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Research Article

On the Computation of the Landmark Generating Function of the Hermite Polynomials

To my grandfather, Xhevdet Progri, and my father, Fiqiri Progri, the men who have supported me the most during my early education all the way to my Graduate School.[i]

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The landmark computation of the exponential function when the exponent is a quadratic function of the variable via a landmark expansion of the Hermite polynomials is a continuation of study of the computation of the exponential function for negative exponents that Progri performed in 2021. Since, for producing the closed form expression of the expansion of the exponential function via series of Hermite polynomials, the power series of the exponential functions was employed for negative exponents, this expansion was shown to lack convergence for values of the exponent main variable larger than one in absolute value. The main purpose of this paper is to explain and fix the computation of the exponent when the exponent is a quadratic function via a landmark expansion of the Hermite polynomials for values of the exponent's main variable larger than one in absolute value. This journal paper offers new insights into the computation of the exponential functions for negative exponents as it applies to the expansion of the Hermite polynomials, its derivative, integral. Two new expressions of the sum of the error functions via a generalized hypergeometric function when one of the Pochhammer symbols is replaced by a Hermite polynomial is obtained. The theoretical results are verified with MATLAB simulation results which offer a tremendous confidence into the computation of the exponential functions when the exponent is a quadratic function, the derivative, and the integral and of the newly obtained equations.

Index Terms—Exponential function, power series expansion, positive exponents, negative exponents, Hermite polynomials, quadratic exponent, landmark generating function, derivative computation, integral computation, error function, generalized hypergeometric functions.

1 Introduction

New results in the computation of the exponential function when the exponent is a quadratic function of variable via the series of Hermite polynomials offers tremendous insights into a specific type of the expansion of the exponential functions for negative exponents of the main variable in which the expansion is performed via series of Hermite polynomials.

Stone (1927, [1]) wrote and published a paper on the Development of the Hermite Polynomial that uses the power series of the exponential functions for negative exponents.

The Stone (1927, [1]) formula has made it all the way to the *Tables of Integrals, Product, and Series* (Gradshteyn, Ryzhik, 2007, [2] pg. 997, ex. 8.997.1).

From 2016 until 2022 I have employed most of the closed form expressions of Gradshteyn, Ryzhik (2007, [2]) in Progri (2016, [3]) to Progri (2022, [15]).

In his paper Stone (1927, [1]) failed to recognize the flaws of the expansion of the exponential function for quadratic exponents when the main variable is large and negative that we find in Progri (2022, [16]).

From Progri (2022, [16]) we know that the power series expansion of the exponential function for negative exponents diverges for large values of magnitude of the variable.

First, why does the existing (or classical or conventional or traditional or typical or common) power series expansion of the exponential function via the series of Hermite polynomials fail for large values of the magnitude of the exponents when the exponent is a quadratic function of the main variable?

Second, in Sect. 2 I argue analytically that the existing (or classical or conventional or traditional or typical or common) generating function of the Hermite polynomials, e^{-x^2+zx}, for large values of the exponent $\forall x \in \mathbb{R}^{0+}$ is either incorrect, erroneous, misleading, inaccurate, or suffers from serious convergence issues.

Third, is it possible to produce a modified closed form expression of the power series expansion of the exponential function, via the series of Hermite polynomials, e^{-x^2+zx} for large values of the exponent $\forall x \in \mathbb{R}^{0+}$? The modified power series expansion of the exponential function via the series of the Hermite polynomials, e^{-x^2+zx}, for negative exponents provides the much-needed correction of the computation of the exponential function for negative exponents, and for large values of the variable $\forall x \in \mathbb{R}^{0+}$.

This paper is organized as follows: in Sect. 2 the computation of the exponential function via series of Hermite polynomials is discussed. The landmark generating function (LGF) of the Hermite Polynomials is presented in Sect. 3. The computation of the derivative of the LGF of the Hermite Polynomials is derived in Sect. 4. The computation of the integral of the LGF of the Hermite Polynomials in discussed in Sect. 5. Section 6 contains (or accommodates, consist of, encloses, encompasses, has, incorporates, involves) numerical, theoretical results; Conclusion is provided in Sect. 7, Acknowledgement in Sect. 8 along with a list of References in Sect. 9. In Appendix A we give a brief review on the power series expansion of the error function and connection with a generalized hypergeometric function when one of the one of the Pochhammer symbols is replaced by a Hermite Polynomial.

2 The Generating Function of the Hermite Polynomials

The conventional computation of the generating function of the Hermite Polynomials as pointed out by Stone (1927, [1]) and (Gradshteyn, Ryzhik, 2007, [2], pg. 997, ex. 8.997.1) is given by

$$e^{-ax'^2+2\frac{z'}{\sqrt{a}}\sqrt{a}x'} = \sum_{k=0}^{\infty} \frac{H_k\left(\frac{z'}{\sqrt{a}}\right)\left(\sqrt{a}x'\right)^k}{k!};\ \left|\begin{matrix} -\infty < x' < \infty \\ a > 0, |z'| < \infty \end{matrix}\right. \tag{1}$$

where the parameter a is a scaling factor.

If we make the substitution

$$x = \sqrt{a}x' \tag{2}$$

$$z = \frac{z'}{\sqrt{a}} \tag{3}$$

Then (1) becomes:

$$e^{-x^2+2zx} = \sum_{k=0}^{\infty} \frac{H_k(z)x^k}{k!};\ \left|\begin{matrix} -\infty < x < \infty \\ |z| < \infty \end{matrix}\right. \tag{4}$$

We can go back and forth from either (1) to (4) or vice versa via the substitutions (2) and (3). However, for the majority of the theoretical developments we will use the definition (4) because it is much easier to write equations.

Why does this generating function of the Hermite polynomial have converge problems?

First, let us take into consideration the easiest case when $z = 0$. From the Hermite polynomial generating function, e^{-x^2+zx}, $\forall x \in \mathbb{R}^{0+}$ as in (1), when $z = 0$ we have the following:

$$e^{-x^2+2(z=0)x} = \sum_{k=0}^{\infty} \frac{H_k(0)x^k}{k!};\ -\infty < x < \infty$$

$$= \sum_{\substack{k=0\\ \text{even}}}^{\infty} \frac{H_{2n}(0)x^{2n}}{(2n)!} + \sum_{\substack{k=1\\ \text{odd}}}^{\infty} \frac{H_{(2n+1)}(0)x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!x^{2n}}{(-1)^{-n}2^{-n}(2n)!} + \frac{H_{(2n+1)}(0)x^{2n+1}}{(2n+1)!}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (2n-1)!!x^{2n}}{(2n)!} + 0$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (2n)!x^{2n}}{(2n)!2^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!};\ -\infty < x < \infty \tag{5}$$

We should not be surprised that (2) is identical to Gradshteyn, Ryzhik (2007, [2], 1.21 1.211 ex. 3., pg. 26)).

