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Research Letter

Reduction Formulae of a 2F2 Hypergeometric Function

To my Prof., Alex Emanuel, a phenomenal teacher, scholar, researcher, friend, the best Master Thesis advisor who I have ever known in my life.[i]

—Ilir F. Progri

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Abstract: In 2020 when I published the research paper on the computation of a 2F2 generalized hypergeometric function (GHGF) for a particular set of parameters the paper generated a lot of interest. However, I felt that the work I published in 2020 needed more clarifications, which is the reason why I am publishing this research letter. The main focus of this research letter is to offer more insights into the nature of the computation of the fundamental 2F2 GHGF, its connection with GHGF and some connections with confluent HGF. Several numerical examples are used to verify the closed form expressions of a 2F2 GHGF.

Index Terms— generalized hypergeometric functions, Confluent hypergeometric functions, Fundamental 2F2, recursive reduction formulae, Gauss power series, Kummer transformations.

1 Introduction

A 2F2 generalized hypergeometric function (GHGF) occurs when an integration of a confluent hypergeometric function (CHGF or CHF) is performed [1]-[17].

Although implicitly the computation of a 2F2 has been discussed in many of my previous publications such as in Progri(2016, [8]-[11]), Progri(2018, [12]), it was not until I worked many years earlier on integrating the incomplete Gamma function (IGF) the integer case of Progri(2019, [13] (143), (145), (189)) and many other forms of the GHGF such as 1F2 or 2F3.

I suspect that one of the main reasons why Progri(2020, [14]) it is because it showed the connections between so many of the instances that if I had known how to bring the computation of a 2F2 into m research I could have; however, the publication of Progri(2020, [14]) accomplished just that since the papers on the computation of a 2F2 are somewhat scarce (not in plentiful quantities).

This letter exploits some of the relationship between a 2F2 GHGF with the Kampé de Fériet functions, CHGF, and 1F2 and 2F3 GHGF.

The main theme in this letter is to present some other methods on the computation of a fundamental 2F2[1,2;2,2;z] GHGF and make a few observations about their strengths and weaknesses. What is the main connection between a 2F2 GHGF and other GHGF and CHGF? of interests: one of them is called the fundamental 2F2 generalized hypergeometric function and the other a 2F2 generalized hypergeometric function. The fundamental ${}_2F_2[1,1;2,2;z]$ hypergeometric function is computed by the means of the Recursive Reduction Formulae (RRF) (15) algorithm *phypergeom13* and split of even and odd components (26). However, RRF (15) algorithm *phypergeom13* is *three to four orders of magnitude faster than* (26) and *two to three orders of magnitude faster than the computation via* Progri(2020, [14] (49)). This is an original result never published before.

This letter is organized as follows: in Sect. 2 reduction formulae of certain CHFs are discussed. In Sect. 3, RRF of a fundamental 2F2 GHGF is presented. In Sect. 4 several numerical examples are considered. Conclusion is provided in Sect. 5 along with Acknowledgments in Sect. 6, and a list of references in Sect. 7.

2 Reduction Formulae of Certain CHFs

From the well-known definition of the M CHF via the *Gauss Power Series* (GPS) [14] we have

$$M(a,b|z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!} \tag{1}$$

$$= 1 + z\sum_{k=0}^{\infty} \frac{(a)_{k+1}}{(b)_{k+1}}\frac{z^k}{(k+1)!}$$

$$= 1 + z\sum_{k=0}^{\infty} \frac{a(a+1)_n}{b(b+1)_n}\frac{z^k}{(k+1)!}$$

$$= 1 + z\frac{a}{b}\sum_{k=0}^{\infty} \frac{(a+1)_n}{(b+1)_n}\frac{z^k}{(k+1)!}$$

$$= 1 + z\frac{a}{b}\sum_{k=0}^{\infty} \frac{(a+1)_n(1)_k}{(b+1)_n(2)_k}\frac{z^k}{k!}$$

$$= 1 + z\frac{a}{b}\,{}_2F_2\begin{bmatrix} a+1,1; \\ b+1,2; \end{bmatrix} z \tag{2}$$

Equations (1) and (2) provide the direct connection between the CHF and a fundamental 2F2 GHGF.

