Cdf 1 Integral, Cdf 2 the EW cdf.



(b) The computation of the Log pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$ and $b_0 = -2$. Cdf 1 Integral, Cdf 2 the Log cdf.

Fig. 7. The computation of EW and Log pdf and cdf for $-20 \leq x \leq 10$

$-20 \leq x \leq 10$, $a = 1.5$, $b = 2$, $d = 2$, $p = 100$, $q = 0$, and $b_0 = 4.05$. Cdf 1 Integral, Cdf 2 EGB3 cdf. Is EGB3 pdf a valid pdf for the values of main variable and of all its six parameters? Certainly yes it is as depicted in Fig. 4(b) in complete agreement with Progri (2016, [45] (1)-(7)). The main computation of EGB3 pdf is performed via Progri (2016, [45] (4)), the values of the limits via Progri (2016, [45] (6)), and the integral by means Progri (2016, [45] (7)). In contrast to EGB3 pdf shown in Fig. 2(a) for $p = 0.7$, for the computation of the EGB3 pdf in Fig. 4(b) the value of $p = 100$. We can say that the parameter p is the sharpness parameter. The higher the value of p the smoother EGB3 pdf and the smaller the value of p the sharper EGB3 pdf is. This is a small example to illustrate the flexibility that EGB3 pdf offers in modeling of the various stock returns. We can also observe that EGB3 pdf, for $p = 100$, is almost as good as a Nor pdf.

Figure 5(a) presents the computation of GL3, it is an acronym that describes the Generalized Logistics of Type 3 Sect. 4.4, pdf and cdf for $-20 \leq x \leq 10$, $d = 2$, $b_0 = 0$. Cdf 1 Integral, Cdf 2 GL3 cdf. Is GL3 pdf a valid pdf as described in (44) through (47)? The main computation of GL3 pdf is performed by means of computation of (44), the inequality (45) is fulfilled for all value of the main variable x from minus twenty to plus ten, the computation of the limits of (46) is approximated with the values of GL3 pdf for the values of the main variable in the edges or equal to minus twenty or plus ten. The calculation of the integral (47) is performed numerically to show that indeed the integral grows monotonically from zero to one. However, GL3 pdf being only a 2-parameter model is really a great approximation of pdfs that are in the middle of the pack when it comes to sharpness of rate of growth; GL3 pdf is very close to the Nor pdf and EGB3 pdf for $p = 100$.

Figure 5(b) illustrates the computation of GL1, an anacronym that describes the Generalized Logistics of Type 1 in Sect. 4.2, pdf and cdf for $-20 \leq x \leq 10$, $d = 2$, $b_0 = -0.85$. Cdf 1 Integral, Cdf 2 GL1 cdf. The main computation of GL1 pdf is performed by computing (36) which appears to indicate that it obeys inequality (37). The computation of the limits values is done via (38). And finally the computation of the integral (39) is shown via the MATALB BIF integral which appears to grow monotonically from zero to one. All of these observations together appear to indicate that GL1 pdf is a valid pdf model. In terms of sharpness for the values of the parameters considered, GL1 pdf is less sharp than GL3 pdf but it is sharper than the remaining pdf models from the EGB family of distributions.

Figure 6(a) depicts the computation of GL2, an anacronym that describes the Generalized Logistics of Type 2 in Sect. 4.3, pdf and cdf for $-20 \leq x \leq 10$, $d = 2$, $b_0 = 0.9$. Cdf 1 Integral, Cdf 2 GL2 cdf. The main computation of GL2 pdf is performed by computing (40) which appears to indicate that it obeys inequality (41). The computation of the limits values is accomplished via (42). And finally the computation of the integral (43) is shown via the MATALB BIF integral which appears to grow monotonically from zero to one. All of these observations together appear to indicate that GL2 pdf is a valid pdf model. In terms of sharpness for the values of the parameters considered, GL2 pdf is less sharp than GL1 pdf but it is sharper than the remaining pdf models from the EGB family of distributions.