Equating the first and the last equality in (5) (which is identical to Gradshteyn, Ryzhik (2007, [2], 8.956 ex. 6., 7. pg. 997)) we obtain:

$$H_k(0) = \begin{cases} (-1)^n 2^n (2n-1)!! & k = 2n = \text{even} \\ 0 & k = 2n+1 = \text{odd} \end{cases} \tag{6}$$

In Gradshteyn, Ryzhik (2007, [2], 8.952 ex. 2. pg. 996)) we make use of the recursive expression later on in the paper as follows:

$$H_{k+1}(0) = -2kH_{k-1}(0) \tag{7}$$

However, as pointed out by Progri (2022, [16]) (5) converges to its true value if and only if, (iff) $0 \le x < 1$ and it diverges if $1 < x < \infty$. Clearly, the generating function of the Hermite polynomial when $z = 0$ is as follows:

$$e^{-x^2+2(z=0)x} = \sum_{k=0}^{\infty} \frac{H_k(0)x^k}{k!};\ 0 \le x < \infty$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} + 0;\ 0 \le x < 1 \tag{8}$$

Second, let us consider the more general case when $z \ne 0$. For this case we obtain the following:

$$e^{-x^2+2zx} = e^{-x^2}e^{2zx};\ -\infty < x < \infty$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} \sum_{n=0}^{\infty} \frac{2^n z^n x^n}{n!}$$

$$= \left| \begin{array}{l} \left(1 - x^2 + \cdots - \frac{x^{2k}}{k!} + \cdots\right) \cdot \\ \left(1 + 2zx + \cdots + \frac{2^n z^n x^n}{n!} + \cdots\right) \end{array} \right.$$

$$= \left| \begin{array}{c} 1 + 2zx + \cdots + \frac{2^n z^n x^n}{n!} + \cdots \\ -x^2\left(1 + 2zx + \cdots + \frac{2^n z^n x^n}{n!} + \cdots\right) \\ \cdots \\ -\frac{x^{2k}}{k!}\left(1 + 2zx + \cdots + \frac{2^n z^n x^n}{n!} + \cdots\right) \end{array} \right.$$

$$= 1 + 2zx + \left(\frac{4z^2}{2!} - 1\right)x^2 + \left(8\frac{z^3}{3!} - 2z\right)x^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{H_k(z)x^k}{k!};\ \left| \begin{array}{l} -\infty < x < \infty \\ |z| < \infty \end{array} \right. \tag{9}$$

Similarly, as indicated in by Progri (2022, [16]) (9) converges to its true value (iff) $0 \le x < 1$ and it diverges if $1 < |x| < \infty$. Clearly, the generating function of the Hermite polynomial when $z \ne 0$ is as follows:

$$e^{-x^2+2zx} = e^{-x^2}e^{2zx};\ 0 \le x < 1$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} \sum_{n=0}^{\infty} \frac{2^n z^n x^n}{n!}$$

$$= 1 + 2zx + \left(\frac{4z^2}{2!} - 1\right)x^2 + \left(8\frac{z^3}{3!} - 2z\right)x^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{H_k(z)x^k}{k!};\ \left| \begin{array}{l} 0 \le x < 1 \\ 0 \le z < \infty \end{array} \right. \tag{10}$$

Having identified that the existing (or classical or conventional or traditional or typical or common) generating function of the Hermite polynomials, e^{-x^2+zx}, for large values of the exponent $\forall x \in \mathbb{R}^{0+}$ is either incorrect (, erroneous, misleading, inaccurate,) or suffers from serious convergence issues. Therefore, I have no other option but to propose a modification of the generating function of the Hermite Polynomials which is performed next.

3 LGF of the Hermite Polynomials

In the previous section we identified that the generating function of the Hermite polynomials for negative exponents needs to be enhanced. In this section we propose a computation of the LGF of for negative exponents for large values of the variable x.

First, let us consider the case when $z = 0$. Let us write the variable x^2 as follows:

$$x = (\lceil x \rceil \equiv n) - y \tag{11}$$

where $\lceil x \rceil \equiv n = \{1,2,\cdots\}$ is the ceiling of x (or the integer portion of the number, such that $x \le n$, and $0 \le y < 1$ is the fractional part of the variable x. Substituting (11) into the expansion of the exponential function, e^{-x^2}, for $\forall x \gg 1 \in \mathbb{R}^+$ we obtain:

$$e^{-x^2} = e^{-(n-y)^2} = e^{-n^2}e^{2ny}e^{-y^2} \tag{12}$$

The computation of the integer part of (12) is done in a way that we discussed in Progri (2022, [16], (23) pg. 17) as

$$e^{-n} = (e^{-1})^n = (1/e)^n \tag{13}$$

The computation of (12) can be performed as follows:

$$e^{-x^2} = e^{-n^2}e^{2ny}e^{-y^2}$$

$$= e^{-n^2}e^{-y^2+2ny}$$

$$= e^{-n^2}\sum_{k=0}^{\infty}\frac{H_k(n)y^k}{k!};\ \begin{vmatrix}0 \le y < 1\\ n = \{1,2,\cdots\}\end{vmatrix} \tag{14}$$

Next, let us consider the more general case when $z \neq 0$. In this case. for the computation of the fractional part we can use the series expansion because $0 \le y < 1$

$$\begin{aligned} e^{-x^2+2zx} &= e^{-x^2+2zx-z^2+z^2} \\ &= e^{-x^2+2zx-z^2}e^{z^2} \\ &= e^{-(x-z)^2}e^{z^2} \\ &= e^{-(n-y)^2}e^{z^2};\ 0 \le y < 1 \\ &= e^{-n^2+2ny-y^2}e^{z^2};\ 0 \le y < 1 \\ &= e^{-n^2}e^{z^2}\sum_{k=0}^{\infty}\frac{H_k(n)y^k}{k!};\ \begin{vmatrix}0 \le y < 1\\ 0 \le z < \infty\end{vmatrix} \end{aligned} \tag{15}$$

In this expression we made use of the expansion of

$$x - z = (\lceil x - z\rceil \equiv n) - y \tag{16}$$

where $\lceil x - z\rceil \equiv n = \{1,2,\cdots\}$ is the ceiling of $x - z$ (or the integer portion of the number, such that $x - z \le n$, and $0 \le y < 1$ is the fractional part of the variable $x - z$.