Well known reduction formula for the Pochhammer symbol (or rising factorial [23]) that were employed in (1) is

$$(q)_{n+1} = q\frac{\Gamma(q+1+n)}{q\Gamma(q)} = q\frac{\Gamma(q+1+n)}{\Gamma(q+1)}\ ;\ q = \{a,b\}$$

$$= q(q+1)_n \tag{3}$$

For a special case when $a = b$ from (2) we find a known CFE of the $M[1,2|z]$

$${}_2F_2\begin{bmatrix} a+1,1; \\ a+1,2; \end{bmatrix} z = M[1,2|z] = \begin{cases} \frac{e^z-1}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases} \tag{4}$$

If we were to employ *Kummer's First Transformation* (KFT) [19], [20] of (3) we obtain,

$$M[1,2|z] = e^z M[1,2|-z] = \begin{cases} \frac{e^z-1}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases} \tag{5}$$

Equations (4) and (5) are identical, which means that Kummer's first transformation of (5) is an identity.

For a special case when $a = 1$, $b = a + 1$ from (2) we find a new CFE of the $M[1,3|z]$

$${}_2F_2\begin{bmatrix} 2,1; \\ 3,2; \end{bmatrix} z = M[1,3|z]$$

$$= \begin{cases} \frac{2[M(1,2|z)-1]}{z} = 2\left[\frac{e^z-1}{z^2} - \frac{1}{z}\right], & z \neq 0 \\ 1, & z = 0 \end{cases} \tag{6}$$

If we were to employ KFT [19], [20] of (6) we obtain,

$${}_2F_2\begin{bmatrix} 2,1; \\ 3,2; \end{bmatrix} z = M[1,3|z]$$

$$= e^z M[2,3|-z] = \begin{cases} 2\left[\frac{e^z-1}{z^2} - \frac{1}{z}\right], & z \neq 0 \\ 1, & z = 0 \end{cases} \tag{7}$$

We can easily derive the CFE of the $M[2,3|-z]$ as follows,

$$M[2,3|-z] = \begin{cases} 2\left[\frac{1-e^{-z}}{z^2} - \frac{e^{-z}}{z}\right], & z \neq 0 \\ 1, & z = 0 \end{cases} \tag{8}$$

And similarly, we can obtain the $M[2,3|z]$

$$M[2,3|z] = \begin{cases} 2\left[\frac{1-e^z}{z^2} + \frac{e^z}{z}\right], & z \neq 0 \\ 1, & z = 0 \end{cases} \tag{9}$$

Another important relation in the computation of the $M[a,b|z]$ is the *Continuous Relation Formula* (CRF) [19] as given below:

$$aM[a_+,b|z] = (a+z)M[a,b|z] + z(a-b)M[a,b_+|z] \tag{10}$$

Where

$$a_+ = a + 1 \tag{11}$$

$$b_+ = b + 1 \tag{12}$$

For example, if we set $a = b \neq 0$ from (10) we obtain

$$M[a_+,a|z] = \frac{(a+z)}{a}M[a,a|z] = \left(1+\frac{z}{a}\right)e^z \tag{13}$$

We were able to produce several reduction formulae of the CHF which are special cases of a fundamental 2F2 GHGF by means of invoking the GPS, KFT, and CRF. This means that there is hope that we can produce more reduction formulae for a fundamental 2F2 GHGF as discussed next.

3 RRF of the Fundamental ${}_2F_2$ GHGF

In this section we discuss the computation of *RRF* of the fundamental ${}_2F_2[1,1;2,2;z]$ GHGF; by means of the *GPS* that is special case of a fundamental ${}_2F_2[a,1;b,b;z]$.

