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Research Letter

The Significance of the Landmark Computations of the Generalized Bessel Function Distribution of the First Kind

For by him were all things created, that are in heaven, and that are in earth, visible and invisible, whether they be thrones, or dominions, or principalities, or powers: all things were created by him, and for him:

—Colossians 1:16

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The landmark computation of the generalized Bessel function distribution of the first kind (GBFD1K) emerged from a thorough technical review of the research that I performed earlier from 2016 until 2019 based on the extensive computational body of wisdom, knowledge, and understanding. As a result, it was fitting to revisit the computation of the GBFD1K given that the closed form expressions that I developed from 2016 until 2019 did not perform to a desired computational performance in both accuracy and speed for large values of the main variable. This journal letter discusses the various technical details and decisions that were involved in producing four outstanding journal papers that enabled the accurate and fast landmark computation of the GBFD1K for integer and non-integer values of the parameter and for both small and large values of the main variable.

Index Terms—Bessel functions, modified Bessel functions, cumulative distribution function, Kampé de Fériet function, Progri L function, incomplete gamma functions, hypergeometric series, landmark computation.

1 Introduction

The landmark computation of the generalized Bessel function distribution of the first kind (GBFD1K) is perhaps one of the most outstanding area of research that I have ever performed in the field of the masterpiece marvels in computational derivations and computational mathematics series because it provided a significant body of landmark computational wisdom, knowledge, and understanding that did not exist before. Before we can go do describe once more in depth the many achievements that were accomplished let me give a short review of how everything got started.

In 2016 I began the investigation of the generalized Bessel function distributions (GBFD) in Progri (2016, [1]). This publication was merely an introduction to the GBFD, to the computation of the incomplete gamma function (IGF), and of the Kampé de Fériet functions. In 2016 I developed many other outstanding journal papers thar deal with the computation of the Kampé de Fériet functions in other distributions of computational mathematics in Progri (2016, [2]) through Progri (2016, [4]).

What are the pros and cons of the research that I performed in 2016? The pros are as follows:

1. I began to produce substantial results from the very little body of computational wisdom, knowledge, and understanding.
2. I was able to show that the computation of the GBFD1k was feasible and it should work for both integer and non-integer values of the parameter, p, and for any values of the main variable x; although, I did not explicitly mention that the parameter p must have either an integer or a non-integer.
3. I also was able to show that embryo of various computational techniques such as the integral or linear approximation, or a closed from of the probability cumulative distribution function (cdf) needed to be developed to provide both accurate and fast computation of the GBFD1K.

The cons are as follows:

1. I had absolutely no idea on how accurate the formulate that I obtained from various references were.
2. I lacked a depth of the computational body of wisdom, knowledge, and understanding on the computation of the Kampé de Fériet functions for example.
3. There was a mountain of computational body of wisdom, knowledge, and understanding that needed to be developed to verify that all of these expressions either were correct or new expressions needed to be developed.

From 2018 until 2020 I produced three outstanding journal papers that started the prosses of investigation of some of the current computational body of wisdom, knowledge, and understanding in Progri (2018, [5]) through Progri (2020, [7]). What are the pros and cons of the research that I performed in 2018 through 2020? The pros are as follows:

1. I began the creation of the computational body of wisdom, knowledge, and understanding necessary to perform the computation of the algorithms that I proposed in Progri (2016, [1]).
2. I was able to produce the first computation of the Kampé de Fériet functions in MATLAB.
3. I also was able to show that there was a methodology that had emerged as a result in the computation of the GBFD1K via four distinct ways: MATLAB integral built in function (BIF), linear approximation, at least two ways via the Kampé de Fériet functions.

The cons are as follows:

1. I assumed that all the formulate that I obtained from various references were somehow accurate for all the values of the variable.
2. The computation of the GBFD1K was only performed for a small range of the variable x.
3. I did not investigate all the formulae that existed in the literature; i.e., the computational accuracy and speed was relative to the computation accuracy and speed of the formulae that existed in the literature that were used to derive the CFE of the GBFD1K via the Kampé de Fériet functions.
4. I did not put together a review or assessment, or examination of the work that I had performed until 2020 to enable me to get a much better understanding of the big picture.

In 2021, however, there was a significant shift in all the work that I had previously performed. In 2021 I began 2 expand the computational body of wisdom, knowledge, and understanding that resulted in significant performance improvements of the generalized parabolic cylinder function distribution (GPCF) that involved a deeper development of the computational body of wisdom, knowledge, and understanding that produced many significant achievements that did not exit prior to these publications in Progri (2021, [7]) to Progri (2021, [12]).