Therefore, the LGF of the Hermite Polynomials for negative exponents for large values of the variable x can be written as

$$e^{-x^2+2zx} = e^{-n^2}e^{z^2}\sum_{k=0}^{\infty}\frac{H_k(n)y^k}{k!};\ \begin{vmatrix}0 \le y < 1\\ 0 \le z < \infty\end{vmatrix} \tag{17}$$

where integer n is subject to (16).

Equation (17) provides a significantly improved understanding of the expansion via the landmark Hermite Polynomials that hopefully should not have any convergence problems. We need, however, to exploit further the computation of the derivative and of the integral which are performed in the upcoming Sects.

4 Computation of the Derivative of the LGF of the Hermite Polynomials

The derivation of the landmark expansion of the Hermite Polynomials that was performed in the pervious Sect. needs to pass the analytical and computational test of the derivative.

For this reason we start with the simplified expression of the exponential function as e^{-x^2}. If we take the derivative of the exponential function e^{-x^2} of the lefthand side of (5) or (8) we obtain:

$$\frac{de^{-x^2}}{dx} = -2xe^{-x^2} \tag{18}$$

Next, let us take the derivative of the righthand side of (5) or (8), employing the series expansion of (5) or (8)

$$\begin{aligned} \frac{de^{-x^2}}{dx} &= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{k!} \\ &= \sum_{k=0}^{\infty}\frac{d}{dx}\frac{(-1)^k x^{2k}}{k!} \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{d}{dx}\frac{x^{2k}}{k!} \\ &= 2\sum_{k=1}^{\infty}(-1)^k\frac{kx^{2k-1}}{k!} \\ &= 2\sum_{k=1}^{\infty}(-1)^k\frac{x^{2k-1}}{(k-1)!} \\ &= 2\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{n!} \\ &= 2(-1)x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!} \\ &= -2xe^{-x^2} \end{aligned} \tag{19}$$

We are not surprised that (19) is identical to (18).

Next, let us repeat (18) using the Hemite Polynomial expansion of (9):

$$\begin{aligned} \frac{de^{-x^2}}{dx} &= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{H_k(0)x^k}{k!} \\ &= \sum_{k=0}^{\infty}H_k(0)\frac{d}{dx}\frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty}H_k(0)k\frac{x^{k-1}}{k!} \\ &= 0 + \sum_{k=1}^{\infty}H_k(0)\frac{x^{k-1}}{(k-1)!} \\ &= \sum_{n=0}^{\infty}H_{n+1}(0)\frac{x^n}{n!} \\ &= -2\sum_{n=1}^{\infty}nH_{n-1}(0)\frac{x^n}{n!}\ \text{(see (4))} \\ &= -2\sum_{k=0}^{\infty}(k+1)H_k(0)\frac{x^{k+1}}{(k+1)!} \\ &= 2(-1)x\sum_{k=0}^{\infty}H_k(0)\frac{x^k}{k!} \\ &= -2xe^{-x^2} \end{aligned} \tag{20}$$

Again, we should not be surprised that (20) is also identical to (19) which is also identical and (18).

Next, let us take the derivative of (8) utilizing the landmark expansion of the Hermite Polynomial (14) yields:

$$\begin{aligned} \frac{de^{-x^2}}{dx} &= -\frac{d}{dy}e^{-n^2}\sum_{k=0}^{\infty}\frac{H_k(n)y^k}{k!} \\ &= -e^{-n^2}\sum_{k=0}^{\infty}\frac{d}{dy}\frac{H_k(n)y^k}{k!} \\ &= -e^{-n^2}\sum_{k=0}^{\infty}H_k(n)\frac{d}{dy}\frac{y^k}{k!} \\ &= -e^{-n^2}\sum_{k=0}^{\infty}H_k(n)\frac{ky^{k-1}}{k!} \end{aligned}$$

$$= 0 - e^{-n^2} \sum_{k=1}^{\infty} H_k(n) \frac{y^{k-1}}{(k-1)!}$$

$$= -e^{-n^2} \sum_{m=0}^{\infty} H_{m+1}(n) \frac{(m+1)y^m}{(m+1)!}$$

$$= -e^{-n^2} \left[2n + \sum_{m=1}^{\infty} H_{m+1}(n) \frac{y^m}{m!}\right]$$

$$= -e^{-n^2} \left\{2n + \sum_{m=1}^{\infty} [2nH_m(n) - 2mH_{m-1}(n)] \frac{y^m}{m!}\right\}$$

$$= -e^{-n^2} 2\left[n \sum_{m=0}^{\infty} H_m(n) \frac{y^m}{m!} - \sum_{m=1}^{\infty} H_{m-1}(n) \frac{y^m}{(m-1)!}\right]$$

$$= -e^{-n^2} 2\left[n \sum_{k=0}^{\infty} H_k(n) \frac{y^k}{k!} - y \sum_{m=1}^{\infty} H_k(n) \frac{y^k}{k!}\right]$$

$$= -e^{-n^2} 2(n - y) \sum_{m=1}^{\infty} H_k(n) \frac{y^k}{k!}$$

$$= 2(-1)x \sum_{k=0}^{\infty} H_k(n) \frac{x^k}{k!}$$

$$= -2xe^{-x^2} \tag{21}$$

Finally, we should not be surprised that (21) is also identical to (20), (19), and (18). It is more likely that the analytical derivations of (14) are correct. We further need to verify the accuracy via the computation of (14) in Sect. 6.

Next, let us exploit the more general case; if we take the derivative of the exponential function e^{-x^2+2zx} or the lefthand side of (9) we obtain:

$$\frac{de^{-x^2+2zx}}{dx} = 2(z - x)e^{-x^2+2zx} \tag{22}$$

Next, let us assume that $0 \leq x < 1$, $0 \leq z < \infty$, let us repeat (22) using the Hemite Polynomial expansion of (10):

$$\frac{de^{-x^2}}{dx} = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{H_k(z)x^k}{k!};$$

$$= \sum_{k=0}^{\infty} H_k(z) \frac{d}{dx} \frac{x^k}{k!}$$

$$= \sum_{k=0}^{\infty} H_k(z) k \frac{x^{k-1}}{k!}$$

$$= 0 + \sum_{k=1}^{\infty} H_k(z) \frac{x^{k-1}}{(k-1)!}$$

$$= \sum_{n=0}^{\infty} H_{n+1}(z) \frac{x^n}{n!}$$

$$= 2z \sum_{n=0}^{\infty} H_n(z) \frac{x^n}{n!} - 2 \sum_{n=1}^{\infty} nH_{n-1}(z) \frac{x^n}{n!} \text{ (see (7))}$$

$$= 2z \sum_{n=0}^{\infty} H_n(z) \frac{x^n}{n!} - 2 \sum_{k=0}^{\infty} (k+1)H_k(z) \frac{x^{k+1}}{(k+1)!}$$

$$= 2z \sum_{n=0}^{\infty} H_n(z) \frac{x^n}{n!} + 2(-1)x \sum_{k=0}^{\infty} H_k(z) \frac{x^k}{k!}$$

$$= 2ze^{-x^2+zx} - 2xe^{-x^2+zx}$$

$$= 2(z - x)e^{-x^2+zx} \tag{23}$$

It is no wonder (or as expected, or as anticipated or only to be predicted or as it was to be expected) that (23) is identical to (22).

