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Research Article

Computation of the Complementary Landmark Generalized Parabolic Cylinder Function Distribution

To my mother, Lumturi Progri, and my sister, Adriana Progri, the two finest women who I have ever known in my life.[i]

—Ilir F. Progri

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The computation of the parabolic cylinder function (PCF) via the series of Hermite polynomials as pointed out by Progri in 2023 diverges for large values of the main variable. The expansion of the PCF for large values of the main variable is defined as the complementary PCF (CPCF). Then the CPCF for large values of the main variable is employed to compute the complementary landmark generalized PCF distribution (CLGPCFD) probability density function (pdf) and commutative probability distribution function (cdf). The CLGPCFD pdf is seen as a product of a normally distributed with a form of a generalized polynomial. The CLGPCFD cdf is on the other hand seen as a generalized function of the complementary error function (GCEF). Both the CLGPCFD pdf and cdf are unique expressions not produced anywhere else. Two special cases are provided for both the CLGPCFD pdf and cdf. The integration of the CPCF leads to the computation of a series of Kampé de Fériet functions.

Index Terms—Parabolic cylinder function, cumulative distribution function, complementary cumulative distribution function, complementary error function, Kampé de Fériet function, complementary pdf cdf analysis, hypergeometric series.

1 Introduction

The computation of the complementary generalized parabolic cylinder function distribution (CGPCFD[ii]) probability density function (pdf) and commutative probability distribution function (cdf) is not new. There is a lot of important material from Gradshteyn, Ryzhik (2007, [1] 9.24-9.25 pp 1028-1031) and ironically enough from Wikipedia [3]-[11] that took me a long time to figure out how to properly employ it and compute it via MATLAB.

In 2016, when Dr. Progri, when investigating of the PCFDs in indoor geolocation applications [12], recognized that the efficient computation of the GPCFD pdf and cdf (Progri (2016, [13])) can only be accomplished by means of the computation of the complementary pdf and cdf via an expansion by means of the Kampé de Fériet function Progri, (2018, [14]), Progri (2019, [15]).

The idea of computing the complementary pdf and cdf is not new since the first publication of (Progri (2016, [13])); however, it was not until Progri (2021, [16] (63) and (64)) that the only way to compute the efficient GPCDFD (or EGPCDFD) pdf and cdf was by means of the complementary GPCDFD (CGPCDFD) pdf and cdf even for small values of the main variable. The EGPCDF may not be as fast as the landmark computation of the GPCDFD (or LGPCDF) pdf and cdf; however, as shown in this journal paper is very robust because it was formulated on the bases of computing the CGPCDFD pdf and cdf.

The computation of the CGPCFD pdf and cdf was also employed in Progri (2021, [18]), however, ironically enough the key word complementary was never used. The complementary error function (CEF), $\Phi_c(x)$, should have been used just to name one example because it was implicitly invoked as one minus the error function, $\Phi(x)$, (see Gradshteyn, Ryzhik (2007, [1] 8.25 pp 887-888), and [4]). It is an illustration that even when the computation of the special cases of the GPCFD pdf and cdf, the computation of the CEF was employed but it was NOT explicitly stated.

However, even in the computational body of work in 2022, in Progri (2022, [19]-[22]), in which in Progri (2022, [21] (36), (39)) is via the computation of the complementary cdf ironically enough the key word complementary was never used again.

Even more surprising is that even in Progri (2023, [23]), which involved the expansion of the Hermite Polynomials which is suitable for the computation of the complementary pdf and cdf, ironically enough again the key word complementary was never mentioned.

Recognizing the critical importance of the fundamental computation of the complementary pdf and cdf in the computation of the GPCFD pdf and cdf since it is the most important concept in the body of work since Progri (2016, [13]), Progri (2021, [16], and [18]), Dr. Progri, decided to bring the keyword complementary to the title page and add the other keyword landmark.

Moreover, this journal paper provides some unique contributions such as the direct relations between:

1. The CEF and CPCF in a way that was never presented before.
2. The fundamental understanding of the CLGPCFD cdf as a more generalized form of the CEF or GCEF; i.e., it provides a significant understanding into generalized functions that contain the CEF as a special case.
3. More insights into the splitting of the CLGPCFD cdf into the even and odd components (or terms or pairs of the Kampé de Fériet functions).

This paper is organized as follows: in Sect. 2 the computation of the CLGPCFD pdf is derived. The efficient computation of the CLGPCFD cdf is discussed in Sect. 3. Section IV contains numerical, theoretical results; Conclusion is provided in Sect. V, Acknowledgment in Sect. VI, along with a list of references in Sect. VII. In Appen. A, the derivation of the Rising Factorial in (4) is presented in Sect. VIII. In Appen. B, the derivation of the CLGPCFD PDF of Progri (2021, [15] (8)) is derived in Sect. IX. In Appen. C, the derivation of the CLGPCFD PDF of Progri (2021, [15] (11)) is depicted. In Appen. D, the derivation of the integral (36) is presented in Sect. X. In Sect. XI the derivation of the integral (86) is derived in Sect. XII. Finally, in Sect. XIII Appen F: the derivation of the integral (37) is given.

2 The CLGPCFD PDF

The CLGPCFD pdf is presented in this Sect. First, we discuss the complementary[iii] PCF (or CPCF) in SubSect. 2.1. Second, we derive the efficient commutation of the CLGPCFD pdf in SubSect. 2.2. Third, a few special cases of the CLGPCFD PDF are presented in SubSect. 2.3.

2.1 The CPCF

The CPCF is the foundation (or the building block) of the computation of the closed form expression (CFE) of the CGPCFD. We will answer two main questions:

1. What is the definition of the CPCF?
2. How is the CPCF computed?

Staring with answering the first question. The CPCF is a sub-section or sub-segment of the PCF, $D_p(z)$, for $|z| \gg (|p| \cap 1) \equiv (|z| \gg |p|) \cap (|z| \gg 1)$; i.e., it is the tail of the PCF for the values of the main variable much greater than both the absolute value of the parameter p and one; i.e.. this is a complementary expansion of the $D_p(z)$; i.e., this expansion and the previous expansion of the PCF complete the definition and the computation of the PCF.

From (Gradshteyn, Ryzhik 2007, [1] 9.246 1. pg. 1029) the CPCF is given by,

$$D_p(z) \cong \frac{e^{-\frac{z^2}{4}} \sum_{k=0}^{\infty} \frac{(-1)^k (p-2k+1)_{2k}}{2^k z^{2k} k!}}{z^{-p}}, \ |z| \gg (|p| \cap 1) \tag{1}$$

where the notation on the righthand side $|z| \gg (|p| \cap 1)$ means that $|z| \gg |p|$ and $|z| \gg 1$ and where,

$$(p-2k+1)_{2k} = (-1)^{2k}(-p)_{2k} = (-p)_{2k} \tag{2}$$

denotes a Pochhammer symbol of a falling factorial [7].

Substituting (2) into (1) yields,

$$D_p(z) \cong \frac{e^{-\frac{z^2}{4}} \sum_{k=0}^{\infty} \frac{(-1)^k (-p)_{2k}}{2^k z^{2k} k!}}{z^{-p}}, \ |z| \gg (|p| \cap 1) \tag{3}$$

Since the requirement for the parameter p is a negative integer, then it is safe to assume that (3) is written for either $p > 0$ or for $p < 0$. However, for $p > 0$ the series terminates for $p = 2k$; as for $p < 0$ then the series does not terminate. With the above discussion in mind we can rewrite (3) to show that

$$D_{-p}(z) \cong \frac{e^{-\frac{z^2}{4}} \sum_{k=0}^{\infty} \frac{(-1)^k (p)_{2k}}{2^k z^{2k} k!}}{z^{p}}, \ |z| \gg (p > 0 \cap 1) \tag{4}$$

For very large values of $|z|$, the series converges vert fast; hence, only a few terms are needed; therefore, we can further rewrite (4) as follows (invoking the results of Appen. A):

$$D_{-p}(z) \cong \frac{e^{-\frac{z^2}{4}} \sum_{k=0}^{K} \frac{(-1)^k 2^{2k} \left(\frac{p}{2}\right)_k \left(\frac{p+1}{2}\right)_k}{2^k z^{2k} k!}}{z^{p}}, \ |z| \gg (p > 0 \cap 1) \tag{5}$$

where K is the total number of terms that we need for fast convergence to achieve the desired accuracy.

For ease of notation let us further define a new function, $U_{-p}(z)$, which is in the form of a generalized hypergeometric function [1]. [11] that is computed as a polynomial, as follows:

$$U_{-p}(z) = z^{-p} \sum_{k=0}^{K} \frac{\left(\frac{p}{2}\right)_k \left(\frac{p+1}{2}\right)_k (-1)^k 2^k}{z^{2k} k!}, \ |z| \gg (p > 0 \cap 1)$$

$$= z^{-p} F\left(\begin{matrix} K \\ 0 \end{matrix} \middle| \left[\frac{p}{2}, \frac{p+1}{2}\right], -; -\frac{2}{z^2}\right) \tag{6}$$

Hence, substituting (6) into (5) yields a very nice, closed form expression of the CPCF:

$$D_{-p}(z) \cong e^{-\frac{z^2}{4}} U_{-p}(z)^{\text{iv}}, \ |z| \gg (p > 0 \cap 1) \tag{7}$$

Equation (7) can now be employed to perform the desired computation of CLGPCFD pdf, which is accomplished in the following SubSect.

This concludes the derivations of the CPCF for large values of the main variable $|z| \gg (p > 0 \cap 1)$.

2.2 *The Computation of the CLGPCFD PDF*

Having discussed the CPCF for large values of the main variable we can consider the computation of the CLGPCFD pdf[v] (see Progri (2016, [13] (1)), Progri (2021, [15] (28)))

$$f_c(x) = \frac{d_0(x) \sum_{r=0}^{N-1} \frac{(p-1)!\left[b_0(x) D_{-p}[c_0(x)] + b_1(x) D_{-p}[c_1(x)]\right]}{H_r(N)^{-1} a^{-p}}}{C} \tag{8}$$

Where the various parameters are given as follows,

$$p = N - r; \ p > 0, \text{ positive integer} \tag{9}$$

$$c_0(x) = a a_{0^-}(x) \tag{10}$$

$$c_1(x) = a a_{0^+}(x) \tag{11}$$

$$b_0(x) = e^{\frac{a^2 a_{0^-}^2(x)}{4}} \tag{12}$$

$$b_1(x) = e^{\frac{a^2 a_{0^+}^2(x)}{4}} \tag{13}$$

$$d_0(x) = e^{-a_0(x)(x-b)} = e^{-\frac{(x-b)^2}{2a^2}} = e^{-2a^2[a_0(x)]^2} \tag{14}$$

For small values of $|x| < p$ the computation of Progri (2021, [15] (28)-(53)) are correct. We have to rewrite Progri (2021, [15] (28)-(53)) for large values of $|x| \gg p$.

First, we simplify the products by substituting (5) and (10)-(13) we obtain:

$$b_0(x) D_{-p}[c_0(x)] \cong e^{\frac{a^2 a_{0^-}^2(x)}{4}} e^{-\frac{[a a_{0^-}(x)]^2}{4}} U_{-p}[c_0(x)]$$

$$= U_{-p}[c_0(x)] \tag{15}$$

$$b_1(x) D_{-p}[c_1(x)] \cong e^{\frac{a^2 a_{0^+}^2(x)}{4}} e^{-\frac{[a a_{0^+}(x)]^2}{4}} U_{-p}[c_1(x)]$$

$$= U_{-p}[c_1(x)] \tag{16}$$

Hence, substituting (14), (15) and (16) into (8) yields,

$$f_c(x) \cong \frac{e^{-2a^2[a_0(x)]^2} \sum_{r=0}^{N-1} \frac{(p-1)!\{U_{-p}[c_0(x)] + U_{-p}[c_1(x)]\}}{H_r(N)^{-1} a^{-p}}}{C}$$

$$\cong \frac{e^{-2a^2[a_0(x)]^2} \sum_{r=0}^{N-1} \frac{(p-1)! V_{-p}(x)}{H_r(N)^{-1} a^{-p}}}{C}$$

$$= \frac{e^{-2a^2[a_0(x)]^2} W_{-p}(x)}{C} \tag{17}$$

where

$$a^2 = N b c^2 \ \text{(Progri 2016, [13] (6), (7))} \tag{18}$$

$$W_{-p}(x) = \sum_{r=0}^{N-1} \frac{(p-1)!V_{-p}(x)}{H_r(N)^{-1}a^{-p}};\ p = \{1,2,\cdots\} \tag{19}$$

$$V_{-p}(x) = U_{-p}[c_0(x)] + U_{-p}[c_1(x)];\ p = \{1,2,\cdots\} \tag{20}$$

The interpretation of the GPCFD or the CLGPCFD pdf is that it is a multiplication of a normally distributed pdf, $e^{-2a^2[a_0(x)]^2}$, with a weighted polynomial function, $W_{-p}(x)$, divided by the GPCFD normalization coefficient, C, (see Progri (2016, [13], (56) pg. 77)).