Figure 6(b) presents the computation of GL4, an anacronym that describes the Generalized Logistics of Type 4 in Sect. 4.5, pdf and cdf for $-20 \leq x \leq 10$, $d = 2$, $p = 0.7$, $b_0 = 0.8$. Cdf 1 Integral, Cdf 2 GL4 cdf. GL2 pdf main computation is performed by computing (48) which appears to indicate that it obeys inequality (49). The computation of the limits values is accomplished via (50). And finally the computation of the integral (51) is shown via the MATALB BIF integral which appears to grow monotonically from zero to one. All of these observations together indicate that GL4 pdf is a valid pdf model. In terms of sharpness for the values of the parameters considered, GL4 pdf is less sharp than GL2 pdf but it is sharper than the remaining pdf models from the EGB family of distributions.

Figure 7(a) depicts the computation of EW, an anacronym that describes the Exponential Weibull in Sect. 3.1, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, and $b_0 = 1.6$. Cdf 1 Integral, Cdf 2 EW cdf. The main computation of EW pdf is performed by computing (11) or (15) which appears to indicate that it obeys inequality (16). The computation of the limits values is accomplished via (17). And finally the computation of the integral (18) is shown via the MATALB BIF integral which appears to grow monotonically from zero to one. All of these observations together appear to indicate that EW pdf is a valid pdf model. In terms of sharpness for the values of the parameters considered, EW pdf is the second less sharp (or the second smoothest) pdf from the EGB family of distributions.

Figure 7(b) presents the computation of Log, an anacronym that describes the Logistics in Sect. 3.4, pdf and cdf for $-20 \leq x \leq 10$, $a = 1.5$, $b = 2$ and $b_0 = -2$. Cdf 1 Integral, Cdf 2 Log cdf. Log pdf main computation is performed by computing (28) which appears to indicate that it obeys inequality (29). The computation of the limits values is accomplished via (30). And finally the computation of the integral (31) is shown via the MATALB BIF integral which appears to grow monotonically from zero to one. All of these observations together indicate that Log pdf is a valid pdf model. In terms of sharpness for the values of the parameters considered, Log pdf is the least sharp pdf model from the EGB family of distributions.

This concluded the discussion on examples and numerical, theoretical results.

6 Conclusions

There are so many great points about this journal paper.

First it is the description of the historical context. This journal-paper includes a detailed description of statistical probability density functions or models and of the people and events that occurred as early as the 1600s to 2024. No other journal paper or other publication of mince comes close to achieving this goal.

The second important accomplishment of this journal paper is the description of the EGB family of distributions in Sect. 3 and Sect. 4. We have a total of thirteen different distributions; five non-generalized pdfs, five generalized pdfs, and three fundamental pdfs. Of the thirteen, ten of them are a direct relation of the EGB of type 3, also known as the 5-paramater EGB model.

The third major accomplishment of this journal paper is the organization, or the preliminary ranking of the various pdf models based on the level of growth and sharpness as shown in the qualitative and quantitative outcome Sect. 5.

The fourth major contribution that comes from this journal paper is where in the preliminary raking of the three fundamental pdfs are, and the ten pdfs from the EGB family of distributions are.

EGB of type 3, a.k.a. the 5-paramater EGB provides a very powerful and flexible way of modeling pdfs that exhibit a wide range of sharpens, growth, and smoothness.

7 Acknowledgement

This work was supported by Giftet Inc. executive office.

I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [61] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications*. *Vol. I* (Not yet available in print) [60].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say I have fallen in love with his masterpiece book, *Golf My Way* and *Mo Golden Lessons*. Moreover, Jack Nicklaus [61] reminds me of my grandfather who I loved very much.

Finally, I am also dedicating this journal paper to our most beloved President Ronald Reagan, who reminds me of my grandfather, [62] who spoke from the heart, who spoke the truth, who was a staunch supporter of personal freedom, the

greatest leader of the free world, and perhaps the greatest critic of the wasteful government control and spending[xxiii].

"*Government is not the answer, government is the problem*."—Ronald Reagan

I would like to express my deepest gratitude to my mother Lumturi Progri, my sister Adriana Dine, and Ms. Elizabeth Demir for their support, loyally, and dedication to me during some of the most difficult times in my life as a boy, man, and scholar.