Next, we perform the derivation of the derivative of (23) based on the landmark Hermite Polynomial expansion (17) as follows:

$$\frac{de^{-x^2+2zx}}{dx} = \frac{d}{dx} e^{-n^2} e^{z^2} \sum_{k=0}^{\infty} \frac{H_k(n)y^k}{k!}$$

$$= -e^{-n^2} e^{z^2} \frac{d}{dy} \sum_{k=0}^{\infty} \frac{H_k(n)y^k}{k!}$$

$$= -e^{-n^2} e^{z^2} \sum_{k=0}^{\infty} \frac{d}{dy} \frac{H_k(n)y^k}{k!}$$

$$= -e^{-n^2} e^{z^2} \sum_{k=1}^{\infty} \frac{H_k(n)y^{k-1}}{(k-1)!}$$

$$= -e^{-n^2} e^{z^2} \sum_{k=0}^{\infty} \frac{H_{k+1}(n)y^k}{k!}$$

$$= -e^{-n^2} e^{z^2} \sum_{k=0}^{\infty} \frac{[2nH_k(n) - 2kH_{k-1}(n)]y^k}{k!}$$

$$= -e^{-n^2} e^{z^2} \left[2n \sum_{k=0}^{\infty} \frac{H_k(n)y^k}{k!} - 2 \sum_{k=0}^{\infty} \frac{kH_{k-1}(n)y^k}{k!}\right]$$

$$= -e^{-n^2} e^{z^2} \left[2n \sum_{k=0}^{\infty} \frac{H_k(n)y^k}{k!} - 2 \sum_{k=1}^{\infty} \frac{kH_{k-1}(n)y^k}{k!}\right]$$

$$= -e^{-n^2} e^{z^2} 2(n - y) \sum_{k=0}^{\infty} \frac{H_k(n)y^k}{k!}$$

$$= -2(x - z)e^{-x^2+zx} \tag{24}$$

As it was predicted (24) is in absolute agreement with (23) and (22), in exactly the same form and produces exactly the same analytical answer as (23) and (22). We may not be able to confirm the same answer about the numerical computation similarity of (24) with (23) and (22).

This ends, (or arrives at the conclusion, or closes or finishes) the analytical discussion of the computation of the derivative of the LGF of the Hermite Polynomials. Next, we continue the discussion with the computation of the integral of the LGF of the Hermite Polynomials.

5 Computation of the Integral of the LGF of the Hermite Polynomials

When I performed the study of the computation of the exponential function in Progri (2022, [16]) I did not spent a lot of time or I did not provide a lot of details on the similarities, differences, of the analytical and computational differences of various techniques that I proposed. I think I performed rather quickly the integration of the exponential function.

Similar to the computation of the derivative of the Hermite Polynomials, the computation of the integral of the exponential function can be performed in at least four different ways based on at least four different analytical representations of the exponential function.

The first approach is similar to the one proposed in Progri (2022, [16]), which is known as the direct integration, if we take the integral of the exponential function e^{-x^2+2zx} we obtain:

$$\begin{aligned}\int_0^x e^{-t^2+2zt}dt &= \int_0^x e^{-t^2+2zt-z^2}e^{z^2}dt\\ &= e^{z^2}\int_0^x e^{-t^2+2zt-z^2}dt\\ &= e^{z^2}\int_0^x e^{-(t-z)^2}dt\\ &= e^{z^2}\int_{-z}^{x-z} e^{-y^2}dy\\ &= e^{z^2}\left[\int_0^{x-z} e^{-y^2}dy + \int_{-z}^{0} e^{-y^2}dy\right]\\ &= e^{z^2}\left[\int_0^{x-z} e^{-y^2}dy + \int_0^{z} e^{-y^2}dy\right]\\ &= e^{z^2}\frac{\sqrt{\pi}[\Phi(x-z)+\Phi(z)]}{2};\ \Phi(x)\equiv \operatorname{erf}(x) \end{aligned} \tag{25}$$

The second approach is based on computation of the righthand side of (25) via (1) as follows:

$$\begin{aligned}\int_0^x e^{-t^2+2zt}dt &= \int_0^x \sum_{k=0}^{\infty}\frac{H_k(z)t^k}{k!}dt\\ &= \sum_{k=0}^{\infty}\frac{H_k(z)\int_0^x t^k dt}{k!}\\ &= \sum_{k=0}^{\infty}\frac{H_k(z)t^{k+1}\big|_0^x}{k!(k+1)}\\ &= \sum_{k=0}^{\infty}\frac{H_k(z)x^{k+1}}{k!(k+1)}\\ &= x\sum_{k=0}^{\infty}\frac{H_k(z)x^k}{(k+1)!}\\ &= x\sum_{k=0}^{\infty}\frac{(1)_k H_k(z)x^k}{(2)_k k!}\\ &= x\,{}_2F_1[H(z),1;2;x]; \end{aligned} \tag{26}$$

Equation (26) is supposed to suffer from the same computational limitations; i.e., it converges (iff) $0 \le x < 1$ and it diverges if $1 < |x| < \infty$.

Next, in the third (and final) approach we have to perform the integral of the exponential function e^{-x^2+2zx} based on (or dependent, contingent, in regard to, pertaining to, stemming from, and connected with) a similar computation as the one discussed in the landmark computation of the Hermite Polynomial (15), or (17) as follows:

$$x = (\lfloor x\rfloor \equiv n) + y, \tag{27}$$

where $\lfloor x\rfloor \equiv n = \{1,2,\cdots\}$ is the floor of x (or the integer portion of the number, such that $n \le x$, and $0 \le y < 1$ is the fractional part of the variable x.