The GPCFD or the CLGPCFD pdf (17) does not contain any singularities as $x \to \pm\infty$ that needs to be eliminated in contrast to (31) in Progri (2021, [15]); indeed if we take the limit as

$$\lim_{x\to\pm\infty} e^{-2a^2[a_0(x)]^2} = \lim_{x\to\pm\infty} e^{-\frac{(x-b)^2}{2a^2}} = 0 \tag{21}$$

$$\lim_{x\to\pm\infty} W_{-p}(x) = \lim_{x\to\pm\infty} V_{-p}(x)$$

$$= \lim_{x\to\pm\infty} U_{-p}[c_i(x)]$$

$$= \lim_{x\to\pm\infty} \frac{\sum_{k=0}^{K} \frac{(p)_{2k}(-1)^k}{2^k[c_i(x)]^{2k}k!}}{[c_i(x)]^p} = 0 \tag{22}$$

Taking into consideration the results of (21) and (22) produces the following:

$$\lim_{x\to\pm\infty} f_c(x) = 0 \tag{23}$$

This concludes the discussion on the computation of the CLGPCFD pdf. Next, we consider a few special cases.

2.3 *Special Cases of the CLGPCFD PDF*

In Progri (2021, [16]) two special cases were proposed for the computation of the CLGPCFD pdf.

The first special case occurs when:

$$N = 1 \Rightarrow p = 1;\ n = 0;\ r = 0 \tag{24}$$

Substituting (24) into (1) and from (Gradshteyn, Ryzhik 2007, [1] pg. 1027 9.236 1.) we obtain the solution for this special case as

$$f_c(x, |x| \gg 1) \cong \frac{e^{-2a^2[a_0(x)]^2} W_{-1}(x)}{\frac{4\sqrt{\pi}e^{\frac{a^2}{3c^4}}\frac{2a}{\sqrt{3}}aH_0(1)\left(\frac{\sqrt{3}}{2a}\right)^0 D_{-1}\left(\frac{2a}{\sqrt{3}c^2}\right)}{\sqrt{2}}}$$

$$= \frac{e^{-2a^2[a_0(x)]^2} \frac{V_{-1}(x)}{H_0(1)^{-1}a^{-1}}}{\frac{4\sqrt{\pi}e^{\frac{a^2}{3c^4}}\frac{2a}{\sqrt{3}}aH_0(1)\left(\frac{\sqrt{3}}{2a}\right)^0 D_{-1}\left(\frac{2a}{\sqrt{3}c^2}\right)}{\sqrt{2}}}$$

$$= \frac{e^{-2a^2[a_0(x)]^2} V_{-1}(x)}{\frac{4\sqrt{2\pi}}{\sqrt{3}}e^{\frac{a^2}{3c^4}} a D_{-1}\left(\frac{2a}{\sqrt{3}c^2}\right)}$$

$$= \frac{e^{-2a^2[a_0(x)]^2} V_{-1}(x)}{\frac{4\pi}{\sqrt{3}}e^{\frac{2a^2}{3c^4}} a \Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)} \tag{25}$$

Equation (25) is an original CFE of the CLGPCFD pdf not published anywhere else for the special case (24).

In Appen. B, it is shown that indeed the CLGPCFD pdf (25) is identical to the CGPCF pdf derived from Progri (2021, [16] (8)).

Equation (25) simplifies even more when $a = 1$,

$$f_c(x, |x| \gg 1) \cong \frac{e^{-2[a_0(x)]^2} V_{-1}(x)}{\frac{4\pi}{\sqrt{3}}e^{\frac{2}{3c^4}} \Phi_c\left(\frac{\sqrt{2}}{\sqrt{3}c^2}\right)} \tag{26}$$

From (25) and (26) we obtain all the insight into the nature of the special case of the CLGPCFD pdf. For this special case the CLGPCFD pdf is reduced to the computation of a normally distributed pdf, $e^{-2[a_0(x)]^2}$, multiplied by the polynomial, $V_{-1}(x)$, and then weighted by the CEF, $\Phi_c(\sqrt{2}/\sqrt{3}c^2)$.

Therefore, not only is (26) very significant as an original expression not published anywhere else, but also because it provides great insight into the nature and the efficient computation of the CLGPCFD cdf.

Significant savings in the computation of the integral function are obtained from (25) because the CLGPCFD pdf is reduced to the computation of MATLAB built in functions (BIF) such as exp, erf, etc.

Next, let us consider another special case when:

$$N = 2 \Rightarrow r = \{0,1\};\ p = \{2,1\};\ n = \{1,0\} \tag{27}$$

then (1) becomes:

$$f_c(x, |x| \gg 1) \cong \frac{e^{-2a^2[a_0(x)]^2} W_{-2}(x)}{C}$$

$$= \frac{e^{-2a^2[a_0(x)]^2} \sum_{k=0}^{1} \frac{(p-1)!V_{-p}(x)}{H_k(2)^{-1}a^{-p}}}{C}$$

$$= \frac{e^{-2a^2[a_0(x)]^2}[H_0(2)aV_{-2}(x) + H_1(2)V_{-1}(x)]}{\frac{8\sqrt{2\pi}}{3}e^{\frac{a^2}{3c^4}}a^2\left[H_0(2)D_{-2}\left(\frac{2a}{\sqrt{3}c^2}\right) + H_1(2)\frac{\sqrt{3}}{2a}D_{-1}\left(\frac{2a}{\sqrt{3}c^2}\right)\right]}$$

$$= \frac{e^{-2a^2[a_0(x)]^2}[aV_{-2}(x) + 4V_{-1}(x)]}{\frac{8\sqrt{2\pi}}{3}e^{\frac{a^2}{3c^4}}a^2\left[D_{-2}\left(\frac{2a}{\sqrt{3}c^2}\right) + \frac{2\sqrt{3}}{a}D_{-1}\left(\frac{2a}{\sqrt{3}c^2}\right)\right]}$$

$$= \frac{e^{-2a^2[a_0(x)]^2}[aV_{-2}(x) + 4V_{-1}(x)]}{\frac{8\pi}{3}e^{\frac{2a^2}{3c^4}}a^2\left[\frac{\sqrt{2}}{\sqrt{\pi}}e^{-\frac{2a^2}{3c^4}} - \frac{2a}{\sqrt{3}c^2}\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right) + \frac{2\sqrt{3}}{a}\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)\right]}$$

$$= \frac{e^{-2a^2a_0^2(x)}[aV_{-2}(x)+4V_{-1}(x)\equiv W_{-2}(x)/a]}{\frac{8\pi}{3}a^2e^{\frac{2a^2}{3c^4}}\left[\sqrt{\frac{2}{\pi}}e^{-\frac{2a^2}{3c^4}}+2\left(\frac{\sqrt{3}}{a}-\frac{a}{\sqrt{3}c^2}\right)\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)\right]} \tag{28}$$

Equation (28) is an original CFE of the CLGPCFD pdf not published anywhere else for the special case (27). The CLGPCFD pdf corresponding to this special case (27) is a product of a normally distributed pdf, $e^{-2a^2a_0^2(x)}$, multiplied by the function, $W_{-2}(x)$, and then weighted by the sum of the CEF and an exponential function.

As a special case when $a = 1$, then (28) simplifies even further as

$$f_c(x) \cong \frac{e^{-2a_0^2(x)}[V_{-2}(x)+4V_{-1}(x)\equiv W_{-2}(x)]}{\frac{8\pi}{3}\left[\sqrt{\frac{2}{\pi}}+2e^{\frac{2}{3c^4}}\left(\sqrt{3}-\frac{1}{\sqrt{3}c^2}\right)\Phi_c\left(\frac{\sqrt{2}}{\sqrt{3}c^2}\right)\right]} \tag{29}$$

Even for this special case we have reduced the computation of the CLGPCFD pdf to a computation of twice as many elementary functions if we compare and contrast (28) or (29) with (25) or (26). Presumably, the same process can be followed for other values of $N = 3, 4, \cdots$ if there is a need to compute the CLGPCFD pdf and there is a tighter requirement for the integration performance than the one that is obtained from employing only the reduction from the savings that we discussed in the computation of the CLGPCFD pdf.

This concludes the discussion of the computation of the CLGPCFD pdf. Next, we discuss the CLGPCFD cdf.

3 The CLGPCFD CDF

In this section the optimized efficient computation of the CLGPCFD cdf is performed. Even though in 2021 I was successful in developing for the first time the efficient computational algorithm of the GPCFD cdf the numerical results indicated that this algorithm was far from being optimized for large values of the main variable. This section contains two subsections: the derivation of the landmark GPCFD cdf and special cases of the landmark cdf.

3.1 Derivation of the CLGPCFD CDF

The derivation of the CLGPCFD cdf is performed by invoking the CPCF in SubSect. 2.1.

The derivation of the CLGPCFD cdf is based on the direct integration method which was discussed in great detail in Appen. B of Progri (2021, [15]). However, for the CLGPCFD cdf to work a slight modification of the direct integration method is required.

First, the ccdf or simply the tail distribution or exceedance, is defined as

$$\begin{aligned} F_c(x) &= 1 - \int_{-\infty}^{x}[f_c(t) \cong f(t)]dt \\ &= \int_{x}^{\infty}[f_c(t) \cong f(t)]dt,\ x \gg (b,0) \\ &= 1 - \int_{-\infty}^{x} f_c(t)dt,\ x \ll (b,0)^{\mathrm{vi}} \end{aligned} \tag{30}$$

where $f_c(x)$ is given by (17), taking advantage of the symmetry around $-\infty < b < \infty$.

Since, we are interested in the computation of the LGPCFD cdf $F(x)$ then we can write it via the computation of the CLGPCFD as follows:

$$F(x) = 1 - F_c(x) \equiv \left| \begin{array}{ll} \int_{-\infty}^{x} f_c(t)dt & x \ll (b,0) \\ 1 - \int_{x}^{\infty} f_c(t)dt & x \gg (b,0) \end{array} \right. \tag{31}$$

Hence, it only makes sense to compute the following integral for reasons explained later in the derivations:

$$F_c(x) = \int_{x}^{\infty} f_c(t)dt;\ x \gg (b,0) \tag{32}$$

Substituting (17) into (32) and after some elementary algebra like changing the order of summation and integration we have

$$\begin{aligned} F_c(x) &= \int_{x}^{\infty} \frac{e^{-2a^2[a_0(t)]^2}W_{-p}(t)}{C}dt \\ &= \int_{x}^{\infty} \frac{e^{-2a^2[a_0(t)]^2}\sum_{r=0}^{N-1}\frac{(p-1)!V_{-p}(t)}{H_r(N)^{-1}a^{-p}}}{C}dt \\ &= \frac{\sum_{r=0}^{N-1}\frac{(p-1)![\mathcal{F}_{c0}(p,x)+\mathcal{F}_{c1}(p,x)]}{H_r(N)^{-1}a^{-p}}}{C} \end{aligned} \tag{33}$$

It remains to solve two integrals,

$$\begin{aligned} \mathcal{F}_{cq}(p,x) &= \int_{x}^{\infty} e^{-2a^2[a_0(t)]^2}U_{-p}\left[c_q(t)\right]dt;\ q = \{0,1\} \\ &= \int_{x}^{\infty} e^{-2a^2[a_0(t)]^2}\frac{\sum_{k=0}^{K}\frac{(-1)^k(p)_{2k}}{2^k[c_q(t)]^{2k}k!}}{[c_q(t)]^p}dt;\ q = \{0,1\} \\ &= \sum_{k=0}^{K}\frac{(-1)^k(p)_{2k}\int_{x}^{\infty}e^{-2a^2[a_0(t)]^2}[c_q(t)]^{-m}dt}{2^k k!} \end{aligned} \tag{34}$$

where

$$m = p + 2k \tag{35}$$

which then produces two integrals,

$$\begin{aligned} \mathcal{G}_{c0}(m,x) &= \int_{x}^{\infty} e^{-2a^2[a_0(t)]^2}[c_0(t)]^{-m}dt \\ &= \int_{x}^{\infty} e^{-2a^2[a_0(t)]^2}[aa_{0^-}(t)]^{-m}dt \\ &= \frac{\int_{x}^{\infty} e^{-2a^2[a_0(t)]^2}\left[\frac{1}{c^2}-a_0(t)\right]^{-m}dt}{a^m} \end{aligned} \tag{36}$$

$$\begin{aligned} \mathcal{G}_{c1}(m,x) &= \int_{x}^{\infty} e^{-2a^2[a_0(t)]^2}[c_1(t)]^{-m}dt \\ &= \int_{x}^{\infty} e^{-2a^2[a_0(t)]^2}[aa_{0^+}(t)]^{-m}dt \end{aligned}$$

$$= \frac{\int_x^{\infty} e^{-2a^2[a_0(t)]^2}\left[\frac{1}{c^2}+a_0(t)\right]^{-m} dt}{a^m} \tag{37}$$

The derivations of (36) and (37) are given in Appen. D and F respectively.