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9 Appendix A: EW PDF is Special Case of EGB PDF

In order to show that EW is Special Case of EGB we set the following parameters in the EGB

$$b \rightarrow a\log(p) + b \tag{52}$$

$$x' \rightarrow e^{\frac{x - a\log(p) + \log(a')}{a}}$$

$$= e^{\frac{1}{a}\left[\log\left(\frac{y}{p^a a'}\right)\right]}$$

$$= e^{\log\left(\frac{y}{p^a a}\right)^{\frac{1}{a}}}$$

$$= \left(\frac{y}{p^a a}\right)^{\frac{1}{a}}$$

$$= \frac{1}{p}\left(\frac{e^x}{e^b}\right)^{\frac{1}{a}}$$

$$= \frac{1}{p} e^{\frac{x-b}{a}}$$

$$= \frac{1}{p} x' \tag{53}$$

The EW is obtained from EGB by letting $p \to \infty$ which leads to $b \to \infty$; in a more compact form the EW is reduced to

$$f_{\mathrm{EW}}(x; a, b, d) = f_{\mathrm{EGB}}(x; a, a\log(p) + b \to \infty, d, p \to \infty, q)$$
$$= |f(x), \quad -\infty < x < \infty \tag{54}$$

where $f(x)$ is obtained from

$$f(x) = \lim_{p\to\infty} \frac{x'^d[1-(1-q)x']^{p-1}}{|a|\mathrm{B}(d,p)[1+qx']^{d+p}}$$

$$= \lim_{p\to\infty} \frac{x'^d\left[1-(1-q)\frac{x'}{p}\right]^{p-1}}{|a|p^d\mathrm{B}(d,p)[1+qx']^{d+p}}$$

$$= \lim_{p\to\infty} \frac{x'^d e^{-(1-q)x'}}{|a|\Gamma(d)e^{qx'}}$$

$$= \lim_{p\to\infty} \frac{x'^d e^{-x'}}{|a|\Gamma(d)}$$

$$= \frac{x'^d e^{-x'}}{|a|\Gamma(d)} \tag{55}$$

Let us evaluate the following limit

$$\lim_{p\to\infty}\left[1-(1-q)\frac{x'}{p}\right]^{p-1} \cong \lim_{p\to\infty}\left[e^{-(1-q)\frac{x'}{p}}\right]^{p-1}$$

$$= \lim_{p\to\infty} e^{-(1-q)\frac{(p-1)}{p}e^{\frac{x-b}{a}}}$$

$$= e^{-(1-q)e^{\frac{x-b}{a}}} \tag{56}$$

Next, we evaluate the following limit

$$\lim_{p\to\infty}\left(1+q\frac{x'}{p}\right)^{d+p} \cong$$

$$= \lim_{p\to\infty}\left(e^{q\frac{x'}{p}}\right)^{p+d}$$

$$= \lim_{p\to\infty} e^{q\frac{(p+d)x'}{p}}$$

$$= \lim_{p\to\infty} e^{q\frac{p+d}{p}x'}$$

$$= \lim_{p\to\infty} e^{q\left(1+\frac{d}{p}\right)x'}$$

$$= \lim_{p\to\infty} e^{q(1+0)x'}$$

$$= e^{qx'} \tag{57}$$

Let us evaluate the following limit of the product of the power function p^d with the beta function $\mathrm{B}(d,p)$ as the parameter p approaches infinity

$$\lim_{p\to\infty} p^d\mathrm{B}(d,p) \cong \lim_{p\to\infty} p^d\frac{\Gamma(d)\Gamma(p)}{\Gamma(d+p)}$$

$$= \Gamma(d)\lim_{p\to\infty} p^d\frac{\Gamma(p)}{\Gamma(d+p)}$$

$$= \Gamma(d)\lim_{p\to\infty}\frac{p^d\Gamma(p)}{p(p+1)\cdots(p+d-1)\Gamma(p)}$$

$$= \Gamma(d)\lim_{p\to\infty}\frac{p^d}{p^d\left(1+\frac{1}{p}\right)\cdots\left(1+\frac{d-1}{p}\right)}$$

$$= \Gamma(d)\lim_{p\to\infty}\frac{1}{1\left(1+\frac{1}{p}\right)\cdots\left(1+\frac{d-1}{p}\right)}$$

$$= \Gamma(d) \tag{58}$$

Next, the gamma function $\Gamma(d+p)$ can be computed from

$$\Gamma(d+p) = p(p+1)\cdots(p+d-1)\Gamma(p)$$
$$= p^d\left(1+\frac{1}{p}\right)\cdots\left(1+\frac{d-1}{p}\right)\Gamma(p) \tag{59}$$

This concludes the brief discussion on how the EW PDF is obtained from a Special Case of EGB PDF.