From the *GPS* definition of a fundamental ${}_2F_2[a,1;b,b;z]$ GHGF, where a, $b=a+1$ are positive integers, we can write,

$$ {}_2F_2\left[\begin{matrix}a,1;\\b,b;\end{matrix}z\right]=\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{(1)_k}{(b)_k}\frac{z^k}{k!} \tag{14}$$

$$=1+\sum_{k=1}^{\infty}\frac{(a)_k}{(b)_k}\frac{(1)_k}{(b)_k}\frac{z^k}{k!}$$

$$=1+z\sum_{n=0}^{\infty}\frac{(a)_{n+1}}{(b)_{n+1}}\frac{(1)_{n+1}}{(b)_{n+1}}\frac{z^n}{(n+1)!}$$

$$=1+\frac{za}{bb}\sum_{n=0}^{\infty}\frac{(a+1)_n}{(b+1)_n}\frac{(2)_n(1)_n}{(b+1)_n}\frac{z^n}{(2)_n n!}\ \text{(see (2))}$$

$$=1+\frac{za}{bb}\sum_{n=0}^{\infty}\frac{(a+1)_n}{(b+1)_n}\frac{(1)_n}{(b+1)_n}\frac{z^n}{n!}$$

$$=1+\frac{za}{bb}\,{}_2F_2\left[\begin{matrix}a_+,1;\\b_+,b_+;\end{matrix}z\right]$$

$$=1+\frac{za}{bb}\left\{1+\frac{za_+}{b_+b_+}\,{}_2F_2\left[\begin{matrix}a_{+2},1;\\b_{+2},b_{+2};\end{matrix}z\right]\right\} \tag{15}$$

Equations (14) and (15) provide the recursive reduction formulae of a fundamental 2F2 GHGF by means of another 2F2 GHGF.

After n iterations the nth terms can be evaluated in limit as n goes to infinity since $b=a+1$ and a,b positive integers as follows,

$$\lim_{n\to\infty}\frac{za_{+n}}{b_{+n}b_{+n}}\,{}_2F_2\left[\begin{matrix}a_{+n+1},1;\\b_{+n+1},b_{+n+1};\end{matrix}z\right]=$$

$$=\lim_{n\to\infty}\frac{z}{b_{+n}}\,{}_1F_1\left[\begin{matrix}1;\\b_{+n+1};\end{matrix}z\right]$$

$$=\lim_{n\to\infty}\frac{z}{b+n}\to 0 \tag{16}$$

Where we made use of the following limit,

$$\lim_{n\to\infty}{}_2F_2\left[\begin{matrix}a_{+n+1},1;\\b_{+n+1},b_{+n+1};\end{matrix}z\right]=$$

$$=\lim_{n\to\infty}{}_1F_1\left[\begin{matrix}1;\\b_{+n+1};\end{matrix}z\right]$$

$$=1 \tag{17}$$

Then the algorithm for computing the fundamental ${}_2F_2[1,1;2,2;z]$ is as follows

```
function h = phypergeom13(a,b,z,n)
% Performs the recursive computation of the 2F2[a,1;b,b;z]
% a and b are integers and b = a + 1 based on (15)
h = 1; % initialize the (n+1)th term 2F2[a,1;b,b;z]
for k=n:-1:0
    h=1+z*(a+k)/(b+k)^2*h; % recursive update
end
end
```

The results of the implementation of this algorithm in MATLAB are given in the next Sect. The other computation of the fundamental ${}_2F_2[1,1;2,2;z]$ is via the Progri phypergeom (see Progri(2019, [13] pg. 19, 31 EN. iv).

Next, the second technique is that of splitting the terms into even and off components as follows,

$${}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right]=\sum_{k=0}^{\infty}\frac{(1)_k}{(2)_k}\frac{(1)_k z^k}{(2)_k k!}$$

$$=\sum_{n=0}^{\infty}\frac{(1)_{2n}}{(2)_{2n}}\frac{(1)_{2n}z^{2n}}{(2)_{2n}(2n)!}+\sum_{n=0}^{\infty}\frac{(1)_{2n+1}}{(2)_{2n+1}}\frac{(1)_{2n+1}z^{2n+1}}{(2)_{2n+1}(2n+1)!}$$

$$=F_e(z)+F_o(z) \tag{18}$$

Well known reduction formulae for the Pochhammer symbol (or rising factorial [23])

