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Research Article

Landmark Computation of the Incomplete Gamma Function

To my Undergraduate Physics University Professor, Niko Thomo, the best professor, and friend I have ever known.

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The landmark[i] computation of the regularized[ii] incomplete gamma function (RIGF) provides a novel, innovative, new, and significantly improved understanding of the regularization of the IGF by means of new integral expressions and describes (or gives a very detailed account) on the scale of validity of the range of several already known closed form expressions of the computation of the RIGF. The lack of the convergence range on the computation of the IGF is the root cause of the computational issues that were described in the quantitative evaluation of the generalized Bessel function distribution of the first kind (GBFD1K). The landmark computation of the RIGF depicts that the landmark RIGF is in fact an integration of the exponential function with a more complicated exponent; therefore, the convergence of the power series expansion of the exponential function integration for negative exponents has directly impacted the computation of the RIGF. In order to compare (or collate and contrast) the computation of the RIGF with the available literature approaches we used three methods: analytical, graphical, and numerical (or computationally numerical via MATLAB simulations). As a result of this work, we provided a well-defined range of convergence for the closed form expressions of the computation of the RIGF in the literature and we have significantly improved the understanding of the computation of the RIGF.

Index Terms—Exponential function, Gamma function, incomplete gamma function, regularized incomplete gamma function, Progri RIGF integrals, hypergeometric series, Kummer First transformation, landmark computation.

1 Introduction

In 2016, in Progri (2016, [1]), I began the investigation of the computation of the incomplete gamma function (IGF). I was under the impression that all I needed to do is plugin some already known series expansion formulas and that should have been enough. Since then, I produced a remarkable number of publications which required the computation of the IGF is some ways Progri (2016, [1]) until Progri (2022, [13]). However, while I was investigating the landmark computation of the generalized Bessel function cumulative distribution functions (cdf) of the first kind (GBFD1K) for integer values of the parameter p in Progri (2022, [13]-[15]) I understood for the first time that there was something not right (wrong, imperfect, erroneous, misleading) that was directly linked to the computation of the IGF based on the closed form expansions of the IGF from the literature [18], [21], [24].

It is well known that the computation of the IGF is based in part due to the power series expansion of the exponential function [18], [21], [24].

First, is it possible to have the computation of the IGF is such as manner that it will be obvious that it will depend entirely on the power series expansion of the exponential function? This issue is analytically investigated in Sect. 2.

The IGF has been studied for a few centuries since, Gauss, Euler, Taylor, etc.

It was the works of Legendre (1826, [16]) for the logarithm of the Gamma function that initiated a serious study of the computation of the IGF.

The works of Kummer ([17], 1837) enabled the study of the confluent hypergeometric functions (CHFs) and much of the needed transformations that improve their region of convergence.

Nielsen ([18], 1906) developed a handbook on theory of the Gamma functions based on the works of Legendre (1826, [16]), Kummer ([17], 1837), and others.

Pearson ([19], 1922) provides tables of IGF for non-integer values of the parameter and then he uses the method of interpolation for the integer and non-integer values of the parameter that are not found in the tables with up to six or seven decimal points. While Pearson ([19], 1922) builds upon the works of Legendre (1826, [16]) for the logarithm of the Gamma function; however, in Pearson ([19], 1922) I did not see any mention of the CHF or even of the Kummer's first transformation.

Nevertheless, it was not until the Second World War (WWII) that it became obvious that the accurate computation of the regularized IGF (or RIGF) was required on may radar, signal processing, inertial navigation, or other communications systems engineering problems.

The three volumes of Batman on *Higher Transcendental Functions* ([20]-[22], 1953-1955) provide the most comprehensive account of closed form expressions of the computation of the IGF that are employed in many other textbooks. Even in the case of the Batman's three monumental volumes, many important details of the computational convergence are missing.

As it is been reported from Arfken and Weber, ([23], 1995, pg. 351), an expansion of the IGF diverges because it fails the d'Alembert ratio test or any other absolute convergence tests described in (see Gradshteyn, Ryzhik, 2007 [24] pg. 6 ex. 0.22 0.222).

Even in the case of Gradshteyn, Ryzhik, ([24], 2007 p. 899-902) as it relates to the content of the RIGF, although the biggest contributors are Nielsen ([18], 1906) and Batman ([20]-[22], 1953-1955), it is obvious that the computational region of convergence for several closed form expressions of the RIGF are lacking.

Due to the accumulation and the wealth of wisdom, knowledge, and understanding I came to the realization that computation of the RIGF needed to be revisited, investigated, and assessed.