Substituting (25) into the integral of the exponential function e^{-x^2+2zx} we obtain, for $\forall x \gg 1 \in \mathbb{R}^+$ we obtain:

$$\begin{aligned}\int_0^x e^{-at^2+2zt}dt &= \int_0^n e^{-at^2+2zt}dt + \int_n^{n+y} e^{-at^2+2zt}dt\\ &= I_n(a,z) + \int_0^y e^{2z(n+w)-a(n+w)^2}dw\\ &= I_n(a,z) + \frac{\int_0^y e^{-aw^2+2(z-n)w}dw}{e^{an^2-2zn}}\\ &= I_n(a,z) + \frac{y\,{}_2F_1\left[H\left(\frac{z}{\sqrt{a}}-n\sqrt{a}\right),1;2;y\sqrt{a}\right]}{e^{an^2-2zn}} \end{aligned} \tag{28}$$

where $I_n(a,z)$ is the integral portion that contains the integer computation as follows:

$$I_n(a,z) = \frac{\sqrt{\pi}\left[\Phi\left(n\sqrt{a}-\frac{z}{\sqrt{a}}\right)+\Phi\left(\frac{z}{\sqrt{a}}\right)\right]}{2\sqrt{a}e^{-z^2/a}} \tag{29}$$

The main reason for showing (29) is because of the scaling factor of $\sqrt{a}$ in the denominator of (29).

Equations (28) and (29) should be free of any computation issues found in (26).

In this section Dr. Progri provided two brilliant, new expressions (or new identities) of the sum of two error functions with the Hermite Polynomials that did not exist before (26) and (28).

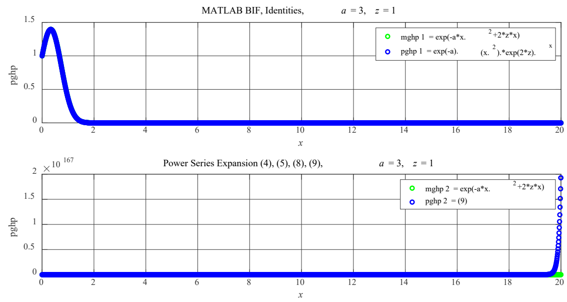
We have been able to clearly illustrate that the integration of the exponential function when the exponent is a quadratic equation of the variable x there are at least three distinct analytical expressions of the integral. We did not include a fourth approach which is via numerical integration. This approach will be discussed in further detail in Sect. 6.

This ends, (or arrives at the conclusion, or closes or finishes) the analytical discussion of the computation of the integral of the LGF of the Hermite Polynomials. Next, we continue the discussion with the computation of the numerical, theoretical results.

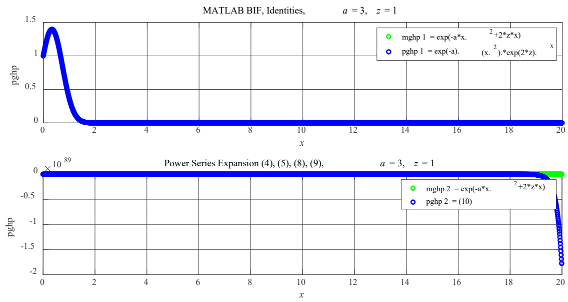
6 Numerical, Theoretical Results

The numerical theoretical results present the computation of the exponential function e^{-x^2+2zx} via either MATLAB identities or serves as a verification of the analytical derivations presented in Sects. 2-5.

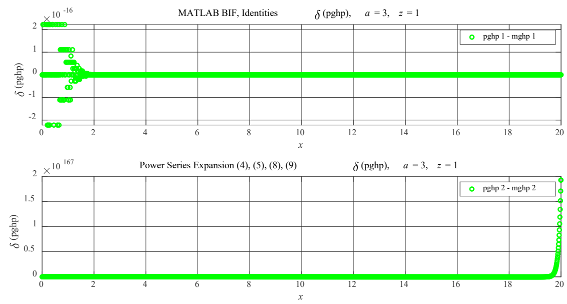
The main purpose of this section is to illustrate numerically what was discussed in Sects. 2-5 is accomplished via MATLAB simulations via the utilization of the MATLAB built in function (BIF) and identities.



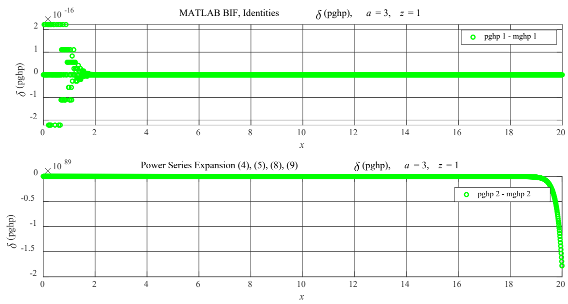
(a) the computation of the exponential function (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Power Series Expansion (9) is option 1.



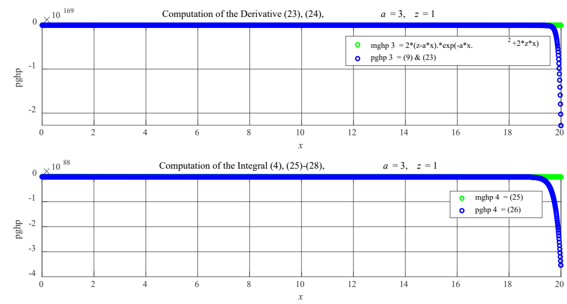
(a) the computation of the exponential function (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Power Series Expansion via (10) is option 2.



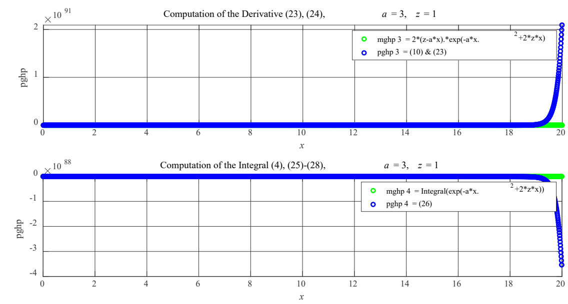
(b) The error δ of (a) MATLAB BIF vs. Power Series Expansion (9) is option 1.



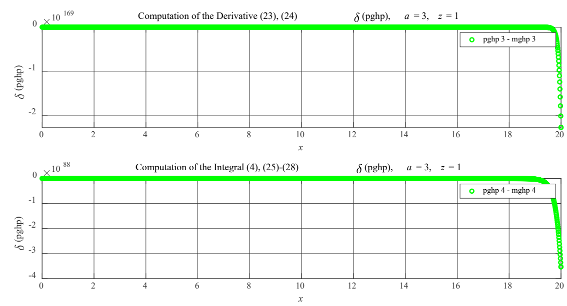
(b) The error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) Power Series Expansion via (10).