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[i] I am dedicating this journal paper to my Ph.D. Studies Academic Advisor Prof. John Orr. I wrote on my 2003 Ph.D. Dissertation: "On a more personal note, I am very grateful of Professor John A. Orr as a great example of an honest and just person, who lives in the "Fear of God," and who treats colleges equally and fairly. I would like to thank you, John, for your wisdom and courage, for believing in me and for being my academic advisor for the early part of my Ph.D. program." Prof. Orr was a phenomenal teacher, scholar, researcher, academic advisor, and Department Head.

[ii] According to McDonald(1988, [18]), the development of formal statistical models [which includes the pdf, cdf, etc.] began in the late seventeenth century some of the early developments were motivated by problems in astronomy and physics the participants in this development represented many disciplines and a list of the contributors reads like a *Who's Who in the World of Science* including Euler, Edgeworth, De Moivre, Galton, Gauss, Laplace, Legendre and Maxwell among others.

McDonald(1988, [18]), one of the first statistical distributions to be observed and mathematically modeled is the uniform distribution due to De Moivre(1725, [18], 0, [39]) formalized the distribution in his treatise on life annuities. The model suggests equally likely outcomes of an event over a finite interval and the average or expected value of the event is at the midpoint of the interval.

In 1733 De Moivre [*] first obtained the normal distribution as an approximation to binomial distribution only later was it found to provide an excellent fit to many types of data this density was often called the law of frequency of error and is one of the most commonly used distributions in statistics—the workhorse of statistics. The mean or average value of a normally distributed variable corresponds to the highest point on the curve, and it is important to note that the distribution is symmetric about the mean McDonald(1988, [18]).

While the concept of pdf developed over time with contributions from various mathematicians, Carl Friedrich Gauss ([40], [42]) is generally credited with the most significant development of the pdf, particularly in relation to the normal distribution, which is often referred to as the "Gaussian distribution" due to his work on it.

[*] H.M. Walker, "De Moivre on the Law of Normal Probability," *Math.*, pp. 75-84, 2006.

In 1823 Gauss (1823, [40], [42]) published his monograph "Theoria combinationis observationum erroribus minimis obnoxiae" where among other things he introduces several important statistical concepts, such as the method of least squares, the method of maximum likelihood, and the normal distribution [42].

Although Gauss was the first to suggest the normal distribution law, Laplace [41], [42] made significant contributions; however, since Laplace [41], [42] contributions are more in line with the normal cdf; his contribution will be mentioned in Progri (2024, [49]).

Normal distribution: Gauss formulated the pdf for the normal distribution, which is a crucial tool in statistics due to its wide applicability in describing naturally occurring phenomena.

Infinite Variance: When the underlying distribution has an infinite variance, the sum of many random variables from that distribution will not converge to a normal distribution, instead tending towards a stable distribution with heavier tails.

Extreme Events: Stable distributions are often seen in scenarios where extreme events are more likely than predicted by a normal distribution, such as financial market fluctuations or natural disasters.

Lévy Flights: A prominent example of a stable distribution with heavy tails is the Lévy flight, which models random walks with large jumps occurring occasionally, relevant in fields like ecology, physics, or the stock market.

Power Law Distributions: Distributions that follow a power law with a heavy tail, like the Pareto distribution, can lead to stable distributions when summed.

Asymmetric Distributions: Even if a distribution has heavy tails but is not symmetric, the resulting stable distribution can be skewed, further deviating from a normal distribution.

[iii] Typically for a function to be a valid pdf, it must satisfy two conditions: (1) it must be non-negative for all values of x ($f(x) \geq 0$) and (2) the integral of the function over its entire domain must equal 1 ($\int f(x)dx = 1$); essentially meaning the area under the curve of the function must be equal to 1.

Non-negativity: This condition ensures that the probability of any event calculated using the PDF will never be negative.

Integral equals 1: This condition guarantees that the total probability of all possible outcomes is equal to 1, which is a fundamental property of probability.

When the main variable in a probability density function approaches either positive or negative infinity, the value of the function approaches zero; essentially, the probability density function becomes infinitesimally small at extreme values like plus or minus infinity.

Key points: Continuous random variables: For a continuous random variable, the probability of a specific value is zero, so the probability is calculated by finding the area under the PDF curve over a specific interval

Notation: The integral condition is usually written as ∫_{-∞}^{+∞} f(x) dx = 1, where the integral is taken over the entire possible range of the random variable.