Substituting the results from Appen. D (95) and (96) we have

$$\mathcal{G}_{c0}(m,x) = ae^{-u_0^2}\frac{K_{c0e}(m,x)+m2^{0.5m-0.5}x_0K_{c0o}(m,x)}{u_0\upsilon^m}$$

$$= \frac{ae^{-u_0^2}}{u_0}\left|\begin{array}{l}\frac{K_{c0e}(p+2k,x)}{\upsilon^{p+2k}}+\\ \frac{(p+2k)2^{0.5(p+2k)-0.5}x_0K_{c0o}(p+2k,x)}{\upsilon^{p+2k}}\end{array}\right.$$

$$= \frac{ae^{-u_0^2}}{u_0\upsilon^p}\left|\begin{array}{l}\frac{K_{c0e}(p+2k,x)}{\upsilon^{2k}}+\\ \frac{p2^{0.5p-0.5}\left(\frac{p}{2}+1\right)_k 2^k x_0K_{c0o}(p+2k,x)}{\left(\frac{p}{2}\right)_k\upsilon^{2k}}\end{array}\right. \tag{38}$$

where the even and odd Kampé de Fériet functions are given by

$$K_{c0e}(p+2k,x) = F_{0:3;0}^{2:2;0}\left[\begin{array}{c}\frac{1}{2},1:\frac{p}{2}+k,\frac{p+1}{2}+k;-;\\ -:\frac{1}{2},\frac{1}{2},1;-;\end{array} x_0,x_2\right] \tag{39}$$

$$K_{c0o}(p+2k,x) = F_{0:3;0}^{2:2;0}\left[\begin{array}{c}1,\frac{3}{2}:\frac{p+1}{2}+k,\frac{p+2}{2}+k;-;\\ -:1,\frac{3}{2},\frac{3}{2};\end{array} x_0,x_2\right] \tag{40}$$

Similarly, substituting the results from Appen. F (110) and (111) yields,

$$\mathcal{G}_{c1}(m,x) = ae^{-u_0^2}\frac{K_{c1e}(m,x)+2^{0.5m-0.5}mx_1K_{c1o}(m,x)}{u_0\upsilon^m}$$

$$= \frac{ae^{-u_0^2}}{u_0}\left|\begin{array}{l}\frac{K_{c1e}(p+2k,x)}{\upsilon^{p+2k}}+\\ \frac{(p+2k)2^{0.5(p+2k)-0.5}x_1K_{c1o}(p+2k,x)}{\upsilon^{p+2k}}\end{array}\right.$$

$$= \frac{ae^{-u_0^2}}{u_0\upsilon^p}\left|\begin{array}{l}\frac{K_{c1e}(p+2k,x)}{\upsilon^{2k}}+\\ \frac{p2^{0.5p-0.5}\left(\frac{p}{2}+1\right)_k 2^k x_1K_{c1o}(p+2k,x)}{\left(\frac{p}{2}\right)_k\upsilon^{2k}}\end{array}\right. \tag{41}$$

Where the even and odd Kampé de Fériet functions are given by

$$K_{c1e}(p+2k,x) = F_{0:3;0}^{2:2;0}\left[\begin{array}{c}\frac{1}{2},1:\frac{p}{2}+k,\frac{p+1}{2}+k;-;\\ -:\frac{1}{2},\frac{1}{2},1;-;\end{array} x_1,x_2\right] \tag{42}$$

$$K_{c1o}(p+2k,x) = F_{0:3;0}^{2:2;0}\left[\begin{array}{c}1,\frac{3}{2}:\frac{p+1}{2}+k,\frac{p+2}{2}+k;-;\\ -:1,\frac{3}{2},\frac{3}{2};\end{array} x_1,x_2\right] \tag{43}$$

Substituting (38) into (34) produces,

$$\mathcal{F}_{c0}(p,x) = \sum_{k=0}^{K}\frac{(-1)^k(p)_{2k}ae^{-u_0^2}\frac{K_{c0e}(m,x)+m2^{0.5m-0.5}x_0K_{c0o}(m,x)}{u_0\upsilon^m}}{2^k k!}$$

$$= \frac{ae^{-u_0^2}}{u_0}\sum_{k=0}^{K}\frac{(-1)^k(p)_{2k}\left[K_{c0e}(m,x)+m\frac{2^{0.5m}}{2^{0.5}}x_0K_{c0o}(m,x)\right]}{2^k\upsilon^m k!}$$

$$= \frac{ae^{-u_0^2}}{u_0\upsilon^p}\left|\begin{array}{l}\sum_{k=0}^{K}\frac{\left(\frac{p}{2}\right)_k\left(\frac{p+1}{2}\right)_k K_{c0e}(p+2k,x)\frac{(-1)^k2^k}{\upsilon^{2k}}}{k!}+\\ \frac{p2^{0.5p}x_0\sum_{k=0}^{K}\left(\frac{p+1}{2}\right)_k\left(\frac{p}{2}+1\right)_k K_{c0o}(p+2k,x)\frac{\frac{(-1)^k2^k}{\upsilon^{2k}}}{k!}}{\sqrt{2}}\end{array}\right.$$

$$= \frac{ae^{-u_0^2}}{u_0\upsilon^p}\left[\mathcal{K}_{c0e}(p,x)+\frac{p2^{0.5p}x_0}{\sqrt{2}}\mathcal{K}_{c0o}(p,x)\right] \tag{44}$$

Similarly, substituting (41) into (34) produces,

$$\mathcal{F}_{c1}(p,x) = \sum_{k=0}^{K}\frac{(-1)^k(p)_{2k}ae^{-u_0^2}\frac{K_{c1e}(m,x)+2^{0.5m-0.5}x_1mK_{c1o}(m,x)}{u_0\upsilon^m}}{2^k k!}$$

$$= \frac{ae^{-u_0^2}}{u_0\upsilon^p}\left|\begin{array}{l}\sum_{k=0}^{K}\frac{\left(\frac{p}{2}\right)_k\left(\frac{p+1}{2}\right)_k K_{c1e}(p+2k,x)\frac{(-1)^k2^k}{\upsilon^{2k}}}{k!}+\\ \frac{p2^{0.5p}x_1\sum_{k=0}^{K}\left(\frac{p+1}{2}\right)_k\left(\frac{p}{2}+1\right)_k K_{c1o}(p+2k,x)\frac{\frac{(-1)^k2^k}{\upsilon^{2k}}}{k!}}{\sqrt{2}}\end{array}\right.$$

$$= \frac{ae^{-u_0^2}}{u_0\upsilon^p}\left[\mathcal{K}_{c1e}(p,x)+\frac{p2^{0.5p}x_1}{\sqrt{2}}\mathcal{K}_{c1o}(p,x)\right] \tag{45}$$

where u_0, υ, x_0, and x_1 are given in Appen. D (80), (82), (88) and Appen. F (108) respectively.

Both (44) and (45) are substituted into (33) to produce the results of the CLGPCFD cdf which is then utilized to produce the LGPCFD cdf via (31).

From (44)-(45) the computation of LGPCFD $F(x)$ requires the computation of four pairs of generalized Kampé de Fériet functions (or double hypergeometric series) (see Progri (2018, [14], Appen. A). Now, since these functions are simultaneously computed in pairs, (32) requires the optimized computation of two pairs of the Kampé de Fériet functions; one pair is (39) with (40) and the other pair is (42) with (43).

It is particularly important to mention here that in (39) the function $F_c(x)$ requires the computation of four pairs of the modified generalized Kampé de Fériet function (or double hypergeometric series)) (see Progri (2018, [14], Appen. A); i.e., it is computationally optimized in contrast to (or significantly and computationally more efficient than) (151) in Progri (2021, [15])) because the computational complexity of $F'(x)$ is equal to $O[F'(x)] = N \times O[F_p{}'(x)]$ and because we have reduced the computational complexity of $O[F_p{}'(x)] = O(8M^3+8M^2)$ for (151) in Progri (2021, [15]) to $O[F_c(x)] = O[2KM^2]$, where $2K \ll M$.

Therefore, Dr. Progri has successfully produced another landmark solution that is considered optimized for speed and memory locations because this solution reduces the computational complexity from $O(8M^3+8M^2)$ to $N \times O[2KM^2]$, $2K \ll M$ and produce a solution that is several orders of magnitude more accurate than any available solution.

3.2 Special Cases of the CLGPCFD CDF

The direct integration of a few special cases of the CLGPCFD cdf is performed here.

The first special case corresponds to (24). For this special case we have

$$F_c(x) = \frac{\mathcal{F}_{c0}(1,x)+\mathcal{F}_{c1}(1,x)}{a^{-1}C}$$

$$= \frac{\mathcal{F}_{c0}(1,x)+\mathcal{F}_{c1}(1,x)}{a^{-1}C}$$

$$= \frac{a^2 e^{-u_0^2}}{u_0} \frac{[\mathcal{K}_{c0}(1,x)+\mathcal{K}_{c1}(1,x)]}{C} \quad (46)$$

Where for $p = 1$ the $\mathcal{K}_{c0}(1,x)$, $\mathcal{K}_{c1}(1,x)$, and $K_{c0e}(m,x)$ and $K_{c1o}(m,x)$ and given by,

$$m = 1 + 2k \quad (47)$$

$$\mathcal{K}_{c0}(1,x) = \sum_{k=0}^{K} \frac{(-1)^k (1)_{2k}\left[K_{c0e}(m,x)+m\frac{2^{0.5m}}{2^{0.5}}x_0 K_{c0o}(m,x)\right]}{2^k \upsilon^m k!} \quad (48)$$

$$\mathcal{K}_{c1}(1,x) = \sum_{k=0}^{K} \frac{(-1)^k (1)_{2k}\left[K_{c1e}(m,x)+m\frac{2^{0.5m}}{2^{0.5}}x_1 K_{c1o}(m,x)\right]}{2^k \upsilon^m k!} \quad (49)$$

$$K_{c0e}(m,x) = F_{0:3;0}^{2:2;0}\left[\begin{matrix}\frac{1}{2}, 1: \frac{m}{2}, \frac{m+1}{2}; -; \\ -: \frac{1}{2}, \frac{1}{2}, 1; -;\end{matrix} x_0, x_2\right] \quad (50)$$

$$K_{c0o}(m,x) = F_{0:3;0}^{2:2;0}\left[\begin{matrix}1, \frac{3}{2}: \frac{m+1}{2}, \frac{m+2}{2}; -; \\ -: 1, \frac{3}{2}, \frac{3}{2};\end{matrix} x_0, x_2\right] \quad (51)$$

$$K_{c1e}(m,x) = F_{0:3;0}^{2:2;0}\left[\begin{matrix}\frac{1}{2}, 1: \frac{m}{2}, \frac{m+1}{2}; -; \\ -: \frac{1}{2}, \frac{1}{2}, 1; -;\end{matrix} x_1, x_2\right] \quad (52)$$

$$K_{c1o}(m,x) = F_{0:3;0}^{2:2;0}\left[\begin{matrix}1, \frac{3}{2}: \frac{m+1}{2}, \frac{m+2}{2}; -; \\ -: 1, \frac{3}{2}, \frac{3}{2};\end{matrix} x_1, x_2\right] \quad (53)$$

From (46) we observe that the optimized computation of $F_c(x)$ requires the optimized computation of two pairs of the Kampé de Fériet functions: one pair (50) with (51) for $\mathcal{K}_{c0}(1,x)$ and the other pair (52) with (53) for $\mathcal{K}_{c1}(1,x)$ of the modified generalized multidimensional hypergeometric functions. This produces tremendous savings in computation.

The optimized computation of (46) requires only two pairs of the modified generalized Kampé de Fériet functions (or double hypergeometric series); therefore, the computational complexity of $F_c(x)$ will be $O(2KM^2)$; i.e., fifty percent faster than for $N = 2$.