(c) computation of the derivative of the exponential function (22) (top) Integral (26); (bottom) MATLAB BIF vs. Power Series (9).

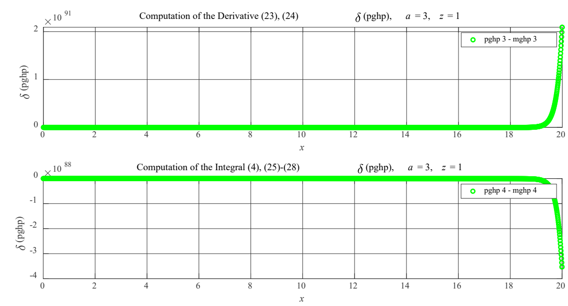


(c) computation of the derivative (23) of the exponential function (top) Integral (26); (bottom) MATLAB BIF vs. Power Series Expansion (10).



(d) The error; δ, (top) derivative (c), (bottom) integral (c) MATLAB BIF vs. Power Series (9) and (22).

FIGURE 1: Numerical calculations of (a) exponential function e^{-3x^2+2x}, (b) error of (a), (c) derivative (22) and integral (26) of (a), (d) error of (c) MATLAB BIF vs. Power Series (9).



(d) Error; δ, (top) Computation of the Derivative (23), (bottom) Integral (26).

FIGURE 2: Numerical calculations of (a) exponential function e^{-3x^2+2x}, (b) error of (a), (c) derivative (23) and integral (26) of (a), (d) error of (c) MATLAB BIF vs. Power Series (10).

There are four main exponential functions that we are comparing and contrasting one vs. another. The first function is via either MATLAB BIF or identities called mhpp. The second function is via the series expansion of the exponential function via (9), (10), and (15) called phpp. The third function is the computation of the derivative of the exponential function via either MATLAB identities or via (22), (23), or (24) called dhpp. And the fourth and final function is the computation of the integral via either MATLAB identities or via (25), (26) or (28) called ihpp.

For the computation of the second function, phpp, we illustrate four cases:

1. the Hermite Polynomial expansion without any modifications (or corrections) (9).
2. the Hermite Polynomial expansion without any modifications (or corrections) (10).
3. the LGF of Hermite Polynomial series expansion via (15) or (17).
4. when appropriate we take the logarithm to illustrate the computation of the modified power series expansion via (20).

The mhpp is considered the truth. This function results either from MATLAB BIF or identities of BIFs. Each function has two outputs: mhp and php. The function mhp is computed wither via MATLAB BIF or identities of BIFs. The php is computed via identities.

The main question we would like to answer from this work is as follows:

How does the computation of the exponential function e^{-x^2+2zx} via php compares to the ones obtained from MATLAB identities of BIFs mhp?

As a performance measure of the comparisons, we define the absolute error, δ, via the following:

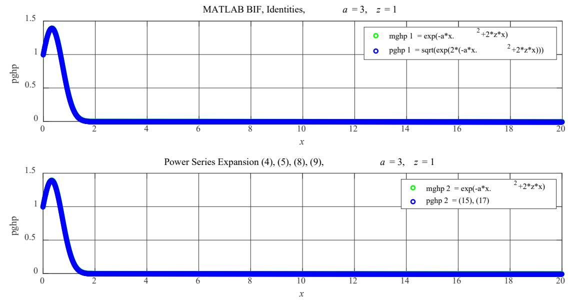
$$\delta = \text{php} - \text{mhp} \quad (30)$$

The numerical results in the following Subsects. will illustrate the degree of similarity of the php from mhp.

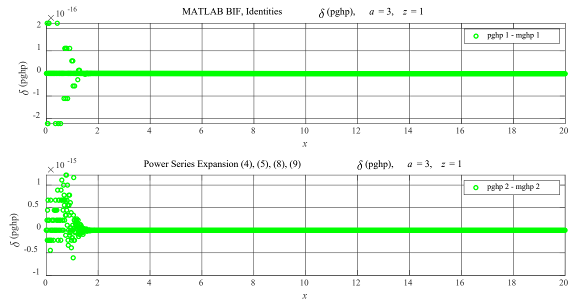
Let us describe the main simulation parameters: the vector of the variable x starts from zero until twenty with a step size of $dx = 1 \times 10^{-2}$. The total number of points from vector x is equal to 2×10^{3}. Other simulation parameters are similar to the ones discussed in Progri (2021, [10])-Progri (2022, [15]),

6.1 Description of the Expansion of the Hermite Polynomials for Negative Exponents

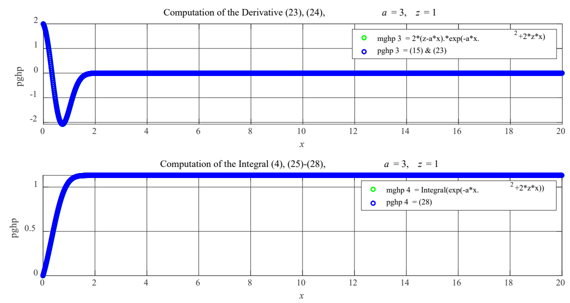
First, we illustrate why the expansion of the exponential function, for a quadratic exponent, needs modification.



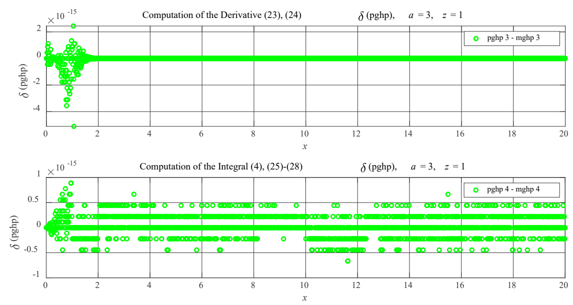
(a) the computation of the exponential function e^{-3x^2+2x} (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Modified Power Series Expansion via (15).



(b) The error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) Modified Power Series Expansion via (15).



(c) computation of the derivative (24) of the exponential function (top) Integral (28); (bottom) MATLAB BIF vs. Power Series Expansion (15).



(d) Error; δ, (top) Computation of the Derivative (24), (bottom) Integral (28).

FIGURE 3: Numerical calculations of (a) exponential function e^{-3x^2+2x}, (b) error of (a), (c) derivative (24) and integral (28) of (a), (d) error of (c) MATLAB BIF vs. Power Series (15).

Figure 1(a) illustrates the plots of (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Power Series Expansion via (9).

Figure 1(b) depicts the computation of the error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) Power Series Expansion via (9).