As a result, I produced an original, landmark, quantum leap expression of the RIGF and I have demonstrated that the computation of the RIGF is nothing more but the integration of an exponential function with a more complicated exponent and that its computational accuracy can only be improved further by means of transformations similar to the Kummer's first Transformation [17], [18], [20], [21].

It was not easy to come to this realization. It took a lot of computational work that gave me more tools and perfected my prediction and the ability to reason based on the numerical results. Further details of how I came to this realization are given in Sects. 2 and 4.

This paper is organized as follows: in Sect. 2 landmark integral expression of the RIGF is discussed. The landmark computation of the RIGF for non-integer values of a parameter is presented in Sect. 3. Landmark computation of the derivative of the RIGF for non-integer values of a parameter is derived in

Sect. 4. Integration of the RIGF for non-integer values of a parameter is discussed in Sect. 5. Section 6 contains numerical, theoretical results; Conclusion are provided in Sect. 7 along with Acknowledgement in Sect. 8 and a list of references in Sect. 9. Finally, the identity of Kummer's first transformation is discussed in Appendix A. The problem of the CHF inverse is presented in Appendix B.

2 Landmark Integral Expression of the RIGF

The computation of the RIGF is important and it turns out it is worth the reader's attention, because certain closed form expression in [24]-[26] lack important information on the region of convergence, which is the main reason why I was searching for a suitable integral expression of the RIGF.

From the Euler integral definition of the (real numbers) IGF, $\gamma'(\alpha, x)$, [19]-[26] for the variable of interest, $x > 0$, and a parameter $\alpha > 0$, we have

$$\gamma'(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt; \ \Gamma'(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt \tag{1}$$

It is worth considering a few limit properties of the IGF, $\gamma'(\alpha, x)$:

$$\gamma'(\alpha, x = 0) = \int_0^0 t^{\alpha-1} e^{-t} dt \equiv 0, \ \alpha > 0 \tag{2}$$

$$\gamma'(\alpha, x \to \infty) = \int_0^\infty t^{\alpha-1} e^{-t} dt \equiv \Gamma(\alpha), \ 0 < \alpha < \infty \tag{3}$$

$$\gamma'(\alpha \to \infty, x < \infty) = \int_0^x t^{\alpha-1} e^{-t} dt \equiv 0 \tag{4}$$

$$\gamma'(\alpha \to \infty, x \to \infty) = \int_0^\infty t^{\alpha-1} e^{-t} dt \equiv \Gamma(\alpha) \to \infty \tag{5}$$

The other important property of the IGF is that it is always positive, and it is monotonically increasing; i.e.,

$$\gamma'(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt \geq 0 \tag{6}$$

And for every pair of two variables x_1 and x_2 such that

$$0 \leq x_1 \leq x_2 < \infty \tag{7}$$

The corresponding IGFs satisfy the monotonically increasing inequality as follows:

$$0 \leq \gamma'(\alpha, x_1) = \gamma'(\alpha, x_2) < \infty \tag{8}$$

Since, we are interested in the computation of the normalized or RIGF then we have from substituting (1) and (3) the following:

$$\gamma(\alpha, x) = \frac{\gamma'(\alpha,x)}{\Gamma(\alpha)} = \frac{\int_0^x t^{\alpha-1} e^{-t} dt}{\int_0^\infty t^{\alpha-1} e^{-t} dt}; \ \Gamma(\alpha, x) = \frac{\int_x^\infty t^{\alpha-1} e^{-t} dt}{\int_0^\infty t^{\alpha-1} e^{-t} dt} \tag{9}$$

First, let us examine the limit properties of the RIGF, $\gamma(\alpha, x)$, as follows:

$$\gamma(\alpha, x = 0) = \frac{\int_0^0 t^{\alpha-1} e^{-t} dt}{\int_0^\infty t^{\alpha-1} e^{-t} dt} \equiv 0, \ \alpha > 0 \tag{10}$$

$$\gamma(\alpha, x \to \infty) = \frac{\int_0^\infty t^{\alpha-1} e^{-t} dt}{\int_0^\infty t^{\alpha-1} e^{-t} dt} \equiv 1, \ 0 < \alpha < \infty \tag{11}$$

$$\gamma(\alpha \to \infty, x < \infty) = \frac{\int_0^x t^{\alpha-1} e^{-t} dt}{\int_0^\infty t^{\alpha-1} e^{-t} dt} \equiv 0 \tag{12}$$

$$\gamma(\alpha \to \infty, x \to \infty) = \frac{\int_0^\infty t^{\alpha-1} e^{-t} dt}{\int_0^\infty t^{\alpha-1} e^{-t} dt} \equiv 1 \tag{13}$$

What are the major strengths[iii] associated with the definition of the RIGF (9)? The RIGF is

1. monotonically increasing with the variable $\forall x \in \mathbb{R}^+$; i.e., it satisfies the monotonically increasing inequality (8) for every pair of variables x_1 and x_2 that satisfy inequality (7).
2. always bounded from zero to one; i.e., it always satisfies (10)-(13) for every positive variable $\forall x \in \mathbb{R}^+$.