As the variable approaches infinity, the pdf curve gets closer and closer to the x-axis, resulting in a value that approaches zero.

[iv] The "exponential generalized beta distribution" is generally attributed to B.C. Arnold (see 2015, [35]) who is credited with developing the concept of the generalized beta distribution, which includes the exponential generalized beta as a specific case; however, the exact attribution can vary depending on the specific form and application of the distribution within the broader family of generalized beta distributions.

Broader concept: The generalized beta distribution is a family of probability distributions that extends the standard beta distribution by adding more parameters, allowing for greater flexibility in modeling different shapes.

Applications: These distributions are often used in fields like hydrology, economics, and reliability analysis due to their ability to model data with skewed or bounded ranges.

[v] "Non-generalized" means not broad or applicable to a wide range, so synonyms could include specific, particular, precise, detailed, narrow, focused, isolated, or contextual depending on the context.

[vi] The genesis of the extreme value distribution is presented in such an outstanding manner in the publication of Johnson et al. (1995, [5] Chap. 22, pp 1-2). However, the introduction here focusses on what is not covered by this publication.

[vii] Google scholar contains about an impressive 14,800 citations until 2023, which is a testament of the quality, depth, and framework of this publication.

[viii] It is safe to assume that Officer (1972, [6]) concluded that in the 1960s until the 1970s the ground work to study the special cases of the EGBD was prepared because it was understood that this distribution will be non-normal, "fat-tailed" and relative to the normal distribution. Officer (1972, [6]) paper contains a great list of references on this very important topic; what as lacking is the mathematical model of the distribution.

[ix] Examples of non-normal distributions include: income distribution within a population (where a few very wealthy individuals skew the data), the number of extreme weather events in a year, the lifespan of a specific product model with a high failure rate early on, the amount of time spent on a website by users (with many short visits and a few very long ones), or the distribution of test scores in a class where a small number of students significantly outperform the majority.

[x] *CLT*: The normal distribution is closely linked to the CLT, which states that the sum of many independent random variables tends to follow a normal distribution, further solidifying its importance.

According to Bookstaber, McDonald (1987, [17]), CLT does not necessarily imply that normal distribution as the limiting distribution. CLT can also lead to other nonnormal stable limits in a number of cases familiar to econometricians (see Bartels (1977, [11])).

The key argument that leads to non-normal stable distributions, in contrast to the typical CLT result of a normal distribution, is the presence of heavy tails in the original distribution, meaning a significant probability of extremely large values occurring, which violates the finite variance condition necessary for the CLT to produce a normal distribution.

[xi] Is EGBD symmetric around its mean? No, an EGBD is generally not symmetric around its mean; it can be skewed to the left or right depending on the values of its parameters, similar to the standard beta distribution, unless the parameters are set specifically to create a symmetric distribution.

[xii] Is EGBD a stable distribution? No, an EGBD is not considered a stable distribution; while it is a flexible distribution that can model various shapes, it does not possess the key property of a stable distribution, which is that the sum of two independent random variables with the same stable distribution also follows the same stable distribution.

Stable distributions: These are a specific class of probability distributions where the sum of two independent random variables with the same stable distribution results in another random variable with the same stable distribution, regardless of scale.

EGBD: This is a flexible distribution derived from the GBD allowing for a wide range of shapes depending on its parameters.

[xiii] According to McDonald (1991, [20]) the GB2, introduced by Gary Venter (1984, [14]) as the transformed Beta distribution [in p. 160], provides the basis for the models considered. Variations of this distribution have also been referred to as the generalized F (see J.D. Kalbfleisch, R.L. Prentice (2002 (first published in 1980 with a book review in (1982, [13]) by John P. Klein Ohio State University it is mentioned the generalized F in Chap. 2, [13])), as Feller-Pareto distribution by Arnold (2015 (first published in 1983), [35]) and the Beta prime by Patil (1984, [16]). However, McDonald(1984, [1]) provides the notation which was later on used in literature. McDonald (1991, [20]) provides a comprehensive literature search of the early contributions on the topic of the GB2.