Figure 1(c) illustrates the plots of (top) derivative of the MATLAB BIF vs MATLAB identities; (bottom) integral of MATLAB BIF vs Power Series Expansion via (9).

Figure 1(d) depicts the computation of the error; δ, Power Series Expansion via (9) minus the MATLAB identities of MATLAB BIFs (top) derivative via (22), (bottom) integral (25).

As indicated from the simulation results of Fig. 1(a)-(d), the Hermite Polynomial expansion of the exponential function, e^{-x^2+2zx}, fails because the error grows exponentially to hundreds of orders of magnitude; therefore, it needs modification for large values of the exponent or variable x. This is option 1 in the simulation.

Figure 2(a)-(d) is identical to Fig. 1(a)-(d). The only difference is in that we employed (10), (23), and (26) to generate the results of Fig. 2(a)-(d) vs. (9), (22), and (25) for Fig. 1(a)-(d).

As indicated from the simulation results of Fig. 2(a)-(d), the Hermite Polynomial expansion of the exponential function, e^{-x^2+2zx}, fails; therefore, it needs modification for large values of the exponent or variable x. This is option 2 in the simulation.

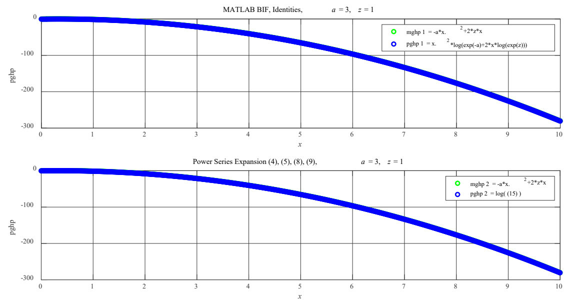
Hence, the root cause of the failure of the computation of the parabolic cylinder function cdf is as a result of the lack of modification or clarification from the Hermite Polynomial series expansion of the exponential function, e^{-x^2+2zx}.

6.2 *Landmark Expansion of the Hermite Polynomial of the Exponential Function for Negative Exponents*

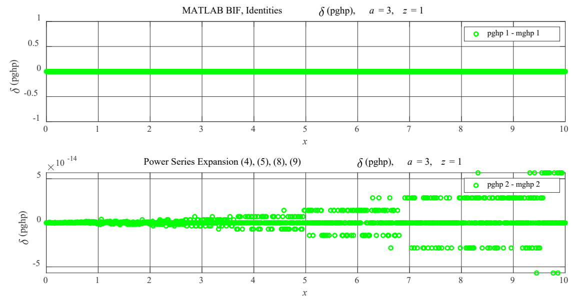
In this Subsect. we repeat exactly the same simulation scenario that we ran in Subsect 6.1 with the only difference we employ the modified power series expansion of the exponential function as given by (20) instead of the standard power series expansion of the exponential function in (2).

Figure 3(a)-(d) is identical to Fig. 1(a)-(d). The only difference is in that we employed (15), (24), and (28) to generate the results of Fig. 3(a)-(d) vs. (9), (22), and (25) for Fig. 1(a)-(d).

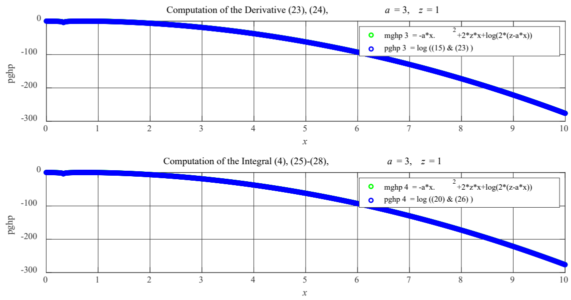
As shown from the results of Fig. 3(a)-(d) the LGF of the Hermite Polynomials does not fail because the error decreases exponentially, it prevails for very large values of the variable x.



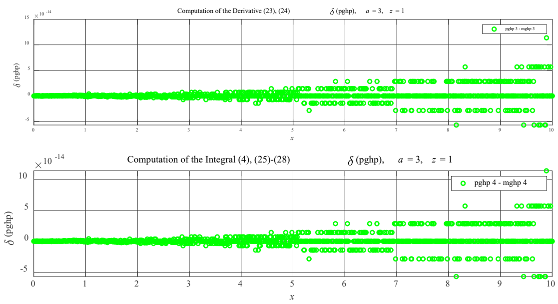
(a) the computation of the function $\log\left(e^{-3x^2+2x}\right)$ (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Modified Power Series Expansion via (15).



(b) The error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) Power Series Expansion of (15).



(c) computation of the logarithm of the derivative of the function e^{-3x^2+2x} (top) Integral; (bottom) MATLAB BIF vs. Power Series Expansion (24).



(d) Error; δ, (top) Computation of the Derivative (24), (bottom) Integral (28).

FIGURE 4: Numerical calculations of (a) function $\log\left(e^{-3x^2+2x}\right)$ and (b) relative error (top) MATLAB BIF vs MATLAB identities; (bottom) Power Series Expansion via (15).

We would have to use a logarithmic scale to see how much the error decreases from 1e-15 to much smaller values. This is the power of the LGF of the Hermite Polynomials.

Figure 4(a)-(d) is identical to Fig. 3(a)-(d). To generate the plots of Fig. 4(a)/(c) we have taken the logarithm of Fig. 3(a)/(c) and Fig. 4(b)/(d) represents the error of Fig. 4(a)/(c) respectively.

As shown from the results of Fig. 4(a)-(d) the LGF of the Hermite Polynomials does not fail, it prevails for very large values of the variable x.[ii]

Furthermore, the results of Fig. 3(c), (d) (bottom) and Fig. 4(c),(d) (bottom) were generated using the generalized Hypergeometric function when one of the Pochhammer symbols is replaced by a Hermite Polynomial as pointed out by Progri in (26) and (28).

7 Conclusions

We have successfully investigated the Hemite Polynomial expansion of the exponential function, e^{-x^2+2zx}, for both small and large values of the variable x.

For large values of the variable x the power series expansion fails, because the error grows exponentially to hundreds of orders of magnitude as indicated in Fig.1(a)-(d) an Fig.2(a)-(d); however, for the LGF of the Hermite Polynomials expansion (2) works well as indicated from Figs. 3, 4.

We would have to use a logarithmic scale to see how much the error decreases from 1e-15 to much smaller values. This is the power of the LGF of the Hermite Polynomials.

Furthermore, the results of Fig. 3(c), (d) (bottom) and Fig. 4(c),(d) (bottom) were generated using the generalized Hypergeometric function when one of the Pochhammer symbols is replaced by a Hermite Polynomial as pointed out by Progri in (26) and (28).