What are the major problems[iv] associated with (9)? The computation of the RIGF requires:

1. a variable (not constant or changing) area of the integral (1).
2. the computation of the product of two functions.
3. the computation of two integrals (1) and (3).

By examining the strengths and weaknesses we conclude that although the definition (9) is very useful it still presents a computation of the RIGF that is complex; i.e., convoluted[v], complicated, sophisticated, and/or perplexing to say the least.

If there was a way to get rid of the major weaknesses associated with the computation of the RIGF, the new closed form expression of the RIGF will require a computation that is neither convoluted nor complicated, sophisticated, perplexing; i.e., it will be more straightforward and it will enable us to see where the root cause of the sophistications, computation of the RIGF is.

We can get rid of the variable area of integration if we make the substitution, assuming that $x > 0$, we have

$$t' = \frac{t}{x} \Rightarrow t = xt' \Rightarrow t^{\alpha-1} = (xt')^{\alpha-1} = x^{\alpha-1} t'^{\alpha-1} \tag{14}$$

$$dt' = \frac{dt}{x} \Rightarrow dt = x dt' \Rightarrow e^{-t} = e^{-xt'} \tag{15}$$

The substitution (14) and (15) when is applied to (9) produces

$$\gamma(\alpha, x) = \frac{x^\alpha \int_0^1 t'^{\alpha-1} e^{-xt'} dt'}{x^\alpha \int_0^\infty t'^{\alpha-1} e^{-xt'} dt'} = \frac{\int_0^1 t'^{\alpha-1} e^{-xt'} dt'}{\int_0^\infty t'^{\alpha-1} e^{-xt'} dt'} \tag{16}$$

As seen from (16) the area of the integration from the top integral is now fixed from zero to one.

Next, in order to reduce the computation of one function instead of two functions, we need to make the following substitution

$$t' = y^{1/\alpha} \Rightarrow t'^{\alpha-1} = y^{1-1/\alpha} \tag{17}$$

$$dt' = \frac{1}{\alpha} y^{1/\alpha-1} dy \Rightarrow e^{-xt'} = e^{-xy^{1/\alpha}} \tag{18}$$

After applying (17) and (18) into (9) yields

$$\gamma(\alpha,x) = \frac{\int_0^1 y^{1-1/\alpha} e^{-xy^{1/\alpha}} \frac{1}{\alpha} y^{1/\alpha-1} dy}{\int_0^\infty y^{1-1/\alpha} e^{-xy^{1/\alpha}} \frac{1}{\alpha} y^{1/\alpha-1} dy}$$

$$= \frac{\int_0^1 e^{-xy^{1/\alpha}} dy}{\int_0^\infty e^{-xy^{1/\alpha}} dy} \text{[vi]}, \; 0 \le x < \infty \tag{19}$$

As seen from (19) we have reduced the computation of the RIGF into the computation of just one function, $e^{-xy^{1/\alpha}}$. *We have also provided another stage of regularization or normalization because both the numerator and denominator in (19) are smaller than the numerator and denominator in (9) assuming that the variable $x > 1$ which is the case of interests in most applications.*

The integration of (19) requires the computation of an exponential function with a complicated exponent; hence, we cannot integrate (19) directly. The only option that is left is to apply the power series expansion of the exponential function.

Equivalently, we find that the upper RIGF would be given by

$$\Gamma(\alpha,x) = \frac{\int_1^\infty e^{-xy^{1/\alpha}} dy}{\int_0^\infty e^{-xy^{1/\alpha}} dy}, \; 0 \le x < \infty \tag{20}$$

Integrals (19) and (20) produce two new entries in the Tables of Integrals, Products, and Series similar to (see Gradshteyn, Ryzhik, 2007 [24]).

It took me six years to finally understand the root cause of all the computational problems associated with the landmark computation of the GBFD1K is the lack of many clarifications on the series expansion of the exponential function, e^{-z}, $\forall z \in \mathbb{R}^+$, (see Progri (2022, [14])) which is why in the following section we will examine this problem in greater detail.