Similarly, the second special case corresponds to (27). For this special case we have

$$F_c(x) = \frac{\frac{\mathcal{F}_{c0}(2,x)+\mathcal{F}_{c1}(2,x)}{H_0(2)^{-1}a^{-2}} + \frac{\mathcal{F}_{c0}(1,x)+\mathcal{F}_{c1}(1,x)}{H_1(2)^{-1}a^{-1}}}{C}$$

$$= \frac{a^2 e^{-u_0^2}}{u_0} \cdot \frac{a[\mathcal{K}_{c0}(2,x)+\mathcal{K}_{c1}(2,x)]+4[\mathcal{K}_{c0}(1,x)+\mathcal{K}_{c1}(1,x)]}{C} \quad (54)$$

Where for $p = 2$, the $\mathcal{K}_{c0}(2,x)$, $\mathcal{K}_{c1}(2,x)$, and $K_{c0e}(m,x)$ and $K_{c1o}(m,x)$ and given by

$$m = 2 + 2k \quad (55)$$

$$\mathcal{K}_{c0}(2,x) = \sum_{k=0}^{K} \frac{(-1)^k (2)_{2k}\left[K_{c1e}(m,x)+m\frac{2^{0.5m}}{2^{0.5}}x_0 K_{c1o}(m,x)\right]}{2^k \upsilon^m k!}; \quad (56)$$

$$\mathcal{K}_{c1}(2,x) = \sum_{k=0}^{K} \frac{(-1)^k (2)_{2k}\left[K_{c1e}(m,x)+m\frac{2^{0.5m}}{2^{0.5}}x_1 K_{c1o}(m,x)\right]}{2^k \upsilon^m k!} \quad (57)$$

$$K_{c0e}(m,x) = F_{0:3;0}^{2:2;0}\left[\begin{matrix}\frac{1}{2}, 1: \frac{m}{2}, \frac{m+1}{2}; -; \\ -: \frac{1}{2}, \frac{1}{2}, 1; -;\end{matrix} x_0, x_2\right] \quad (58)$$

$$K_{c0o}(m,x) = F_{0:3;0}^{2:2;0}\left[\begin{matrix}1, \frac{3}{2}: \frac{m+1}{2}, \frac{m+2}{2}; -; \\ -: 1, \frac{3}{2}, \frac{3}{2};\end{matrix} x_0, x_2\right] \quad (59)$$

$$K_{c1e}(m,x) = F_{0:3;0}^{2:2;0}\left[\begin{matrix}\frac{1}{2}, 1: \frac{m}{2}, \frac{m+1}{2}; -; \\ -: \frac{1}{2}, \frac{1}{2}, 1; -;\end{matrix} x_1, x_2\right] \quad (60)$$

$$K_{c1o}(m,x) = F_{0:3;0}^{2:2;0}\left[\begin{matrix}1, \frac{3}{2}: \frac{m+1}{2}, \frac{m+2}{2}; -; \\ -: 1, \frac{3}{2}, \frac{3}{2};\end{matrix} x_1, x_2\right] \quad (61)$$

Where for $p = 1$ we have the results from the first special case (45)-(51).

The landmark computation of (52) requires only four pairs of the modified generalized Kampé de Fériet functions (or double hypergeometric series) Progri (2018, [15]) and Progri (2019, [16]): two pairs come from twice (58) with (59) and the other pair comes from (60) and (61) and then the other pairs coming from (50)-(53).

From (54) we observe that the landmark computation of $F_c(x)$ requires the optimized computation of four pairs: two pairs for $p = 2$ (58)-(61) and two pairs for $p = 1$ for (50)-(53) of the modified generalized Kampé de Fériet functions. This simplification delivers tremendous savings in computation. Therefore, the computational complexity of $F_c(x)$ will be $2 \times O(2KM^2)$; i.e., fifty percent faster than for $N > 2$.

For these two special cases we can achieve more than fifty percent savings in computational complexity.

This concludes the discussion on the special cases of the CLGPCDF cdf. Next, we discuss in great detail a few numerical examples.

4 Numerical, Theoretical Results

In this section we compare and contrast the performance of the CLGPCFD cdf with the EGPCFD cdf discussed in Progri (2021, [15]) and LGPCFD cdf Progri (2021, [16]). Since we have made significant improvements to the three algorithms this section validates the claims already discussed in Sects. 2, 3 and Appen. B-F.

4.1 Examples

For the numerical examples we employ exactly the same setup as the one in Numerical Examples Sect. Progri (2021, [15]) and Progri (2021, [16]).

We ran the simulations on a Dell computer Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz using MATLAB 2020b [25] using a vector of $0 \leq x \leq 100$ with a sample increment of $\Delta x = 0.05$ that leads to 2000 points. First, notice that in this paper we increased the sample increment and have increased the number of points so as to increase the computation time and accuracy. Second, I retired the linear approximation cdf from the calculations, because it served its purpose in Progri (2016, [13]), Progri (2021, [16]-[18]). Since accuracy is the most important goal for this journal paper, it was the right decision at the right time to retire linear approximation cdf. Third, Tab. 1 presents the computational duration times in seconds of the cdf 1 or the integral, cdf 2 the LGPCFD cdf (49) plus EGPCFD cdf (151) of Progri (2021, [15]) plus CLGPCFD (31), cdf 3 via EGPCFD (151) of Progri (2021, [15]) plus CLGPCFD (31), and cdf 4 is the same as cdf 2.

The first four rows in Tab. 1 correspond to the computation times for values of $a = 1$; $b = 1$; $N = 1$ for $[140, 130]$ terms for option four that is used for the computation of the four pairs of the Kampé de Fériet functions (or double hypergeometric series) Progri (2018, [14] Appen. A) and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appen. E of Progri (2019, [15]); $[80, 80]$ terms employed in the computation of three pairs of the Kampé de Fériet functions (or double hypergeometric series) in Progri (2021, [16]); and $[30,30]$ terms employed in the computation of the two (or four) pairs of the Kampé de Fériet functions in (30). In all of the run that we performed the computation for cdf 1-3 I used option 4 in both the computation of the Kampé de Fériet functions and Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appen. E of Progri (2019, [15]).

Inside Tab. 1 there are four sub tables for various values of $a = \{1,2,1,1\}$; $b = \{1,1,0.5,1\}$; $N = \{1,1,1,2\}$ and $\sum(\cdot) = [140, 130], [80, 80], [30,30]$ the values for the Kampé de Fériet functions and Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ corresponding to cdf 1-3. The first column corresponds to the integral cdf, the second to the fourth columns correspond to cdfs 1-3, and the fifth column corresponds to the total computational time. For every scenario I performed four runs to see any variation in the variance of computational times. As seen from the results in each column there is little variation in the variance of the computational times.

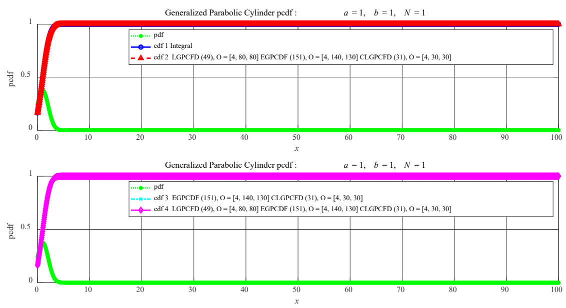
As seen from the results in Tab. 1 the computational time is a function of N and changes very little as we change either or both a and/or b. The computational times for $N = 1$ are the smallest with the integral cdf having the lowest of the four computational times, the cdf 2 having the second lowest. For $N = 2$ the computational times are somewhat revered. For cdfs 1-3 the computational times are doubled as explained in the earlier Sect.; however, for the integral the computational times are one order of magnitude higher than for $N = 1$. The main reason why that occurs is because as explained in the special cases pdf, for $N = 1$, the pdf is quite simple, as for $N = 2$, the pdf is far more complicated than for $N = 1$. Since the pdf is used for integration to produce the integral cdf we can see this dramatic increase in computational time.

The computational times for cdf 1 and 3 are almost identical as they should be. Since, either cdf 1 or 3 is made only of three subsegments as opposed to cdf 2 which is only made of two subsegments, I was hoping to see that the computational times for both cdf 1 and 3 to be lower than the computational times for cdf 2.

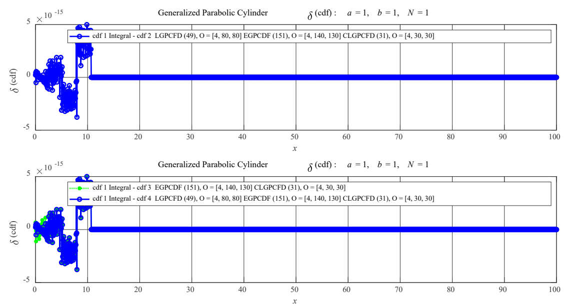
I have to conclude that in the current form the combined algorithm is not optimized in terms of computational speed; it is however optimized in terms of performance accutracy. This is a topic for a separate journal paper.

Figure 1 (a) presents the GPCFD pdf and four GPCFD cdfs cdf 1 corresponds to the integral, cdf 2 the LGPCFD cdf (49) plus EGPCFD cdf (151) of Progri (2021, [16]) plus CLGPCFD (31). (top), cdf 3 via EGPCFD (151) of Progri (2021, [15]) plus CLGPCFD (31), and cdf 4 is the same as cdf 2 (bottom). In Fig. 1(b) the absolute error of the cdf 1 with cdf 2 is shown (top); and cdf 1 with cdf 3 and cdf 3 with cdf 4 (bottom).

I have produced for the first time the most accurate results of



(a) The computation of the GPCFD pdf and cdf for $0 \leq x \leq 100,\ a = 1,\ b = 1,\ N = 1$.



(b) The absolute error the GPCFD cdf for $0 \leq x \leq 100,\ a = 1,\ b = 1,\ N = 1$.

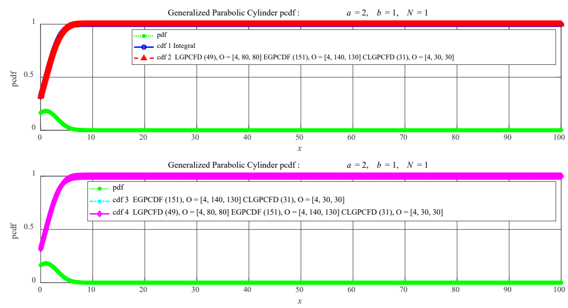
Fig. 1. The computation of the GPCFD pdf and cdf for $0 \leq x \leq 100,\ a = 1,\ b = 1,\ N = 1$.

Cdf 1 Integral

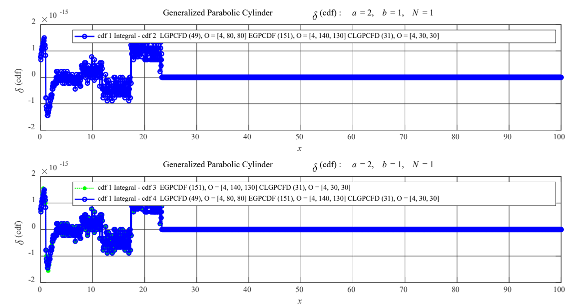
Cdf 2 the LGPCFD cdf (49) plus EGPCFD cdf (151) of Progri (2021, [15]) plus CLGPCFD (31)

Cdf 3 via EGPCFD (151) of Progri (2021, [15]) plus CLGPCFD (31)

Cdf 4 is the same as Cdf 2.



(a) The computation of the GPCFD pdf and cdf for $0 \leq x \leq 100,\ a = 2,\ b = 1,\ N = 1$.



(b) The absolute error the GPCFD cdf for $0 \leq x \leq 100,\ a = 2,\ b = 1,\ N = 1$.

Fig. 2. The computation of the GPCFD pdf and cdf for $0 \leq x \leq 100,\ a = 2,\ b = 1,\ N = 1$.

Cdf 1 Integral

Cdf 2 the LGPCFD cdf (49) plus EGPCFD cdf (151) of Progri (2021, [15]) plus CLGPCFD (31)

Cdf 3 via EGPCFD (151) of Progri (2021, [15]) plus CLGPCFD (31)

Cdf 4 is the same as Cdf 2.

the GPCFD cdf for values of $0 \leq x \leq 100$. The relative error is on the orders of fifteen digits precision for $N = 1$.

Figure 2(a) and (b) present the same result as Fig. 1(a) and (b) but for values of $a = 2;\ b = 1;\ N = 1$.

The results of Figs. 3 and 4 are self-explanatory. We have also removed a bias in the computation of the CLGPCFD cdf for values of $N = 2$. The relative error is on the orders of thirteen digits precision for $N = 2$.

This concluded the discussion on examples and numerical, theoretical results.