I believe that this LGF expansion of the Hermite Polynomials should resolve the issues of convergence of the computation of cdf the Parabolic Cylinder function for large values of the variable x.

8 Acknowledgement

This work was supported by Giftet Inc. executive office.

I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [22] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications*. *Vol. I* (Not yet available in print) [19].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say, I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [20] reminds me of my grandfather who I loved him very much.

Finally, I am also dedicating this journal paper to our most beloved President Ronald Reagan, who reminds me of my grandfather, [21] who spoke from the heart, who spoke the truth, who was a staunch supporter of personal freedom, the greatest leader of the free world, and perhaps the greatest critic of the wasteful government control and spending[iii].

"*Government is not the answer, government is the problem*."—Ronald Reagan

I would like to express my deepest gratitude to my high school math teacher, Gjergji Papanikolla. When I was in high school, Themistokli Germenji, Korcë, Albania, from 1985-1989, I used to visit him at his home once every week and show him my progress in my math exercises. He inspired me, unlike any other teacher. As a result of my persistent work under his direction, I won prizes in three consecutive Math National Competitions in Albania from 1986-1989.

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10 Appendix A: Review on the Power Series Expansion of the Error Function

The series representation of the error function occurs in the computation of the integral of the landmark expansion of the Hermite Polynomials.

Therefore, it is only fitting that I provide a detailed review on the series expansion of the error function, $\Phi(z)$, or $\mathrm{erf}(z)$, as follows (see Gradshteyn, Ryzhik (2007, 3.32-3.34 3.321 ex. 1 pg. 336 [2])):

$$\Phi(z) \equiv \mathrm{erf}(z)$$

$$= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!(2k+1)}$$

$$= \frac{2z}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k! 2\left(k+\frac{1}{2}\right)}$$

$$= \frac{z}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k!\left(k+\frac{1}{2}\right)}$$

$$= \frac{z}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_k z^{2k}}{k! \frac{1}{2} \left(\frac{3}{2}\right)_k}$$

$$= \frac{2z}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_k z^{2k}}{k! \left(\frac{3}{2}\right)_k}$$

$$= \frac{2z}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(-z^2\right)^k}{\left(\frac{3}{2}\right)_k k!}$$

$$= \frac{2z}{\sqrt{\pi}} \Phi\left(\frac{1}{2}, \frac{3}{2}; -z^2\right)$$

$$= \frac{2z}{\sqrt{\pi}} {}_1\mathrm{F}_1\left(\frac{1}{2}; \frac{3}{2}; -z^2\right) \tag{31}$$

The second series expansion of the error function comes from the application of the Kummer's first transformation Progri (2022, [17], (64) pg. 38) along the following line:

$\Phi(z) \equiv \mathrm{erf}(z)$

$$= \frac{2z}{\sqrt{\pi}} \Phi\left(\frac{1}{2}, \frac{3}{2}; -z^2\right)$$

$$= \frac{2z}{\sqrt{\pi}} e^{-z^2} \Phi\left(1, \frac{3}{2}; z^2\right)$$

$$= \frac{2z}{\sqrt{\pi}} e^{-z^2} {}_1\mathrm{F}_1\left(1; \frac{3}{2}; z^2\right) \tag{32}$$

Next, let us show the relation of the Hermite Polynomials and the error function (see Gradshteyn, Ryzhik (2007, 8.953 ex. 1 pg. 997 [2]))

$$\mathrm{H}_{2n}(z) \equiv (-1)^n \frac{(2n)!}{n!} \Phi\left(-n, \frac{1}{2}; z^2\right)$$

$$= (-1)^n \frac{(2n)!}{n!} {}_1\mathrm{F}_1\left(-n; \frac{1}{2}; z^2\right) \tag{33}$$

$$\mathrm{H}_{2n+1}(z) \equiv (-1)^n 2 \frac{(2n+1)!}{n!} \Phi\left(-n, \frac{1}{2}; z^2\right)$$

$$= (-1)^n 2 \frac{(2n+1)!}{n!} z \, {}_1\mathrm{F}_1\left(-n; \frac{3}{2}; z^2\right) \tag{34}$$

It is not intuitively obvious what this relation is. Hence, let us consider the following expression:

$\Phi(x-z) + \Phi(z) \equiv \mathrm{erf}(x-z) + \mathrm{erf}(z)$

$$= \frac{2(x-z)\Phi\left[\frac{1}{2}, \frac{3}{2}; -(x-z)^2\right]}{\sqrt{\pi}} + \frac{2z\Phi\left(\frac{1}{2}, \frac{3}{2}; -z^2\right)}{\sqrt{\pi}}$$

$$= \frac{2x}{\sqrt{\pi}} e^{-z^2} {}_2F_1[H(z), 1; 2; x] \tag{35}$$

The complete proof of (35) is laborious; hence, it is not provided here. However, numerically as shown in Sect. 6 (35) is indeed an identity.

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[i] I have spoken a lot about my grandfather, Xhevdet Progri, and my father, Fiqiri Progri in my previous journal papers. Both my grandfather, Xhevdet Progri, and my father, Fiqiri Progri, were people with amazing hands. After I got my Ph.D. and worked both in Academia and Industry, I felt compelled to give back in remembrance of my grandfather, Xhevdet Progri, and my father, Fiqiri Progri; so, I started playing golf in summer of 2006. I was always clumsy with my hands; so, I struggled for a lot of years in my early golfing career. But I seem to have overcome the biggest obstacle of my golfing career by learning from Jack Nicklaus who reminds me of my grandfather, Xhevdet Progri, Arnold Palmer who reminds me of my father, Fiqiri Progri, and of Gary Player who reminds me of myself. However, if I were in golf as good as I am in math I would have been a Master's Champion by now. I am really thankful to have known all the golf greats via books, YouTube Videos, Movies, TV documentaries, etc. I am also particularly thankful of the USGA for its tremendous support via the USGA championships.

[ii] The only issue is that is not given in any convenient way, and it may not always be practical to employ in its current form. The computation of the logarithm for values of $x > 10$ is practically infinity; hence, we have restricted the computation for values of $0 \leq x \leq 10$.

[iii] There is a false impression that only the research that is funded by the government bureaucrats is worthy of financial support, awards, & recognition. This is not to say that President Ronald Reagan wanted to demonize the role of the government as an enterprise, or resource, or originator of many inventions. But, when the government spending is abused for wasteful spending of the taxpayers trillions of $ dollars, and moreover, when it controls and suppresses innovation and ignores or completely denies funding to innovators and incubators based on political beliefs, then said President Ronald Reagan this type of government is the problem.