3 Landmark Computation of the RIGF for Non-Integer Values of a Parameter

The new integral closed form expression (19), (20) of the RIGF points out that the power series expansion of the exponential function for negative exponents (see Progri (2022, [14])) is the root-cause of computational issues associated with numerical evaluation of the RIGF, $\gamma(\alpha,x)$.

In Progri (2022, [14]) the power series expansion of the e^{-x}, $\forall x \in \mathbb{R}^+$ works well for small values of $0 \le x < 1$.

Let us compute the first integral based on the power series expansion of the exponential function Progri (2022, [14]) (19) as follows:

$$\int_0^1 e^{-xy^{\frac{1}{\alpha}}} dy = \int_0^1 \sum_{k=0}^\infty \frac{(-1)^k x^k y^{k/\alpha}}{k!} dy$$

$$= \sum_{k=0}^\infty \frac{(-1)^k x^k \int_0^1 y^{k/\alpha} dy}{k!}$$

$$= 1 + \sum_{k=1}^\infty \frac{(-1)^k x^k \int_0^1 y^{k/\alpha} dy}{k!} \tag{21}$$

Solving the integral in (21) for values of $k = \{1,2,\cdots\}$ we have

$$\int_0^1 y^{h/\alpha} dy = \frac{1}{k/\alpha+1} y^{k/\alpha+1} \Big|_0^1 = \frac{\alpha}{k+\alpha} \tag{22}$$

Substituting (22) into (21) yields,

$$\int_0^1 e^{-xy^{\frac{1}{\alpha}}} dy = 1 + \sum_{k=1}^\infty \frac{(-1)^k \alpha x^k}{(k+\alpha)k!}$$

$$= 1 + \alpha \sum_{k=1}^\infty \frac{(-1)^k x^k}{(k+\alpha)k!}$$

$$= \alpha \sum_{k=0}^\infty \frac{(-1)^k x^k}{(k+\alpha)k!}$$

$$= \sum_{k=0}^\infty \frac{(-1)^k (\alpha)_k x^k}{(\alpha+1)_k k!} = \Phi\left[\begin{matrix}\alpha \\ \alpha+1\end{matrix}; -x\right] \text{[vii]},$$

$$\cong \sum_{k=0}^N \frac{(-1)^k (\alpha)_k x^k}{(\alpha+1)_k k!}; \; N = 200, \; 0 \le x < 1 \text{[viii]} \tag{23}$$

It is no surprise that if we multiply (23) by x^α/α we obtain exactly the same as result as in Batman, ([21], pg. 133 (3) 1953) and (see Gradshteyn, Ryzhik, 2007 [24] pg. 899 ex. 8.351 2.). Equation (23) provides up to sixteen digits of precision accuracy for values of $0 \le x < 1$. For values of $1 \le x < \infty$, the CHF oscillates and the much-needed precision is lost.

If we apply the Kummer's first transformation [8], [17], [18], [30] (see Appendix A for a detailed proof of the Kummer's Identity) we obtain:

$$\int_0^1 e^{-xy^{\frac{1}{\alpha}}} dy = e^{-x} \Phi\left[\begin{matrix}1 \\ \alpha+1\end{matrix}; x\right]$$

$$\cong e^{-x} \sum_{k=0}^N \frac{x^k}{(\alpha+1)_k} \text{[ix]}, \; N = 200, \; 0 \le x < 20 \text{[x]} \tag{24}$$

As a result of the Kummer's first transformation [8], [17], [18], [30], (24) (see Appendix A for a detailed proof of the Kummer's Identity) has a wider range of convergence for $0 \le x < 20$ for $\alpha > 0$. Equation (24) is the product of one monotonically increasing function $\Phi[1, \alpha+1; x]$ and one monotonically decreasing function, e^{-x}. Since the lefthand side, the integral (24) is in itself a monotonically decreasing function, it turns out that the Kummer's first transformation is a really good transformation for widening the region of convergence up to sixteen decimal points. The problem of the inverse of $\Phi[1, \alpha+1; x]$ as asked in Appendix B. This is a very difficult problem but a problem that needs 2 be asked and perhaps solved.