$$(1)_{2n}=\frac{\Gamma\left[2\left(\frac{1}{2}+n\right)\right]}{\Gamma(1)}=\frac{2^{2n}\Gamma\left[\frac{1}{2}+n\right]\Gamma[1+n]}{\Gamma\left[\frac{1}{2}\right]\Gamma(1)}=2^{2n}\left(\frac{1}{2}\right)_n(1)_n \tag{19}$$

$$(2)_{2n}=\frac{\Gamma[2(1+n)]}{\Gamma(2)}=\frac{2^{2n}\Gamma[1+n]\Gamma\left[\frac{3}{2}+n\right]}{\Gamma\left[\frac{3}{2}\right]\Gamma(2)}=2^{2n}(1)_n\left(\frac{3}{2}\right)_n \tag{20}$$

$$(2n)!=\Gamma\left[2\left(\frac{1}{2}+n\right)\right]=2^{2n}\left(\frac{1}{2}\right)_n n! \tag{21}$$

Substituting (19)-(21) into the even component $F_e(z)$

$$F_e(z)=\sum_{n=0}^{\infty}\frac{(1)_{2n}}{(2)_{2n}}\frac{(1)_{2n}z^{2n}}{(2)_{2n}(2n)!}=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}\frac{\frac{z^{2n}}{2^{2n}}}{n!}$$

$$={}_1F_2\left[\frac{1}{2};\frac{3}{2},\frac{3}{2};\frac{z^2}{4}\right] \tag{22}$$

Similarly, well known reduction formulae for the Pochhammer symbol (or rising factorial [23])

$$(1)_{2n+1}=\frac{\Gamma[2(1+n)]}{\Gamma(1)}=\frac{2^{2n}\Gamma[1+n]\Gamma\left[\frac{3}{2}+n\right]}{\Gamma\left[\frac{3}{2}\right]\Gamma(2)}=2^{2n}(1)_n\left(\frac{3}{2}\right)_n \tag{23}$$

$$(2)_{2n+1}=\frac{\Gamma\left[2\left(\frac{3}{2}+n\right)\right]}{\Gamma(2)}=\frac{\Gamma\left[\frac{3}{2}+n\right]\Gamma[2+n]}{2^{-2n-1}\Gamma\left[\frac{3}{2}\right]\Gamma(2)}=2^{2n+1}\left(\frac{3}{2}\right)_n(2)_n \tag{24}$$

Substituting (23) and (24) into odd component or $F_o(z)$ produces,

$$F_o(z)=\sum_{n=0}^{\infty}\frac{(1)_{2n+1}}{(2)_{2n+1}}\frac{(1)_{2n+1}z^{2n+1}}{(2)_{2n+1}(2n+1)!}=\frac{z}{4}\sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(2)_n(2)_n\left(\frac{3}{2}\right)_n}\frac{\frac{z^{2n}}{2^{2n}}}{n!}$$

$$=\frac{z}{4}\,{}_2F_3\left[1,1;2,2,\frac{3}{2};\frac{z^2}{4}\right] \tag{25}$$

Next, substituting (22) and (25) into (18) produces a new reduction formula of a fundamental ${}_2F_2[1,1;2,2;z]$ GHGF which I overlooked in Progri(2020, [14]) by means of a 1F2 and a 2F3 GHGF as follows,

$$ {}_2F_2\begin{bmatrix}1,1; \\ 2,2;\end{bmatrix}z\Big] = {}_1F_2\left[\tfrac{1}{2};\tfrac{3}{2},\tfrac{3}{2};\tfrac{z^2}{4}\right] + \tfrac{z}{4}\,{}_2F_3\left[1,1;2,2,\tfrac{3}{2};\tfrac{z^2}{4}\right] \quad (26) $$

What other reduction formulae can be obtained for the fundamental ${}_2F_2[1,1;2,2;z]$ GHGF? One technique was employed in Progri(2020, [14] (49)). However, there is another technique that I did not exploit in Progri(2020, [14]), which is based on Kummer Transformation of Type II, Paris(2005, [5] (3)).