5 Conclusions

From the title and the content of the paper, Dr. Progri has produced more than a *masterpiece* he has produced an *iconic* derivation[vii] of the closed form expression of the CLGPCFD by means of the complementary expansion of the PCF and has made use of all the tools in the toolbox to improve the computational accuracy by thirteen to fifteen orders of magnitude.

This is the first journal paper that presents a holistic approach to the computation of the GPCFD cdf as a combination of either the LGPCFD cdf (49) plus EGPCFD cdf (151) of Progri (2021, [16]) plus CLGPCFD (31) or EGPCFD (151) of Progri (2021, [16]) plus CLGPCFD (31). How exactly this combination is done is the topic of a separate journal paper. The combined GPCFD cdf is the most accurate algorithm that needs to be optimized in terms of computational performance.

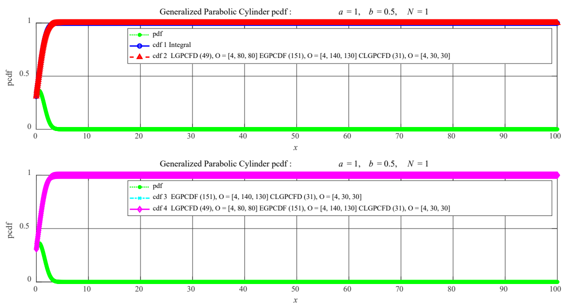
The CLGPCFD cdf is the best option for computing the GPCFD cdf for large values of the main variable which made possible the computational accuracy on thirteen to fifteen orders of magnitude. How was this made possible? This is

TABLE I: THE QUANTITATIVE COMPUTATIONAL PERFORMANCE

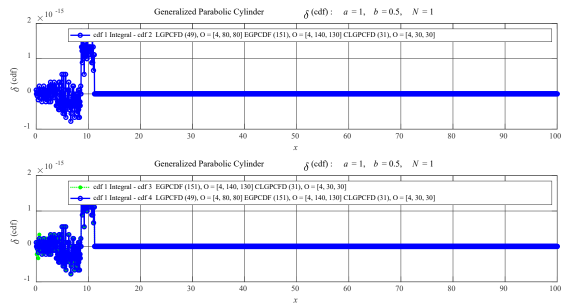
GPCDF cdf:
CFE 1 LGPCFD (49) + EGPCFD (151) + CLGPCFD (30)
vs. CFE 2 EGPCFD (151) + CLGPCFD (30) vs CFE 3 same as CFE 1

$a = 1,\ b = 1,\ N = 1,\ \sum(\cdot) = [140, 130], [80, 80], [30,30]$				
Integral (sec)	CFE 1 (sec)	CFE 2 (sec)	CFE 3 (sec)	Total (sec)
17.73	51.17	39.38	50.98	159.26
16.94	51.06	39.36	50.78	158.14
17.23	52.08	39.42	50.94	159.66
17.32	51.02	39.34	50.87	158.54
$a = 2,\ b = 1,\ N = 1,\ \sum(\cdot) = [140, 130], [80, 80], [30,30]$				
Integral (sec)	CFE 1 (sec)	CFE 2 (sec)	CFE 3 (sec)	Total (sec)
17.06	52.42	40.87	52.84	163.19
17.66	52.64	41.02	53.22	164.55
17.53	53.48	42.05	53.60	166.65
17.20	52.41	40.71	52.26	162.58
$a = 1,\ b = 0.5,\ N = 1,\ \sum(\cdot) = [140, 130], [80, 80], [30,30]$				
Integral (sec)	CFE 1 (sec)	CFE 2 (sec)	CFE 3 (sec)	Total (sec)
17.21	50.98	39.48	51.18	158.85
17.89	51.14	39.47	51.15	159.65
17.60	51.20	39.81	51.09	159.71
17.58	51.35	40.16	51.39	160.49
$a = 1,\ b = 1,\ N = 2,\ \sum(\cdot) = [140, 130], [80, 80], [30,30]$				
Integral (sec)	CFE 1 (sec)	CFE 2 (sec)	CFE 3 (sec)	Total (sec)
113.31	104.63	79.94	106.02	403.90
111.21	101.99	79.03	102.21	394.45
112.30	101.91	78.43	101.65	394.30
112.17	106.83	82.16	103.12	404.27

another test of Dr. Progri's vision that a complementary expansion is needed for large values of the main variable to achieve the computational performance accuracy in Progri (2022, [19]-[22]) and then with Progri (2023, [23]).



(a) The computation of the GPCFD pdf and cdf for $0 \le x \le 100,\ a = 1,\ b = 0.5,\ N = 1$.



(b) The absolute error the GPCFD cdf for $0 \le x \le 100,\ a = 1,\ b = 0.5,\ N = 1$.

Fig. 3. The computation of the GPCFD pdf and cdf for $0 \le x \le 100,\ a = 1,\ b = 0.5,\ N = 1$.
Cdf 1 Integral
Cdf 2 the LGPCFD cdf (49) plus EGPCFD cdf (151) of Progri (2021, [15]) plus CLGPCFD (31)
Cdf 3 via EGPCFD (151) of Progri (2021, [15]) plus CLGPCFD (31)
Cdf 4 is the same as Cdf 2.

6 Acknowledgement

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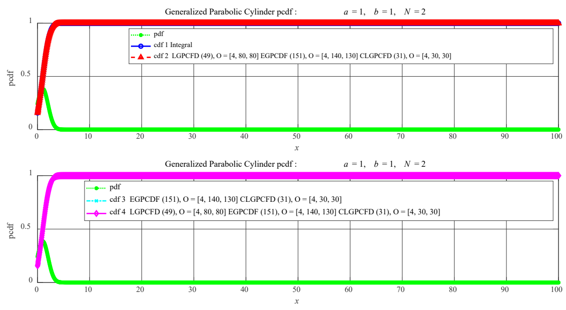
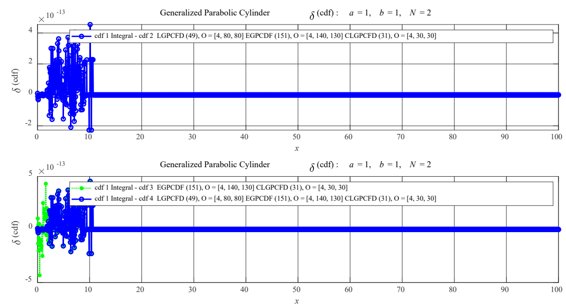
This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [24] reminds me of my grandfather who I loved him very much.



(a) The computation of the GPCFD pdf and cdf for $0 \le x \le 100,\ a = 1,\ b = 1,\ N = 2$.



(b) The absolute error the GPCFD cdf for $0 \le x \le 100,\ a = 1,\ b = 1,\ N = 2$.

Fig. 4. The computation of the GPCFD pdf and cdf for $0 \le x \le 100,\ a = 1,\ b = 1,\ N = 2$.

Cdf 1 Integral

Cdf 2 the LGPCFD cdf (49) plus EGPCFD cdf (151) of Progri (2021, [15]) plus CLGPCFD (31)

Cdf 3 via EGPCFD (151) of Progri (2021, [15]) plus CLGPCFD (31)

Cdf 4 is the same as Cdf 2.

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8 Appendix A: Derivation of the Rising Factorial in (4)

The rising factorial, (or Pochhammer symbol [7]), $(p)_{2k}$ needs to be simplified to enable the computation of either (4) or (7) via the hypergeometric function.

We write the rising factorial, (or Pochhammer symbol Gradshteyn, Ryzhik (2007, [1] Notation pg. xliii), [7]), $(p)_{2k}$ can be extended to real values of p invoking the gamma function provided p and $p+2k$ are real (or integer) numbers that are not negative integers, then we use the definition of the doubling formula, Gradshteyn, Ryzhik (2007, [1] 8.335 1. pp 896), a.k.a. Legendre duplication formula, [8], and then simplify after some algebra,

$$(p)_{2k} = \frac{\Gamma(p+2k)}{\Gamma(p)} = \frac{\Gamma\left[2\left(k+\frac{p}{2}\right)\right]}{\Gamma\left[2\left(\frac{p}{2}\right)\right]}$$

$$= \frac{2^{2k+p-1}\Gamma\left(k+\frac{p}{2}\right)\Gamma\left(k+\frac{p}{2}+\frac{1}{2}\right)}{2^{p-1}\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)}\text{ viii}$$

$$= 2^{2k}\left(\frac{p}{2}\right)_k\left(\frac{p+1}{2}\right)_k \tag{62}$$

Equation (62) is the desired equation. This concludes the derivations of Appen. A.

9 Appendix B: The Derivation of the CLGPCFD PDF of Progri (2021, [16] (8))

In this Sect. we derive the closed form expression of the CLGPCFD PDF from Progri (2021, [16] (8)).

Rewriting Progri (2021, [16] (8)) we have the following:

$$f_c(x) = \frac{\sqrt{3}}{4a}\frac{e^{-0.5g_0^2(x)}\Phi_c(v_0)+e^{-0.5g_1^2(x)}\Phi_c(v_1)}{\sqrt{2\pi}\Phi_c\left(\frac{2a}{\sqrt{6}c^2}\right)} \tag{63}$$

where

$$g_0(x) = \frac{\sqrt{3}a(x-b)}{2a^2} + \frac{a}{\sqrt{3}c^2} = a\sqrt{3}\left[a_0(x)+\frac{1}{3c^2}\right] \tag{64}$$

$$g_1(x) = \frac{\sqrt{3}(x-b)}{2a} - \frac{a}{\sqrt{3}c^2} = a\sqrt{3}\left[a_0(x)-\frac{1}{3c^2}\right] \tag{65}$$

$$\frac{1}{2}g_0^2(x) = \frac{3}{2}a^2\left[a_0^2(x)+\frac{2a_0(x)}{3c^2}+\frac{1}{9c^4}\right] \tag{66}$$

$$\frac{1}{2}g_1^2(x) = \frac{3}{2}a^2\left[a_0^2(x)-\frac{2a_0(x)}{3c^2}+\frac{1}{9c^4}\right] \tag{67}$$

$$\frac{1}{2}g_0^2(x) - \frac{2a^2}{3c^4} - 2a^2a_0^2(x) = -\frac{a^2}{2}\left[a_0^2(x)-\frac{2a_0(x)}{c^2}+\frac{1}{c^4}\right]$$

$$= -\frac{a^2}{2}\left[a_0(x) - \frac{1}{c^2}\right]^2$$

$$= -\frac{a^2}{2}a_{0-}^2(x)$$

$$= -\frac{c_0^2(x)}{2} \tag{68}$$

$$\frac{1}{2}g_1^2(x) - \frac{2a^2}{3c^4} - 2a^2a_0^2(x) = -\frac{1}{2}a^2a_0^2(x) - \frac{a^2a_0(x)}{c^2} - \frac{a^2}{2c^4}$$

$$= -\frac{a^2}{2}\left[a_0(x) + \frac{1}{c^2}\right]^2$$

$$= -\frac{a^2}{2}a_{0+}^2(x)$$

$$= -\frac{c_1^2(x)}{2} \tag{69}$$

Substituting (68) and (69) into (63) yields:

$$f_c(x) = \frac{\sqrt{3}}{4a}\frac{\frac{\Phi_c\left[\frac{c_0(x)}{\sqrt{2}}\right]}{e^{-\frac{c_0^2(x)}{2}}}+\frac{\Phi_c\left[\frac{c_1(x)}{\sqrt{2}}\right]}{e^{-\frac{c_1^2(x)}{2}}}}{e^{\frac{2a^2}{3c^4}}e^{2a^2a_0^2(x)}\sqrt{2\pi}\Phi_c\left(\frac{2a}{\sqrt{6}c^2}\right)} \tag{70}$$

From (Gradshteyn, Ryzhik 2007, [1] 8.254 1 pg. 889) the definition of the complementary error function, $\Phi_c(z)$, [4]

$$\Phi_c(z) = 1 - \Phi(z)$$

$$= \frac{e^{-z^2}}{\sqrt{\pi}z}\left[\sum_{k=0}^{K}\frac{(-1)^k(2k-1)!!}{(2z^2)^k} + O(|z|^{-2K-z})\right]$$

$$\cong \frac{e^{-z^2}}{\sqrt{\pi}z}\sum_{k=0}^{K}\frac{(-1)^k(2k-1)!!}{(2z^2)^k},\ (2k-1)!! = \frac{(2k)!}{2^kk!} = \frac{(1)_{2k}}{2^kk!}$$

$$= \frac{e^{-z^2}}{\sqrt{\pi}z}\sum_{k=0}^{K}\frac{(-1)^k(1)_{2k}}{2^k(2z^2)^kk!}$$

$$= \frac{\sqrt{2}e^{-z^2}}{\sqrt{\pi}(\sqrt{2}z)}\sum_{k=0}^{K}\frac{(-1)^k(1)_{2k}}{2^k\left[(\sqrt{2}z)^2\right]^kk!}$$

$$= \frac{\sqrt{2}e^{-z^2}}{\sqrt{\pi}}U_{-1}(\sqrt{2}z) \tag{71}$$

Equation (71) is very significant because it shows the direct relations between the CEF and CPCF in a way that was never presented before.