[xiv] According to McDonald (1988, [18]) the GB2 has been considered before by Mathai and Saxena (1967, [1]) and Prentice (1975, [10]) however, this distribution was not widely known at that time, was independently obtained by several researchers in the early 1980s and has received considerable attention during the last couple of years. I believe I was able to fill in the vacuum that was left in the McDonald(1988, [18]) introduction because at that time it was hard to obtain all the information due to the lack of collaboration among researchers and research institutions.

[xv] "A generalization of the Beta of the First and Second Kind with an Application" refers to a statistical concept where a new probability distribution is created by extending the properties of both the standard Beta distribution (first kind) and the Beta Prime distribution (second kind), allowing for greater flexibility in modeling data with diverse shapes and skewness, particularly when dealing with situations where traditional Beta distributions might not be suitable; this expanded distribution can then be applied to analyze real-world data across various fields like finance, reliability,

or ecology depending on the specific application.
Combined flexibility: By incorporating aspects of both Beta distributions, the generalized version can model scenarios with both *left-skewed and right-skewed data*, unlike the standard Beta which is primarily used for data concentrated around a certain probability range.
Additional parameters: This generalized form usually involves introducing extra parameters compared to the standard Beta, allowing for finer tuning of the distribution's shape and spread.
Applications: Such a generalized Beta distribution could be used to model phenomena like:

1. Stock market returns with varying volatility
2. Failure times in reliability analysis with skewed distribution
3. Environmental data with extreme values
4. Proportions in complex systems where the probability range might be wider than the usual 0-1 interval

[xvi] In the stock market, "beta estimation" refers to the process of calculating a stock's beta, which is a measure of how volatile a stock is compared to the overall market, essentially indicating how much risk a particular stock adds to a portfolio relative to the market as a whole; a beta of 1 means the stock moves in line with the market, while a beta above 1 suggests higher volatility and a beta below 1 indicates less volatility than the market.
Key points about beta estimation:
*Calculation***:** Beta is calculated by comparing the historical price movements of a stock to the movements of a market index (like the S&P 500) using a statistical formula that involves calculating the covariance between the stock returns and market returns divided by the variance of the market returns.
*Interpretation***:**

1. *Beta > 1***:** The stock is considered more volatile than the market, meaning it will likely experience larger price swings both upwards and downwards.
2. *Beta = 1***:** The stock moves in line with the overall market.
3. *Beta < 1***:** The stock is considered less volatile than the market, meaning it will have smaller price swings.