$$ {}_2F_2\left[\begin{matrix}a,d; \\ b,c;\end{matrix} z\right] = e^z \sum_{k=0}^{\infty} \frac{(c-d)_k}{(c)_k}\frac{(-z)^k}{k!}\,{}_2F_2\left[\begin{matrix}b-a,d; \\ b,c+k;\end{matrix} -z\right] \quad (27) $$

If either $b = a$ or $c = d$ then (27) gets reduced to the well-known KFT. If however $b = d$, then we obtain another layer of reduction as follows,

$$ {}_2F_2\left[\begin{matrix}b-a,d; \\ b,c+n;\end{matrix} -z\right] = {}_1F_1\left[\begin{matrix}b-a; \\ c+n;\end{matrix} -z\right] \quad (28) $$

Next, the left-hand side of (26) becomes

$$ {}_2F_2\left[\begin{matrix}a,d; \\ b,c;\end{matrix} z\right] = {}_1F_1\left[\begin{matrix}a; \\ c;\end{matrix} z\right] \quad (29) $$

The righthand side must be equal to a CHF based on KFT as follows,

$$ \sum_{k=0}^{\infty} \frac{(c-b)_k}{(c)_k}\frac{(-z)^k}{k!}\,{}_1F_1\left[\begin{matrix}b-a; \\ c+k;\end{matrix} -z\right] = {}_1F_1\left[\begin{matrix}c-a; \\ c;\end{matrix} -z\right] \quad (30) $$

What happens when $a = d$ and $b = c$,

$$ {}_2F_2\left[\begin{matrix}a,a; \\ b,b;\end{matrix} z\right] = e^z \sum_{k=0}^{\infty} \frac{(b-a)_k}{(b)_k}\frac{(-z)^k}{k!}\,{}_2F_2\left[\begin{matrix}b-a,a; \\ b,b+k;\end{matrix} -z\right] \quad (31) $$

We can easily see that the fundamental ${}_2F_2[1,1;2,2;z]$ can be obtained by the means of the Kummer's Transformation of Type II Paris(2005, [5])

$$ \begin{aligned} {}_2F_2\left[\begin{matrix}1,1; \\ 2,2;\end{matrix} z\right] &= e^z \sum_{k=0}^{\infty} \frac{(1)_k}{(2)_k}\frac{(-z)^k}{k!}\,{}_2F_2\left[\begin{matrix}1,1; \\ 2,2+k;\end{matrix} -z\right] \\ &= e^z \left| \begin{matrix} {}_2F_2\left[\begin{matrix}1,1; \\ 2,2;\end{matrix} -z\right] + \\ \sum_{k=1}^{\infty} \frac{(1)_k}{(2)_k}\frac{(-z)^k}{k!}\,{}_2F_2\left[\begin{matrix}1,1; \\ 2,2+k;\end{matrix} -z\right] \end{matrix} \right. \end{aligned} \quad (32) $$

In Progri(2020, [14]) I made the following bold statement: "*We cannot really compute the ${}_2F_2[1,1;2,2;z]$ any other way than either (31) or (49) or it will lead to singularities*"; hence, this statement is somewhat true because either/both (26) or/and (32) do not provide any real significant gains for the computation of the ${}_2F_2[1,1;2,2;z]$.

In summary, the recursive reduction formulae of the fundamental GHGF, ${}_2F_2[1,1;2,2;z]$, via (14)-(17) implemented in Progri's function *phypergeom13* is one the most efficient computational algorithm and it should be more accurate and much faster than the computation of the ${}_2F_2[1,1;2,2;z]$ via the sum or the difference of two Kampé de Fériet functions.

4 Numerical Examples

Before we conclude this letter, we consider several numerical examples similar to the examples in Progri(2020, [14]).

Example 1: The first numerical example considers the computation of a $M[a;b;z]$ based on (1), (2) and (3) for $a = 1$, $b = a + 1 = 2$ and $z = 0.5$. The results of the direct computation are shown in Tab. I and of the absolute error of the direct computation are shown in Tab. II.