Furthermore, substituting (71) into (70) produces:

$$f_c(x) \cong \frac{e^{-2a^2a_0^2(x)}\{U_{-1}[c_0(x)]+U_{-1}[c_1(x)]\equiv V_{-1}(x)\equiv W_{-1}(x)/a\}}{\frac{4a}{\sqrt{3}}e^{\frac{2a^2}{3c^4}}\sqrt{2\pi}\Phi_c\left(\frac{2a}{\sqrt{6}c^2}\right)} \tag{72}$$

Equation (72) is identical to (25) which completes the proof. This concludes the derivations of Appen. B.

10 Appendix C: The Derivation of the CLGPCFD PDF of Progri (2021, [16] (11))

In this Appen. we revisit the computation of the special cases of the efficient GPCFD cdf in order to perform the derivation of derivation of the CLGPCFD PDF of Progri (2021, [16] (11)). First let us review the efficient computation of the GPCFD cdf as given by (151) and (152) and then (81) in Progri (2021, [15]).

From Progri (2021, [16] (11)) we have the following:

$$f_c(x) = \frac{\frac{\left[\frac{2\sqrt{2}}{a}-v_0\right]\Phi_c(v_0)+\frac{e^{-v_0^2}}{\sqrt{\pi}}}{e^{\frac{1}{2}g_0^2(x)}}+\frac{\left[\frac{2\sqrt{2}}{a}-v_1\right]\Phi_c(v_1)+\frac{e^{-v_1^2}}{\sqrt{\pi}}}{e^{\frac{1}{2}g_1^2(x)}}}{3^{-1}2^3a\sqrt{\pi}\left[\sqrt{\frac{2}{\pi}}e^{-\frac{2a^2}{3c^4}}+2\left(\frac{\sqrt{3}}{a}-\frac{a}{\sqrt{3}c^2}\right)\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)\right]}$$

$$= \frac{\frac{\left[\frac{2\sqrt{2}}{a}-v_0\right]\Phi_c(v_0)+\frac{e^{-\frac{c_0^2(x)}{2}}}{\sqrt{\pi}}}{e^{\frac{2a^2}{3c^4}}e^{2a^2a_0^2(x)}e^{-\frac{c_0^2(x)}{2}}}+\frac{\left[\frac{2\sqrt{2}}{a}-v_1\right]\Phi_c(v_1)+\frac{e^{-\frac{c_1^2(x)}{2}}}{\sqrt{\pi}}}{e^{\frac{2a^2}{3c^4}}e^{2a^2a_0^2(x)}e^{-\frac{c_1^2(x)}{2}}}}{3^{-1}2^3a\sqrt{\pi}\left[\sqrt{\frac{2}{\pi}}e^{-\frac{2a^2}{3c^4}}+2\left(\frac{\sqrt{3}}{a}-\frac{a}{\sqrt{3}c^2}\right)\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)\right]}$$

$$\frac{\left|\begin{array}{l}\frac{\left[\frac{2\sqrt{2}}{a}-\frac{c_0(x)}{\sqrt{2}}\right]\frac{\sqrt{2}e^{-\frac{c_0^2(x)}{2}}}{\sqrt{\pi}}U_{-1}[c_0(x)]+\frac{e^{-\frac{c_0^2(x)}{2}}}{\sqrt{\pi}}}{e^{-\frac{c_0^2(x)}{2}}}+\\ \frac{\left[\frac{2\sqrt{2}}{a}-\frac{c_1(x)}{\sqrt{2}}\right]\frac{\sqrt{2}e^{-\frac{c_1^2(x)}{2}}}{\sqrt{\pi}}U_{-1}[c_1(x)]+\frac{e^{-\frac{c_1^2(x)}{2}}}{\sqrt{\pi}}}{e^{-\frac{c_1^2(x)}{2}}}\end{array}\right.}{e^{\frac{2a^2}{3c^4}}e^{2a^2a_0^2(x)}\frac{2^3a\sqrt{\pi}}{3}\left[\sqrt{\frac{2}{\pi}}e^{-\frac{2a^2}{3c^4}}+2\left(\frac{\sqrt{3}}{a}-\frac{a}{\sqrt{3}c^2}\right)\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)\right]}$$

$$= \frac{\left[\frac{2\sqrt{2}}{a}-\frac{c_0(x)}{\sqrt{2}}\right]\frac{\sqrt{2}}{\sqrt{\pi}}U_{-1}[c_0(x)]+\frac{1}{\sqrt{\pi}}+\left[\frac{2\sqrt{2}}{a}-\frac{c_1(x)}{\sqrt{2}}\right]\frac{\sqrt{2}}{\sqrt{\pi}}U_{-1}[c_1(x)]+\frac{1}{\sqrt{\pi}}}{e^{\frac{2a^2}{3c^4}}e^{2a^2a_0^2(x)}\frac{2^3a\sqrt{\pi}}{3}\left[\sqrt{\frac{2}{\pi}}e^{-\frac{2a^2}{3c^4}}+2\left(\frac{\sqrt{3}}{a}-\frac{a}{\sqrt{3}c^2}\right)\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)\right]}$$

$$= \frac{[4-ac_0(x)]U_{-1}[c_0(x)]+a+[4-ac_1(x)]U_{-1}[c_1(x)]+a}{e^{\frac{2a^2}{3c^4}}e^{2a^2a_0^2(x)}\frac{8a^2\pi}{3}\left[\sqrt{\frac{2}{\pi}}e^{-\frac{2a^2}{3c^4}}+2\left(\frac{\sqrt{3}}{a}-\frac{a}{\sqrt{3}c^2}\right)\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)\right]}$$

$$= \frac{e^{-2a^2a_0^2(x)}\left|\begin{array}{l}a\{1-c_0(x)U_{-1}[c_0(x)]\}+a\{1-c_1(x)U_{-1}[c_1(x)]\}+\\ 4\{U_{-1}[c_0(x)]+U_{-1}[c_1(x)]\}\end{array}\right.}{e^{\frac{2a^2}{3c^4}}\frac{8a^2\pi}{3}\left[\sqrt{\frac{2}{\pi}}e^{-\frac{2a^2}{3c^4}}+2\left(\frac{\sqrt{3}}{a}-\frac{a}{\sqrt{3}c^2}\right)\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)\right]}$$

$$= \frac{e^{-2a^2a_0^2(x)}\{a[U_{-2}[c_0(x)]+U_{-2}[c_1(x)]]+4[U_{-1}[c_0(x)]+U_{-1}[c_1(x)]]\}}{e^{\frac{2a^2}{3c^4}}\frac{8a^2\pi}{3}\left[\sqrt{\frac{2}{\pi}}e^{-\frac{2a^2}{3c^4}}+2\left(\frac{\sqrt{3}}{a}-\frac{a}{\sqrt{3}c^2}\right)\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)\right]}$$

$$= \frac{e^{-2a^2a_0^2(x)}[aV_{-2}(x)+4V_{-1}(x)\equiv W_{-2}(x)/a]}{e^{\frac{2a^2}{3c^4}}\frac{8a^2\pi}{3}\left[\sqrt{\frac{2}{\pi}}e^{-\frac{2a^2}{3c^4}}+2\left(\frac{\sqrt{3}}{a}-\frac{a}{\sqrt{3}c^2}\right)\Phi_c\left(\frac{\sqrt{2}a}{\sqrt{3}c^2}\right)\right]} \tag{73}$$

From the definition of the $U_{-1}(z)$ in (6) we have

$$zU_{-1}(z) = \sum_{k=0}^{K} \frac{(-1)^k (1)_{2k}}{2^k z^{2k} k!},\ |z| \gg (p > 0, 1)$$

$$= 1 + \sum_{k=1}^{K} \frac{(-1)^k (1)_{2k}}{2^k z^{2k} k!},\ |z| \gg (p > 0, 1)$$

$$= 1 - \frac{1}{2z^2} \sum_{n=0}^{K} \frac{(-1)^n (1)_{2(n+1)}}{2^n z^{2n} (n+1) n!},\ |z| \gg (p > 0, 1)$$

$$= 1 - \frac{1}{2z^2} \sum_{n=0}^{K} \frac{(-1)^n (2n+2)(2)_{2n}}{2^n z^{2n} (n+1) n!},\ |z| \gg (p > 0, 1)$$

$$= 1 - \frac{1}{z^2} \sum_{n=0}^{K} \frac{(-1)^n (2)_{2n}}{2^n z^{2n} n!},\ |z| \gg (p > 0, 1)$$

$$= 1 - U_{-2}(z),\ |z| \gg (p > 0, 1) \qquad (74)$$

In (74) we made use of the Pochhammer symbol, $(1)_{2(n+1)}$, as follows,

$$\begin{aligned}(1)_{2(n+1)} &= (2n+2)! \\ &= (2n+2)(2n+1)! = (2n+2)(2)_{2n} \\ &= (2n+2)(2)_{2n}\end{aligned} \qquad (75)$$

Equation (73) is identical to (27) which completes the proof.

This concludes the derivation of the complementary GPCFD PDF of Progri (2021, [16] (11)).

11 Appendix D: The Derivation of the Integral (36)

In Appen. D we perform the derivations for the computation of the integral (36).

We make the second substitution,

$$u = \sqrt{2} a a_0(t) = \frac{t-b}{\sqrt{2}a} \text{ or} \qquad (76)$$

Then we have

$$t = u\sqrt{2}a + b \qquad (77)$$

$$dt = \sqrt{2} a du \qquad (78)$$

$$t \to |_x^\infty \ \Rightarrow\ u \to |_{u_0}^\infty \qquad (79)$$

$$u_0 = \frac{x-b}{\sqrt{2}a} \qquad (80)$$

Substituting (76)-(80) into (42) we have

$$\mathcal{G}_{c0}(m,x) = \frac{2a^2 \int_{u_0}^{\infty} e^{-u^2} \left(\frac{1}{c^2} - \frac{u}{\sqrt{2}a}\right)^{-m} du}{\sqrt{2} a a^m}$$

$$= \frac{\sqrt{2}a \int_{u_0}^{\infty} e^{-u^2} \left(\frac{1}{c^2} - \frac{u}{\sqrt{2}a}\right)^{-m} du}{a^m}$$

$$= a\sqrt{2}^{m+1} \int_{u_0}^{\infty} e^{-u^2} \left(\frac{\sqrt{2}a}{c^2} + u\right)^{-m} du \qquad (81)$$

Let us define further,

$$v = \frac{\sqrt{2}a}{c^2} \qquad (82)$$

Hence, substituting (82) into (81) yields,

$$\mathcal{G}_{c0}(m,x) = a\sqrt{2}^{m+1} \int_{u_0}^{\infty} e^{-u^2} (u+v)^{-m} du \qquad (83)$$

Invoking the Binomial theorem [9] we have,

$$(v-u)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} (-1)^{-n} u^{-n} v^{-m-n}$$

$$= \sum_{n=0}^{\infty} \binom{m+n-1}{n} u^{-n} v^{-m-n} \qquad (84)$$

Next, substituting (88) into (81) produces,

$$\mathcal{G}_{c0}(m,x) = a \frac{\int_{u_0}^{\infty} e^{-u^2} \sum_{n=0}^{\infty} \binom{m+n-1}{n} u^{-n} v^{-m-n} du}{2^{-\frac{m+1}{2}}}$$

$$= a \frac{\sum_{n=0}^{\infty} \binom{m+n-1}{n} v^{-m-n} \int_{u_0}^{\infty} e^{-u^2} u^{-n} du}{2^{-\frac{m+1}{2}}} \qquad (85)$$

Next, let us define the following integral,

$$\mathcal{H}_{0,n}(u_0) = \int_{u_0}^{\infty} e^{-u^2} u^{-n} du,\ n = \{0,1,2,\cdots\}$$

$$= \frac{e^{-u_0^2}}{2u_0^{n+1}} \sum_{l=0}^{L} \frac{(-1)^l (n+1)_{2l}}{2^l (2u_0^2)^l l!},\ n = \{0,1,2,\cdots\} \qquad (86)$$

The derivations for the solution of this integral are given in Appen. E.