The pros and cons of the work that I had performed in 2021 were somewhat summarized in Progri (2021, [12]). The work that I performed in Progri (2021, [7]) to Progri (2021, [12]) provided an outstanding opportunity because it produced for the first time a landmark computation as I explained in Progri (2021, [10]).

In 2022 began the investigation of the GBFD1K for both integer and non-integer values of a parameter p.

The main purpose of this journal letter is to examine the various CFE that were employed to produce the landmark computation of the GBFD1K for both integer and non-integer values of a parameter p and for both small and large values of the main variable x as described in Progri (2022, [13]) through Progri (2022, [16]). What were the pros and cons of this investigation? The pros are underlined as follows:

1. I produced landmark computations of the GBFD1K for both integer and non-integer values of a parameter p and for both small and large values of the main variable x.
2. I performed a thorough investigation of all the formulae in the literature and I provided a range of expected convergence in both computational performance and speed.
3. The body of the computational wisdom, knowledge, and understanding is so significant that has led to the throughout computation of the GBFD1K via the thoroughly tests formulae that were lacking in any of the previous publications.

The cons are depicted below:

1. It took too long for me to realize that to produce a work at the caliber of the landmark computation of the GBFD1K required far more computational wisdom, knowledge, and understanding that I initially envisioned in 2016.
2. It really eroded my confidence on the validity of the formulae that are founded in the literature.
3. It was laborious. This was not a simple problem that it required just a few steps. It required a lot of work; I mean a lot more that I had ever envisioned.

This letter is organized as follows: in Sect. 2 the description of the landmark computation of the GBFD1K for integer values of a parameter is discussed. A study of the computation of the exponential function is presented in Sect. 3. Landmark computation of the incomplete gamma function is presented in Sect. 4. Section 5 discusses the landmark computation of the GBFD1K for non-Integer values of a parameter. Discussion on numerical, theoretical results is presented in Sect. 6; Conclusion is provided in Sect. 7, Acknowledgement is given in Sect. 8, along with a list of references in Sect. 8.

2 Landmark Computation of the GBFD1K for Integer Values of a Parameter

Before I developed the landmark computation of the GBFD1K for integer values of a parameter, p, I developed the asymptotic computation of the GBFD1K for both integer and non-integer values of a parameter, p as in Progri (2022, [13]).

The asymptotic computation of the GBFD1K was designed to expand the computation of the GBFD1K for large values of the main variable x.

What are the pros and cons of the asymptotic computation of the GBFD1K for both integer and non-integer values of a parameter, p? The pros are as follows:

1. It enabled the computation of the GBFD1K for large values of main variable x as it was designed for.
2. It exposed the vulnerabilities of the computation of the GBFD1K for both integer and non-integer values of a parameter, p based on Progri (2016, [1]) and Progri (2018, [5]).
3. It provided the necessary evidence that fundamentally, there was something wrong inside the equations that caused the error to increase with the main variable x. There is nothing more frustrating as a mathematician, scientist, scholar, and/or engineer than when you develop something and the results are not predictable. This was the case here.

The cons are depicted below:

1. I finally understood that the CFE of the computation of the GBFD1K for both integer and non-integer values of a parameter, p based on Progri (2016, [1]) and Progri (2018, [5]) were only useful for a small range of the main variable, x.
2. What was wrong with the computation of the GBFD1K it was not because my derivations were incorrect, but because the equations that I relied upon did not have the necessary explanations to justify the range of convergence that was needed to perform the computation for large values of the main variable, x.
3. It became clear to me that I needed to developed something out of the ordinary perform the computation of the GBFD1K for both integer and non-integer values of a parameter, p.

From everything I leaned as a result of the asymptotic computation of the GBFD1K for both integer and non-integer values of a parameter, p; I decided to split the computation of the GBFD1K into two parts: one that dealt with the integer values of a parameter, p; and the other for non-integer values of a parameter, p.

The first approach that I developed for the computation of the GBFD1K for integer values of a parameter was blend of my previous equations and of the new equations that I developed based again on an equation from Gradshteyn, Ryzhik (2007, 2.71, 2.711 pg. 237 [18]) that had a typo as it is shown in the endnote v in Progri (2022, [13]). It was not enough that I was applying equations that were lacking in both the region of convergence but now I had to deal with another typo.

What are the pros and cons of the landmark computation of the GBFD1K for both integer a parameter, p? The pros are as follows:

1. I finally produced a CFE of the computation of the GBFD1K for integer values of the main parameter, p, based on Progri (2016, [1]) and Progri (2018, [5]) and the new equations that I developed in Progri (2022, [13]) for small and large values of the main variable, x.
2. It gave me a lot of confidence that everything I developed made sense. This was literally the turning point that I had developed something that was literally landmark in nature. It was predictable, it was fast, it was accurate, it worked like clockwork.
3. It provided enormous benefits and a renewed vision to the development of the computational body of wisdom, knowledge, and understanding.