Example 2: The second numerical example considers the computation of a $M[a;c;z]$ based on (1) and (6) for $a = 1$, $c = a + 2 = 3$ and $z = 0.5$. The results of the direct computation are shown in Tab. III and of the absolute error of the direct computation are shown in Tab. IV.

Example 3: The third numerical example considers the computation of a ${}_2F_2$ based on (14) and (15) for $a = 1$, $b = a + 1 = 2$ and $z = 0.5$. The results of the direct computation are shown in Tab. V and of the absolute error of the direct computation are shown in Tab. VI.

As indicated by the numerical results in table VI, the absolute error via (15) – (16) is zero.

Example 4: The fourth numerical example considers the computation of a ${}_2F_2$ based on (14) and (26) for $a = 1$, $b = a + 1 = 2$ and $z = 0.5$. The results of the direct computation are shown in Tab. VII and of the absolute error of the direct computation are shown in Tab. VIII.

As indicated by the numerical results in Tab. VIII, the absolute error due is zero; hence, (14) and (26) are numerically identical. This is not a surprise because splitting the terms into even and odd components should yield an identity both analytically and numerically. However, the computation time for (26) is *21.76 milliseconds*. This approach is too slow.

Example 5: The fifth numerical example considers the computation of a ${}_2F_2$ based on (14) and Progri(2020, [14] (49)) for $a = 1$, $b = 2$, and $z = 0.5$. The results of the direct

TABLE I: DIRECT COMPUTATION

$M\left[{a; \atop a+1;}z\right]$	(1)	(2)
$a=1,\ z=0.5$	1.297442541400256	1.297442541400256
$a=2,\ z=0.5$	1.405114917199487	1.405114917199487

TABLE II: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

$M\left[{a; \atop a+1;}z\right]$	(2)/(3) – (1)	tsim (ms)
$a=1,\ z=0.5$	0	0.181
$a=2,\ z=0.5$	0	16.31

TABLE II: DIRECT COMPUTATION

$M\left[{a; \atop a+2;}z\right]$	(1)	(6)
$a=1,\ z=0.5$	1.189770165601025	1.189770165601026

TABLE IV: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

$M\left[{a; \atop a+2;}z\right]$	(6) – (1)	tsim (ms)
$a=1,\ z=0.5$	4.440892098500626e-16	0.2697

TABLE V: DIRECT COMPUTATION

${}_2F_2\left[{a,1; \atop a+1,a+1;}z\right]$,	(14)	(15)
$a=1,\ z=0.5$	1.140302841043172	1.297442541400256
$a=2,\ z=0.5$	1.122422728345376	1.122422728345376

TABLE VI: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

${}_2F_2\left[{a,1; \atop a+1,a+1;}z\right]$	(15) – (14)	tsim (ms)
$a=1,\ z=0.5$	0	0.2144
$a=2,\ z=0.5$	0	0.2277

TABLE VII: DIRECT COMPUTATION

${}_2F_2\left[{a,1; \atop a+1,a+1;}z\right]$	(14)	(26)
$a=1,\ z=0.5$	1.257117602856674	1.257117602856674

TABLE VIII: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

${}_2F_2\left[{a,1; \atop a+1,a+1;}z\right]$	(26) – (14)	tsim (ms)
$a=1,\ z=0.5$	0	21.76

TABLE IX: DIRECT COMPUTATION

${}_2F_2\left[{a,1; \atop a+1,a+1;}z\right]$	(14)	(49) [14]
$a=1,\ z=0.5$	1.140302841043172	1.140302841043169

TABLE X: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

${}_2F_2\left[{a,1; \atop a+1,a+1;}z\right]$	(49) [14] – (14)	tsim (ms)
$a=1,\ b=4,\ z=0.5$	-3.10e-15	2.17

TABLE XI: DIRECT COMPUTATION

${}_2F_2\left[{a,1; \atop a+1,a+1;}z\right]$	(14)	RRF (15)
$a=1,\ z=0.5,\ n=12$	1.140302841043172	1.140302841043172
$a=1,\ z=0.5,\ n=20$	1.140302841043172	1.140302841043172

TABLE XII: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

${}_2F_2\left[{a,1; \atop a+1,a+1;}z\right]$	RRF (15) – (14)	tsim (ms)
$a=1,\ z=0.5,\ n=12$	0	0.0083
$a=1,\ z=0.5,\ n=20$	0	0.0097

computations are shown in Tab. IX and of the absolute error of the direct computation are shown in Tab. X.