Next, substituting (86) into (85) yields.

$$\mathcal{G}_{c0}(m,x) = a \frac{\sum_{n=0}^{\infty} \binom{m+n-1}{n} v^{-m-n} \frac{e^{-u_0^2}}{2u_0^{n+1}} \sum_{l=0}^{L} \frac{(-1)^l (n+1)_{2l}}{2^l (2u_0^2)^l l!}}{2^{-\frac{m+1}{2}}}$$

$$= a e^{-u_0^2} \frac{\sum_{n=0}^{\infty} \binom{m+n-1}{n} v^{-n} u_0^{-n} \sum_{l=0}^{L} \frac{(-1)^l (n+1)_{2l}}{(2^2 u_0^2)^l l!}}{2u_0 v^m \times 2^{-\frac{m+1}{2}}}$$

$$= a e^{-u_0^2} \frac{\sum_{n=0}^{\infty} \binom{m+n-1}{n} x_1^n \sum_{l=0}^{L} \frac{(n+1)_{2l} x_2'^l}{l!}}{2u_0 v^m \times 2^{-\frac{m+1}{2}}} \qquad (87)$$

where

$$x_0 = (v u_0)^{-1} \qquad (88)$$

$$x_2' = -(2u_0)^{-2} \qquad (89)$$

The derivations, similar to Appen. A, were invoked to simplify the following binomial coefficients, [10], or Pochhammer symbols, [1], [7],

$$\binom{m+n-1}{n} = \frac{(m)_n}{n!} = \frac{(m)_{2i}}{2i!} = \frac{\Gamma(m+2i)}{\Gamma(m)\Gamma(2i+1)}$$

$$= \frac{\Gamma\left[2\left(\frac{m}{2}+i\right)\right]}{\Gamma\left(2\frac{m}{2}\right)\Gamma\left[2\left(i+\frac{1}{2}\right)\right]} = \frac{\sqrt{\pi}\Gamma\left(\frac{m}{2}+i\right)\Gamma\left(\frac{m}{2}+i+\frac{1}{2}\right)}{2^{\frac{m}{2}-\frac{1}{2}}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)\Gamma\left(i+\frac{1}{2}\right)\Gamma(i+1)}$$

$$= \frac{\left(\frac{m}{2}\right)_i\left(\frac{m}{2}+\frac{1}{2}\right)_i}{2^{\frac{m}{2}-\frac{1}{2}}\left(\frac{1}{2}\right)_i i!} \tag{90}$$

$$\binom{m+n-1}{n} = \frac{(m)_n}{n!} = \frac{(m)_{2i+1}}{(2i+1)!} = \frac{\Gamma(m+2i+1)}{\Gamma(m)\Gamma(2i+2)}$$

$$= \frac{\Gamma\left[2\left(\frac{m}{2}+i+\frac{1}{2}\right)\right]}{\Gamma\left(2\frac{m}{2}\right)\Gamma[2(i+1)]} = \frac{\sqrt{\pi}\Gamma\left(\frac{m}{2}+i+\frac{1}{2}\right)\Gamma\left(\frac{m}{2}+i+1\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)\Gamma(i+1)\Gamma\left(i+\frac{3}{2}\right)}$$

$$= \frac{m\sqrt{\pi}\left(\frac{m}{2}+\frac{1}{2}\right)_i\left(\frac{m}{2}+1\right)_i}{2i!\left(\frac{3}{2}\right)_i\Gamma\left(\frac{3}{2}\right)} = \frac{m\left(\frac{m}{2}+\frac{1}{2}\right)_i\left(\frac{m}{2}+1\right)_i}{\left(\frac{3}{2}\right)_i i!} \tag{91}$$

$$(n+1)_{2l} = \frac{\Gamma(n+2l+1)}{\Gamma(n+1)} = \frac{\Gamma(2i+2l+1)}{\Gamma(2i+1)} = \frac{\Gamma\left[2\left(i+l+\frac{1}{2}\right)\right]}{\Gamma\left[2\left(i+\frac{1}{2}\right)\right]}$$

$$= \frac{2^{2i+2l}\Gamma\left(i+l+\frac{1}{2}\right)\Gamma(i+l+1)}{2^{2i}\Gamma\left(i+\frac{1}{2}\right)\Gamma(i+1)} = 2^{2l}\left(i+\frac{1}{2}\right)_l (i+1)_l$$

$$= 2^{2l}\frac{\left(\frac{1}{2}\right)_{l+i}}{\left(\frac{1}{2}\right)_i}\frac{(1)_{l+i}}{(1)_i} \tag{92}$$

$$(n+1)_{2l} = \frac{\Gamma(n+2l+1)}{\Gamma(n+1)} = \frac{\Gamma(2i+2l+2)}{\Gamma(2i+2)} = \frac{\Gamma[2(i+l+1)]}{\Gamma[2(i+1)]}$$

$$= \frac{2^{2i+2l+1}\Gamma(i+l+1)\Gamma\left(i+l+\frac{3}{2}\right)}{2^{2i+1}\Gamma(i+1)\Gamma\left(i+\frac{3}{2}\right)} = 2^{2l}(i+1)_l\left(i+\frac{3}{2}\right)_l$$

$$= 2^{2l}\frac{(1)_{l+i}}{(1)_i}\frac{\left(\frac{3}{2}\right)_{l+i}}{\left(\frac{3}{2}\right)_i} \tag{93}$$

Next, substituting (90)-(93) into (87) yields.

$$\mathcal{G}_{c0}(m,x) = ae^{-u_0^2}\frac{\begin{matrix}\sum_{i=0}^{\infty}\frac{\left(\frac{m}{2}\right)_i\left(\frac{m}{2}+\frac{1}{2}\right)_i}{2^{\frac{m}{2}-\frac{1}{2}}\left(\frac{1}{2}\right)_i i!}x_0^{2i}\sum_{l=0}^{L}\frac{2^{2l}\frac{\left(\frac{1}{2}\right)_{l+i}}{\left(\frac{1}{2}\right)_i}\frac{(1)_{l+i}}{(1)_i}x_2'^l}{l!}+\\ \sum_{i=0}^{\infty}\frac{m\left(\frac{m}{2}+\frac{1}{2}\right)_i\left(\frac{m}{2}+1\right)_i}{\left(\frac{3}{2}\right)_i i!}x_0^{2i+1}\sum_{l=0}^{L}\frac{2^{2l}\frac{(1)_{l+i}}{(1)_i}\frac{\left(\frac{3}{2}\right)_{l+i}}{\left(\frac{3}{2}\right)_i}x_2'^l}{l!}\end{matrix}}{2u_0 v^m\times 2^{-\frac{m+1}{2}}}$$

$$= ae^{-u_0^2}\frac{\begin{matrix}\frac{\sqrt{2}}{2^{\frac{m}{2}}}\sum_{i,l=0}^{\infty,L}\frac{\left(\frac{1}{2}\right)_{l+i}(1)_{l+i}\left(\frac{m}{2}\right)_i\left(\frac{m}{2}+\frac{1}{2}\right)_i}{\left(\frac{1}{2}\right)_i\left(\frac{1}{2}\right)_i(1)_i}\frac{x_0^{2i}}{i!}\frac{2^{2l}x_2'^l}{l!}+\\ mx_0\sum_{i,l=0}^{\infty,L}\frac{(1)_{l+i}\left(\frac{3}{2}\right)_{l+i}\left(\frac{m}{2}+\frac{1}{2}\right)_i\left(\frac{m}{2}+1\right)_i}{(1)_i\left(\frac{3}{2}\right)_i\left(\frac{3}{2}\right)_i}\frac{x_0^{2i}}{i!}\frac{2^{2l}x_2'^l}{l!}\end{matrix}}{2u_0 v^m\times 2^{-\frac{m+1}{2}}}$$

$$= ae^{-u_0^2}\frac{K_{c0e}(m,x)+m2^{0.5m-0.5}x_0K_{c0o}(m,x)}{u_0 v^m} \tag{94}$$

where the two Kampé de Fériet functions (see Progri (2018, [14]), Appen. A) are given by

$$K_{c0e}(m,x) = \sum_{i,l=0}^{\infty,L}\frac{\left(\frac{1}{2}\right)_{l+i}(1)_{l+i}\left(\frac{m}{2}\right)_i\left(\frac{m}{2}+\frac{1}{2}\right)_i}{\left(\frac{1}{2}\right)_i\left(\frac{1}{2}\right)_i(1)_i}\frac{x_0^{2i}}{i!}\frac{2^{2l}x_2'^l}{l!}$$

$$= F_{0:3;0}^{2:2;0}\left[\begin{matrix}\frac{1}{2},1:\frac{m}{2},\frac{m+1}{2};-;\\ -:\frac{1}{2},\frac{1}{2},1;-;\end{matrix}\ x_0,x_2\right] \tag{95}$$

$$K_{c0o}(m,x) = \sum_{i,l=0}^{\infty,L}\frac{(1)_{l+i}\left(\frac{3}{2}\right)_{l+i}\left(\frac{m}{2}+\frac{1}{2}\right)_i\left(\frac{m}{2}+1\right)_i}{(1)_i\left(\frac{3}{2}\right)_i\left(\frac{3}{2}\right)_i}\frac{x_0^{2i}}{i!}\frac{2^{2l}x_2'^l}{l!}$$

$$= F_{0:3;0}^{2:2;0}\left[\begin{matrix}1,\frac{3}{2}:\frac{m+1}{2},\frac{m+2}{2};-;\\ -:1,\frac{3}{2},\frac{3}{2};\end{matrix}\ x_0,x_2\right] \tag{96}$$

Where

$$x_2 = 2^2x_2' = -u_0^{-2} \tag{97}$$

Integral (36) is reduced to the computation of two Kampé de Fériet functions (see Progri (2018, [14]), Appen. A). This concludes the derivations of the computation of the integral (36).

12 Appendix E: The Derivation of the Integral (86)

Special cases of integral (86) occur a lot in many fields of mathematics, physics, engineering, economics, etc. Here I will provide the details for solving this integral for the more general case.

Let us start with the special case for $n = 0$. For this special case integral (86) is reduced to the computation of the CEF, (see Gradshteyn, Ryzhik (2007, [1] 3.321 2. 3. pg. 336 and pg. 8.250 4. 888)

$$\mathcal{H}_{0,0}(z) = \int_z^{\infty} e^{-u^2}du,\ n = 0$$

$$= \int_0^{\infty} e^{-u^2}du - \int_0^{z} e^{-u^2}du,\ n = 0$$

$$= \frac{\sqrt{\pi}[1-\Phi(z)]}{2},\ n = 0$$

$$= \frac{\sqrt{\pi}}{2}\Phi_c(z),\ n = 0 \tag{98}$$

Let us try to solve (98) in a slightly different way.

If we make the substitution,

$$e^{-u^2} = -\frac{1}{2u}\frac{d\left(e^{-u^2}\right)}{du} \tag{99}$$

and then integrate by parts, we have (see other details in [4])

$$\mathcal{H}_{0,0}(z) = \int_z^{\infty} e^{-u^2}du,\ n = 0$$

$$= -\int_z^{\infty}\frac{1}{2u}\frac{d\left(e^{-u^2}\right)}{du}du,\ n = 0$$

$$= -\left[\frac{e^{-u^2}}{2u}\Big|_z^{\infty} + \frac{1}{2}\int_z^{\infty}\frac{1}{u^2}e^{-u^2}du\right],\ n = 0$$

$$= \frac{e^{-z^2}}{2z} - \frac{1}{2}\int_z^{\infty}\frac{1}{u^2}e^{-u^2}du,\ n = 0$$

$$= \frac{e^{-z^2}}{2z} + \frac{1}{4}\int_z^{\infty}\frac{1}{u^3}\frac{d\left(e^{-u^2}\right)}{du}du,\ n = 0$$

$$= \frac{e^{-z^2}}{2z} + \frac{1}{4}\left[\frac{e^{-u^2}}{u^3}\Big|_z^\infty - 3\int_z^\infty \frac{e^{-u^2}}{u^4}du\right]\ n = 0$$

$$= \frac{e^{-z^2}}{2z} - \frac{1}{4}\frac{e^{-z^2}}{z^3} - \frac{3}{4}\int_z^\infty \frac{e^{-u^2}}{u^4}du\ \ n = 0 \text{ (via induction)}$$

$$= \frac{e^{-z^2}}{2z}\sum_{l=0}^{L} \frac{(-1)^l (1)_{2l}}{2^l (2z^2)^l l!} + R_{L+1}(z),\ n = 0 \tag{100}$$

where the reminder of the integration by parts and invoking induction is equal to,

$$R_{L+1}(z) = \frac{(-1)^{L+1}[2(L+1)]!}{2^{2L+2}(L+1)!}\int_{u_0}^\infty u^{-2L-2}e^{-u^2}du \tag{101}$$