The cons are depicted below:

1. Now I have to live to a new reality that I will have to question all the equations that I used to derive the equations in my previous publications in Progri (2016, [1]), Progri (2018, [5]), and Progri (2019, [6]).
2. It was going to be a very laborious approach to produce significantly improved approaches that were going to require a laborious review of everything I had developed earlier and propose significantly improved algorithms that were going to work for non-integer values of a parameter p and for large values of the main variable x.
3. I was going to developed novel, original, innovative, and new equations that did not exist before and I was going to blend them in a way bever seen before. The production of the landmark computation of the GBFD1K for non-integer values of the main parameter, p was going to require something out of the box to achieve the desired computational performance in both accuracy and speed.

Once I developed the landmark computation of the GBFD1K for non-integer values of parameter, p, it became transparent that the landmark computation of the GBFD1K for integer values of the parameter, p was going to be a very short derivation which it did in Appendix A of Progri (2022, [16]).

This concludes the insights that I have provided on the pros and cons of the landmark computation of the GBFD1K for integer values of a parameter p.

3 A Study of the Computation of the Exponential Function

Why study the computation of the exponential function in Progri (2022, [14])? The exponential function is perhaps one of the most intriguing mathematical functions that shows up in so many integral computations. As rightly depicted in Progri (2022, [14]) the power series expansion of the exponential function is the root cause of all the computational problems related to the landmark computation of the regularized incomplete gamma function in Progri (2022, [15]) and of the landmark computation of the GBFD1K for non-integer values of a parameter in Progri (2022, [16]).

What are the pros and cons of the study of the computation of the exponential function? The pros are as follows:

1. It became crystal clear that indeed the power series expansion of the exponential function was indeed the root cause of all the computational problems related to the landmark computation of the regularized incomplete gamma function in Progri (2022, [15]) and of the landmark computation of the GBFD1K for non-integer values of a parameter in Progri (2022, [16]).
2. It provided the means by which the computation of the exponential function for negative exponents can be performed for simple mathematical operations based on the expansion (20) in Progri (2022, [14]).
3. Indeed, the simulation results validated the numerical calculations and provided the necessary evidence that was needed to perform the computation of more complicated mathematical operations based on the computation of the exponential functions for negative exponents.

The cons are depicted below:

1. Equation (20) in Progri (2022, [14]) is not a permanent fix of the problem related to the computation of the exponential functions for negative exponents.
2. I was not able to provide the complete solution of the computation of the exponential functions for negative exponents.
3. Hopefully in the near future I will be able to provide the added insights for the computation of the exponential functions for negative exponents.

This concludes the insights that I have provided on a study of the computation of the exponential function (mainly focused on the negative exponents).

4 Landmark Computation of the Incomplete Gamma Function

Why study the computation of the incomplete gamma function? There are many computational problems that rely on the accurate evaluation of the IGF, more precisely, the regularized IGF or RIGF, which is the case of the landmark computation of the GBFD1K for non-integer values of a parameter α.

What are the pros and cons of the study of the landmark computation of the RIGF for non-integer values of a parameter α? The pros are as follows:

1. It was this journal paper that showed that the computation of the RIGF is indeed an integral of an exponential function of a more complicated exponent.
2. It provided enormous results on the Kummer's first transformation.
3. It also provided tremendous insights on the computation of the RIGF for large values of the main variable, x.

The cons are depicted below:

1. Equation (23), (24), and (27) in Progri (2022, [15]) are not a permanent fix of the problem related to the computation of the RIGF for non-integer values of a parameter, α.
2. Even though it was not a permanent fix it offered tremendous insights into computational body of wisdom, knowledge, and understanding that was related to (23), (24), and (27) in Progri (2022, [15]).
3. More work was needed to provide the permanent fix for the landmark computation of the RIGF for non-integer values of a parameter, α.

This concludes the insights that I have provided on the pros and cons of the landmark computation of the RIGF for non-integer values of a parameter, α.

5 Landmark Computation of the GBFD1K for non-Integer Values of a Parameter

The landmark computation of the GBFD1K for non-integer values of a parameter, p, came as a result of a laborious work that took many years, many derivations, and a much needed significantly improved methodology that significantly expanded the computational wisdom, knowledge, and understanding and produced many useful equations as beautifully illustrated in Progri (2022, [16]).