Use in investing: Investors use beta to assess the risk of a particular stock within their portfolio and to make informed investment decisions based on their risk tolerance.
[xvii] Skewness is a statistical measure that indicates how asymmetric a data distribution is, essentially showing how much the data is skewed to one side of the average, and in the context of stock returns, it highlights the potential for extreme gains or losses, with a "positively skewed" distribution meaning there's a higher probability of large positive returns compared to large negative ones, which is generally considered favorable for investors.
Key points about skewness and stock returns:
*Interpretation***:** *Positive Skewness***:** A distribution with a longer tail on the right side, indicating a higher chance of large positive returns (most common in stock markets).
*Negative Skewness***:** A distribution with a longer tail on the left side, implying a higher chance of large negative returns.
*Zero Skewness***:** A perfectly symmetrical distribution, where the data is evenly distributed on both sides of the mean.
Importance for Investors:
*Risk Assessment***:** Skewness helps investors understand the potential for extreme events and the overall risk profile of an investment.
*Portfolio Construction***:** Investors can use skewness to select stocks that offer a favorable balance between potential returns and downside risk.
Real-world Example: A stock with a positively skewed return distribution might experience frequent small losses but occasionally have large price jumps, leading to potentially high long-term returns.
[xviii] Leptokurtosis refers to a statistical distribution with "fat tails," meaning it has a higher likelihood of extreme events occurring on either side of the mean, compared to a normal distribution; in the context of stock returns, it indicates a higher potential for large price swings, signifying increased risk for investors, but also the possibility of significantly higher returns.
Key points about Leptokurtosis:
*Distribution shape***:** A leptokurtic distribution appears "peaked" in the center with significantly wider tails compared to a normal distribution.
Impact on stock returns: *High volatility*: A leptokurtic stock return distribution suggests a higher chance of experiencing large, sudden price movements, both positive and negative.
Risk and reward potential: While the potential for large losses exists, there is also a higher chance of experiencing substantial gains with a leptokurtic distribution.
Investor considerations: *Risk tolerance*: Investors with a high-risk tolerance might seek out investments with leptokurtic distributions, as they offer the possibility of significant returns, but also come with a higher risk of extreme losses.
Diversification: To mitigate the risk associated with leptokurtic distributions, investors often diversify their portfolios across different asset classes.
[xix] Kurtosis, in the context of stock returns, is a statistical measure that indicates how "peaked" a distribution is and how frequently extreme outliers occur in the tails of that distribution, essentially signifying the likelihood of experiencing large, sudden price movements in a stock, which translates to higher risk for investors when kurtosis is high.
Key points about kurtosis and stock returns:
*Measuring extreme events***:** A high kurtosis value suggests a higher probability of experiencing extreme positive or negative returns (outliers) compared to a normal distribution, indicating potential for large price swings in either direction.
Risk assessment: Investors generally prefer investments with lower kurtosis as it implies a more stable distribution with fewer extreme events, reducing overall risk.
*Types of kurtosis: Leptokurtic***:** A distribution with high kurtosis, meaning "fat tails" and a higher probability of extreme outliers.
Platykurtic: A distribution with low kurtosis, indicating fewer extreme outliers and flatter tails.
*Mesokurtic***:** A normal distribution with moderate kurtosis.
How to use kurtosis in stock analysis:
Comparing investments: When analyzing multiple stocks, comparing their kurtosis values can help identify which ones are more likely to experience sudden price fluctuations.
Risk management: Investors can use kurtosis to adjust their portfolio allocation, potentially allocating less capital to stocks with high kurtosis if they are risk-averse.
[xx] An ARIMA model, which stands for "Autoregressive Integrated Moving Average," is a statistical method used in time series analysis to predict future stock returns by analyzing past price data, essentially forecasting future trends based on patterns identified in historical stock price movements; it's a popular tool in finance to understand and predict short-term fluctuations in stock returns by considering the relationships between past values and the current value of a stock price.
Key points about ARIMA models in stock returns:
How it works: ARIMA models use three components:
*Autoregressive (AR)***:** This part of the model looks at the relationship between a current value and its own past values.
Integrated (I): This component involves differencing the data to remove trends and make it stationary, which is necessary for accurate forecasting.
Moving Average (MA): This component considers the past errors (the difference between actual and predicted values) to improve the forecast.
Application: By analyzing historical stock prices, an ARIMA model can be used to predict future price movements, identify trends, and estimate potential volatility.
Limitations: While useful for identifying patterns, ARIMA models can be less accurate during periods of significant market volatility or when major news events occur, as they rely on the assumption that past trends will continue into the future.
[xxi] Bookstaber, McDonald (1987, [17]) states the following:
"There is no law of nature that demands that the security returns fit the GB2, anymore that there is that the returns follow a log-*t* or a stable Paretian distribution. Like these distributions, the GB2 has attractive properties that should make it useful in empirical estimation and in some specialized theoretical work in which the specification of the distribution is of higher importance."
Other added benefits of the GB2 according to Bookstaber, McDonald (1987, [17]) are its flexibility due to generalization of many other distributions, makes use of all the parameters involved in (or facilitates the development of) the option price model and other models that depend on the specifications and mathematical manipulation of distributions, the existence of higher moments,
What is the option price model? An option pricing model is a mathematical formula used to calculate the theoretical value of a stock option, considering factors like the current stock price, strike price, time until expiration, volatility, and risk-free interest rate, essentially estimating the fair market value of an

option contract based on these variables; the most well-known option pricing model is the Black-Scholes model.

[xxii] The mathematicians primarily credited with developing the concept of the pdf and its application to calculating probabilities at infinity are Pierre-Simon Laplace and Carl Friedrich Gauss; particularly, Gauss is often associated with the specific case of the normal distribution, where the pdf approaches zero as the variable approaches positive or negative infinity.

According to McDonald(1988, [18]), in 1774 Laplace derived the double exponential or Laplace distribution which satisfies the properties under consideration.

Laplace's contribution: He played a key role in establishing the CLT, which explains why many naturally occurring phenomena tend to follow a normal distribution, where the probability at infinity is essentially zero.

Gauss's contribution: Gauss further developed the concept of the normal distribution, often called the "Gaussian distribution" due to his significant contributions to its theory and application.

[xxiii] Ditto Progri(2020, [64]) EN vii.