Next, let us consider the case when $n \neq 0$

$$\mathcal{H}_{0,n}(z) = \int_z^\infty u^{-n}e^{-u^2}du,\ n \neq 0$$

$$= -\int_z^\infty \frac{1}{2u^{n+1}}\frac{d\left(e^{-u^2}\right)}{du}du,\ n \neq 0$$

$$= -\left[\frac{e^{-u^2}}{2u^{n+1}}\Big|_z^\infty + \frac{(n+1)}{2}\int_z^\infty \frac{1}{u^{n+2}}e^{-u^2}du\right],\ n \neq 0$$

$$= \frac{e^{-z^2}}{2z^{n+1}} - \frac{(n+1)}{2}\int_z^\infty \frac{1}{u^{n+2}}e^{-u^2}du,\ n \neq 0$$

$$= \frac{e^{-z^2}}{2z^{n+1}} + \frac{(n+1)}{4}\int_z^\infty \frac{1}{u^{n+3}}\frac{d\left(e^{-u^2}\right)}{du}du,\ n \neq 0$$

$$= \frac{e^{-z^2}}{2z^{n+1}} + \frac{(n+1)}{4}\left[\frac{e^{-u^2}}{u^{n+3}}\Big|_z^\infty - (n+3)\int_z^\infty \frac{e^{-u^2}}{u^{n+4}}du\right]$$

$$= \frac{e^{-z^2}}{2z^{n+1}} - \frac{(n+1)}{4}\frac{e^{-z^2}}{z^{n+3}} - \frac{(n+1)(n+3)}{4}\int_z^\infty \frac{e^{-u^2}}{u^{n+4}}du$$

$$= \frac{e^{-z^2}}{2z^{n+1}}\sum_{l=0}^{\infty} \frac{(-1)^l (n+1)_{2l}}{2^l (2z^2)^l l!}\ \text{ix} \tag{102}$$

Since, we will only take a few terms; hence, we have the following,

$$\mathcal{H}_{0,n}(z) \cong \frac{e^{-z^2}}{2z^{n+1}}\sum_{l=0}^{L} \frac{(-1)^l (n+1)_{2l}}{2^l (2z^2)^l l!} \tag{103}$$

If we were to take $n = 0$ in (103) then (100) is obtained; i.e., the computation of the CEF. This is really the most important information about the computation of the CLGPCFD cdf in that it involves the integration of a generalized CEF or GCEF. This concludes the derivations of Appen. E.

13 Appendix F: The Derivation of the Integral (37)

In Appen, F we perform the derivations for the computation of (7)

We make the second substitutions (76)-(80) into (37) we have,

$$\mathcal{G}_{c0}(m,x) = \frac{2a^2\int_{u_0}^\infty e^{-u^2}\left(\frac{u}{\sqrt{2}a} + \frac{1}{c^2}\right)^{-m}du}{\sqrt{2}aa^m}$$

$$= \frac{\sqrt{2}a\int_{u_0}^\infty e^{-u^2}\left(\frac{u}{\sqrt{2}a} + \frac{1}{c^2}\right)^{-m}du}{a^m}$$

$$= a\sqrt{2}^{m+1}\int_{u_0}^\infty e^{-u^2}\left(u + \frac{\sqrt{2}a}{c^2}\right)^{-m}du$$

$$= a\sqrt{2}^{m+1}\int_{u_0}^\infty e^{-u^2}(v+u)^{-m}du \tag{104}$$

Invoking the Binomial theorem [9] we have,

$$(v+u)^{-m} = \sum_{n=0}^{\infty}\binom{-m}{n}u^{-n}v^{-m-n}$$

$$= \sum_{n=0}^{\infty}(-1)^{-n}\binom{m+n-1}{n}u^{-n}v^{-m-n} \tag{105}$$

Next, substituting (105) into (104) produces,

$$\mathcal{G}_{c1}(m,x) = a\frac{\int_{u_0}^\infty e^{-u^2}\sum_{n=0}^{\infty}(-1)^{-n}\binom{m+n-1}{n}u^{-n}v^{-m-n}du}{2^{-\frac{m+1}{2}}}$$

$$= a\frac{\sum_{n=0}^{\infty}(-1)^{-n}\binom{m+n-1}{n}v^{-m-n}\int_{u_0}^\infty e^{-u^2}u^{-n}du}{2^{-\frac{m+1}{2}}} \tag{106}$$

Next, substituting (86) into (106) yields.

$$\mathcal{G}_{c1}(m,x) = a\frac{\sum_{n=0}^{\infty}(-1)^{-n}\binom{m+n-1}{n}v^{-m-n}\frac{e^{-u_0^2}}{2u_0^{n+1}}\sum_{l=0}^{L}\frac{(-1)^l(n+1)_{2l}}{2^l\left(2u_0^2\right)^l l!}}{2^{-\frac{m+1}{2}}}$$

$$= ae^{-u_0^2}\frac{\sum_{n=0}^{\infty}(-1)^{-n}\binom{m+n-1}{n}v^{-n}u_0^{-n}\sum_{l=0}^{L}\frac{(-1)^l(n+1)_{2l}}{\left(2^2u_0^2\right)^l l!}}{2u_0v^m\times 2^{-\frac{m+1}{2}}}$$

$$= ae^{-u_0^2}\frac{\sum_{n=0}^{\infty}\binom{m+n-1}{n}x_1^n\sum_{l=0}^{L}\frac{(n+1)_{2l}x_2'^l}{l!}}{2u_0v^m\times 2^{-\frac{m+1}{2}}} \tag{107}$$

where

$$x_1 = -(vu_0)^{-1} \tag{108}$$

Substituting (90)-(93) into (107) yields.

$$\mathcal{G}_{c1}(m,x) = ae^{-u_0^2}\frac{\left|\begin{array}{l}\sum_{i=0}^{\infty}\frac{\left(\frac{m}{2}\right)_i\left(\frac{m}{2}+\frac{1}{2}\right)_i}{2^{\frac{m}{2}-\frac{1}{2}}\left(\frac{1}{2}\right)_i i!}x_1^{2i}\sum_{l=0}^{L}\frac{2^{2l}\frac{\left(\frac{1}{2}\right)_{l+i}}{\left(\frac{1}{2}\right)_i}\frac{(1)_{l+i}}{(1)_i}x_2'^l}{l!} + \\ \sum_{i=0}^{\infty}\frac{m\left(\frac{m}{2}+\frac{1}{2}\right)_i\left(\frac{m}{2}+1\right)_i}{\left(\frac{3}{2}\right)_i i!}x_1^{2i+1}\sum_{l=0}^{L}\frac{2^{2l}\frac{(1)_{l+i}}{(1)_i}\frac{\left(\frac{3}{2}\right)_{l+i}}{\left(\frac{3}{2}\right)_i}x_2'^l}{l!}\end{array}\right.}{2u_0v^m\times 2^{-\frac{m+1}{2}}}$$

$$= ae^{-u_0^2}\frac{\left|\begin{array}{l}\frac{\sqrt{2}}{2^{\frac{m}{2}}}\sum_{i,l=0}^{\infty,L}\frac{\left(\frac{1}{2}\right)_{l+i}(1)_{l+i}\left(\frac{m}{2}\right)_i\left(\frac{m}{2}+\frac{1}{2}\right)_i}{\left(\frac{1}{2}\right)_i\left(\frac{1}{2}\right)_i(1)_i}\frac{x_1^{2i}}{i!}\frac{2^{2l}x_2'^l}{l!} + \\ mx_1\sum_{i,l=0}^{\infty,L}\frac{(1)_{l+i}\left(\frac{3}{2}\right)_{l+i}\left(\frac{m}{2}+\frac{1}{2}\right)_i\left(\frac{m}{2}+1\right)_i}{(1)_i\left(\frac{3}{2}\right)_i\left(\frac{3}{2}\right)_i}\frac{x_1^{2i}}{i!}\frac{2^{2l}x_2'^l}{l!}\end{array}\right.}{2u_0v^m\times 2^{-\frac{m+1}{2}}}$$

$$= ae^{-u_0^2}\frac{K_{c1e}(m,x) + 2^{0.5m-0.5}mx_1K_{c1o}(m,x)}{u_0v^m} \tag{109}$$

where

$$K_{c1e}(m,x) = \sum_{i,l=0}^{\infty,L} \frac{\left(\frac{1}{2}\right)_{l+i}(1)_{l+i}\left(\frac{m}{2}\right)_i\left(\frac{m}{2}+\frac{1}{2}\right)_i}{\left(\frac{1}{2}\right)_i\left(\frac{1}{2}\right)_i(1)_i}\frac{x_1^{2i}}{i!}\frac{2^{2l}x_2'^{l}}{l!}$$

$$= F_{0:3;0}^{2:2;0}\left[\begin{matrix}\frac{1}{2},1:\frac{m}{2},\frac{m+1}{2};-; \\ -:\frac{1}{2},\frac{1}{2},1;-;\end{matrix} x_1, x_2\right] \tag{110}$$

$$K_{c1o}(m,x) = \sum_{i,l=0}^{\infty,L} \frac{(1)_{l+i}\left(\frac{3}{2}\right)_{l+i}\left(\frac{m}{2}+\frac{1}{2}\right)_i\left(\frac{m}{2}+1\right)_i}{(1)_i\left(\frac{3}{2}\right)_i\left(\frac{3}{2}\right)_i}\frac{x_1^{2i}}{i!}\frac{2^{2l}x_2'^{l}}{l!}$$

$$= F_{0:3;0}^{2:2;0}\left[\begin{matrix}1,\frac{3}{2}:\frac{m+1}{2},\frac{m+2}{2};-; \\ -:1,\frac{3}{2},\frac{3}{2};\end{matrix} x_1, x_2\right] \tag{111}$$

Integral (37) is reduced to the computation of two pairs Kampé de Fériet functions (see Progri (2018, [14]), Append. A). This concludes the derivations of (37).

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[i] Sine this journal paper deals with the topic of a complementary cdf then it is only suitable to talk about women as complementary to men. And the two finest women who I have ever know are my mother, Lumturi Progri, and my sister, Adriana Progri (Dine). Both my mother and my sister wanted me to succeed in school. My sister Adriana Progri (Dine) is almost four years older than I am. Both my mother and my sister complement me on a lot of areas that I am not as good as I would like to be. For example both my mother and my sister are better cooks than I am. They regularly keep in contact with our friends and relatives than I do. They are more hospitable than I am. This is a way of me showing my love, respect, and gratitude to both my mother and my sister.

[ii] The CGPCFD is a type of the complementary cdf or ccdf and it is a measure of computing the tail distribution or exceedance of the cdf [2]. The definition of the ccdf is given as follows: the sum of both cdf and ccdf is equal to one for all values of the main variable,

What are the advantages and disadvantages of computing the ccdf vs the cdf?

The main advantage for computing the ccdf is given below. Say for example that we would like to compute the cdf from the pdf of a function? If the pdf has a symmetry, then it is more convenient to compute the ccdf because (1) it for certain functions the integration is possible and (2) it takes less amount of time to perform the computation.

The main disadvantage for computing the ccdf is that it requires a separate expansion of the pdf because of the integration that is involved. This is exactly what is illustrated in this journal paper in the computation of the CLGPCFD pdf and cdf.

[iii] The meaning of the *complementary* here is that of combining in such a way as to *enhance or emphasize the qualities of each other or another*; i.e., harmonizing, harmonious, complementing, supportive, supporting, reciprocal, interdependent, interrelated, compatible, corresponding, matching, twin, completing, finishing, perfecting, complemental. The Opposite is incompatible, contrasting. This definition of the PCF is such that is only suitable for the computation of the CLGPCFD cdf.

[iv] In Gradshteyn, Ryzhik (2007, [1] 9.246 pg. 1029) the sign $\sim$ is used because only three terms are shown instead of $\cong$ used in (7) in which an unlimited number of terms are shown. Moreover, in (6) and (7) the computation of the CPCF is explained in greater details than in Gradshteyn, Ryzhik (2007, [1] 9.246 pg. 1029) as explained from Appen. A.

[v] The complementary pdf is exactly the same as the normal pdf but is employed for large values of the main variable; i.e., this pdf is suitable for the computation of the ccdf or CLGPCDFD cdf. Ditto ii.

[vi] Ditto i.

[vii] Ditto Progri (2021, [16], iv)

[viii] It is also called the Legendre duplication formula or Legendre relation, in honor of Adrien-Marie Legendre.

[ix] The complete proof of (102) can be given by means of the total mathematical induction.