What are the pros and cons of the study of the landmark computation of the GBFD1K for non-integer values of a parameter, p? The pros are as follows:

1. This journal paper left no stones unturned and it exceeded the computational performance requirements in both accuracy and speed.
2. It provided all the necessary details that was lacking from the references that I employed from Progri (2016, [1]) until Progri (2022, [16]).
3. The computational body of wisdom, knowledge, and understanding expanded even further to the point that everything made sense and everything was predictable.

The cons are depicted below:

1. It was going to rely on various equations and various approximations, not one CFE.
2. The actual computation was going to rely on too many optimization parameters.
3. I could not envision one equation that would be a permanent fix of the landmark computation of the GBFD1K for non-integer values of a parameter, p.

This concludes the discussion on the landmark computation of the GBFD1K for non-integer values of a parameter, p.

6 Discussion on Numerical, Theoretical Results

The numerical, theoretical results provided the framework for validating the computational body of wisdom, knowledge, and understanding of the work that was performed in Progri (2022, [13]) to Progri (2022, [16]).

The decision was made to significantly improve the MATLAB computational tools and make a much more prominent case for the use of the already developed MATLAB BIF such as the *integral*, *gammainc*, etc.

Along the way, the method of linear approximation was retired, because there was really no need to use a computational method based on linear approximation when other computational methods were several orders of magnitude more accurate.

What are the pros and cons of numerical, theoretical results? The pros are as follows:

1. The number of independent computational methodologies, and independent computational scenarios exceeded the expectations that was required to produce sufficient evidence in both computational performance accuracy and speed.
2. The simulation results made use of both MATLAB BIF and of the Giftet Inc. proprietary BIF.
3. The simulation, theoretical results were predictable; i.e.,

the output input relationship was indeed very predictable.

The cons are depicted below:

1. It took an enormous amount of work to produce all the simulation results for Progri (2022, [13]) until Progri (2022, [16]).
2. The amount of work was significantly more complex than when I produced the first simulation results in Progri (2016, [1]) until Progri (2019, [6]).
3. It put a lot of pressure on the way of how to select the proper optimization parameters that would enable the simulation to perform at the desired computational performance accuracy and speed.

This concludes the insights that I have provided on the discussion on numerical, theoretical results.

7 Conclusions

The research that I performed in Progri (2022, [13]) to Progri (2022, [16]) was nothing short of exceptional work based on the depth and amount of significant computational wisdom, knowledge, and understanding.

This work has many benefits that included my:

1. Making it crystal clear as to the usefulness of the expressions that came from various references.
2. Producing landmark computation of the GBFD1K for both integer and non-integer of the parameter, p, and for a much wider range of the main variable, x.
3. Producing simulation results that were both predictable and exceeded the expectations in both computational performance accuracy and speed.

What was some of the drawback of this work?

1. It took significantly more effort than when I first envisioned in Progri (2016, [1]) to Progri (2019, [6]).
2. I felt compelled to generate a lot of material to achieve the desired results in both computational performance accuracy and speed.
3. It was not going to be easy to digest all the material and select the optimization simulation parameters.

I sincerely hope that readers will consider that both the benefits and well as some of the drawbacks when reading through Progri (2022, [13]) to Progri (2022, [16]).

8 Acknowledgement

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I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [21] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications. Vol. I* (Not yet available in print) [17].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say, I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [19] reminds me of my grandfather who I loved him very much.

Finally, I am also dedicating this journal paper to our most beloved President Ronald Reagan, who reminds me of my grandfather, [20] who spoke from the heart, who spoke the truth, who was a staunch supporter of personal freedom, the greatest leader of the free world, and perhaps the greatest critic of the wasteful government control and spending[i].

"*Government is not the answer, government is the problem*."—Ronald Reagan

I would like to express my deepest gratitude to my high school math teacher, Gjergji Papanikolla. When I was in high school, Themistokli Germenji, Korcë, Albania, from 1985-1989, I used to visit him at his home once every week and show him my progress in my math exercises. He inspired me unlike any other teacher. As a result of my persistent work under his direction, I won prizes in three consecutive Math National Competitions in Albania from 1986-1989.

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[i] There is a false impression that only the research that is funded by the government bureaucrats is worthy of financial support, awards, & recognition. This is not to say that President Ronald Reagan wanted to demonize the role of the government as an enterprise, or resource, or originator of many inventions. But, when the government spending is abused for wasteful spending of the taxpayers trillions of $ dollars, and moreover, when it controls and suppresses innovation and ignores or completely denies funding to innovators and incubators based on political beliefs, then said President Ronald Reagan this type of government is the problem.