As indicated by the numerical results in table X, the absolute error due to the difference of two Kampé de Fériet functions is within numerical computational error 10^{-16}; hence, (14) and Progri(2020, [14] (49)) are numerically equivalent; however, the simulation time is equal to 2.17 milliseconds.

Example 6: The sixth numerical example considers the computation of a ${}_2F_2$ based on (14) and RRF (15) for $a=1$, $b=2$, and $z=0.5$. The numerical results of the direct computations are shown in Tab. XI and of the absolute error of the direct computation are shown in Tab. XII.

As indicated by the numerical results in table XII, the absolute error due to the RRF (15) is zero; hence, (14) and (15) are numerically identical but the simulation time is almost *ten microseconds*. The RRF (15) is two orders of magnitude faster than the Progri(2020, [14] (49)).

5 Conclusions

In conclusion, I have produced several reduction formulae for the computation of a fundamental ${}_2F_2$ GHGF either my means of direct computation or via the RRF (15) algorithm *phypergeom13*.

I have shown the connection of several special cases of a fundamental ${}_2F_2$ which leads to partial fraction expansion via ${}_1F_1$ in (2). This is an original result never published before.

The fundamental ${}_2F_2[1,1;2,2;z]$ hypergeometric function is computed by the means of RRF (15) algorithm *phypergeom13* and split of even and odd components (26). However, RRF (15) algorithm *phypergeom13* is three to four orders of magnitude faster than (26) and two to three orders of magnitude faster than the computation via Progri(2020, [14] (49)). This is an original result never published before.

This paper is based almost entirely on the creation of original analytical derivations that are supported by numerical results special cases that completely validate the accuracy and computational speed of the algorithms that are proposed.

In summary the RRF (15) algorithm *phypergeom13 is the fastest and the most accurate means by which the fundamental 2F2 can be computed*.

I was wrong when I said In Progri(2020, [14]): "*We cannot really compute the* ${}_2F_2[1,1;2,2;z]$ *any other way than either (31) or (49) or it will lead to singularities*"; certainly, we can and I have learned not to take 'NO' for an answer; RRF (15) algorithm *phypergeom13 is the right answer so far*.

6 Acknowledgement

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I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [27] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications. Vol. I* (Not yet available in print) [1].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [25] reminds me of my grandfather who I loved very much.

Finally, I am also dedicating this journal paper to our most beloved President Ronald Reagan, who reminds me of my grandfather, [26] who spoke from the heart, who spoke the truth, who was a staunch supporter of personal freedom, the greatest leader of the free world, and perhaps the greatest critic of the wasteful government control and spending[ii].

"*Government is not the answer, government is the problem*."—Ronald Reagan

I would like to express my deepest gratitude to my mother Lumturi Progri, my sister Adriana Dine, and Ms. Elizabeth Demir for their support, loyally, and dedication to me during some of the most difficult times in my life as a boy, man, and scholar.

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[i] I am dedicating this letter to my Master Thesis Advisor Prof. Alex Emanuel. I wrote on my 1997 Master Thesis: "I am indebted to Prof. Emanuel for frequently revising my drafts, clarifying my thoughts, and improving my technical writing skills without any complaints. Moreover, he has been very inspiring and objective with his endless and fruitful advice. The success of this project to date is yet another sign of his tremendous knowledge and intuition." Prof. Emanuel was a phenomenal teacher, scholar, researcher, and friend. He was so funny. I immensely admired his Jewish accent. He was always joking with me about my accent. He was saying to me "You and I will never have the sense humor and speak English as good as some of the American-born students and Professors do." Alex wanted me to me like him. He was without a doubt a bright light in my life and I owe him a lot for teaching me so many things that I took them for granted when I was with him.

[ii] Ditto Progri(2020, [14]) EN vii.