Next, for large values of the variable x, such that $\forall x \in 20 \leq x < \infty$ we propose the following expansion based in Batman, ([21], pg. 139 1953 based on Nielsen ([18], 1906))

$$\gamma(\alpha, x+y) - \gamma(\alpha, x) = \Gamma(\alpha, x) - \Gamma(\alpha, x+y)$$

$$= \frac{\int_0^{x+y} t^{\alpha-1} e^{-t} dt - \int_0^x t^{\alpha-1} e^{-t} dt}{\Gamma(\alpha)}$$

$$= \frac{\int_x^{x+y} t^{\alpha-1} e^{-t} dt}{\Gamma(\alpha)}$$

$$= \frac{\int_0^y (x+u)^{\alpha-1} e^{-(x+u)} dt}{\Gamma(\alpha)}$$

$$= \frac{e^{-x} x^{\alpha-1} \int_0^y \left(1+\frac{u}{x}\right)^{\alpha-1} e^{-u} dt}{\Gamma(\alpha)}; \ |y| < |x|$$

$$= \frac{e^{-x} x^{\alpha-1} \int_0^y \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} \frac{u^k}{x^k} e^{-u} dt}{\Gamma(\alpha)}$$

$$= \frac{x^{\alpha-1} \sum_{k=0}^{\infty} \frac{\binom{\alpha+k-1}{k}}{x^k} \int_0^y \frac{y u^k}{e^u} dt}{e^x \Gamma(\alpha)} \ (k \text{ int.})$$

$$= \frac{x^{\alpha-1} \sum_{k=0}^{\infty} \frac{\binom{\alpha+k-1}{k}}{x^k} k! \left[1 - \frac{e_n(y)}{e^y}\right]}{e^x \Gamma(\alpha)} \tag{25}$$

Then we have for $\forall x \in, 20 \leq x < \infty$, the following is a very good approximation

$$\Gamma(\alpha, x) = \frac{e^{-x} x^{\alpha-1} \sum_{k=0}^{M} \binom{\alpha+k-1}{k} x^{-k} k!}{\Gamma(\alpha)}; \ M = 20, \ 20 \leq x < \infty$$ [xi]

$$= \frac{e^{-x} x^{\alpha-1} \sum_{k=0}^{M} (-1)^k (1-\alpha)_k x^{-k}}{\Gamma(\alpha)} \tag{26}$$

Substituting (26) into (25) yields,

$$\gamma(\alpha, x) = 1 - \Gamma(\alpha, x) = 1 - \frac{e^{-x} x^{\alpha-1} \sum_{k=0}^{M} (-1)^k (1-\alpha)_k x^{-k}}{\Gamma(\alpha)} \tag{27}$$ [xii]

If we compare and contrast (23), (24) and (27) with what is currently being reported in the literature we have provided a much-needed region of convergence for up to sixteen decimal points of precision accuracy for these closed form expressions so that they may be employed appropriately.

This concludes the discussion on computation of the RIGF for non-integer values of a parameter.

4 Landmark Computation of the Derivative of the RIGF for Non-Integer Values of a Parameter

Now, that we have produced the range of convergence for all the major closed form expressions that deal with the computation of the RIGF, let us do the same with the those that compute the derivative.

If we apply the Leibniz integral rule [31], or differentiation under the integral sign rule [32], or Reynolds transport theorem [33] on the RIGF we obtain a very familiar expression

$$\frac{\partial \gamma(\alpha, x)}{\partial x} = \frac{\frac{\partial \int_0^x t^{\alpha-1} e^{-t} dt}{\partial x}}{\Gamma(\alpha)} = \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} + 0 + 0 =$$

$$= \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} \tag{28}$$

On the other hand, if we use the righthand side expression of (23) we obtain

$$\frac{\partial \gamma(\alpha, x)}{\partial x} = \frac{1}{\alpha \Gamma(\alpha)} \frac{\partial}{\partial x} \left\{ x^\alpha \Phi \left[\begin{matrix} \alpha \\ \alpha+1 \end{matrix}; -x \right] \right\}$$

$$= \frac{\alpha x^{\alpha-1} \Phi\left[\begin{matrix} \alpha \\ \alpha+1 \end{matrix}; -x\right] + x^\alpha \frac{\partial}{\partial x} \Phi\left[\begin{matrix} \alpha \\ \alpha+1 \end{matrix}; -x\right]}{\alpha \Gamma(\alpha)}$$

$$= \frac{\alpha x^{\alpha-1} \Phi\left[\begin{matrix} \alpha \\ \alpha+1 \end{matrix}; -x\right] + x^\alpha \frac{\partial}{\partial x} \sum_{k=1}^{\infty} \frac{(-1)^k (\alpha)_k x^k}{(\alpha+1)_k k!}}{\alpha \Gamma(\alpha)}$$

$$= \frac{\alpha x^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha)_k x^k}{(\alpha+1)_k k!} + x^\alpha \sum_{k=1}^{\infty} \frac{(-1)^k (\alpha)_k k x^{k-1}}{(\alpha+1)_k k!}}{\alpha \Gamma(\alpha)}$$

$$= \frac{\alpha x^{\alpha-1} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k (\alpha)_k x^k}{(\alpha+1)_k k!} \left(\frac{\alpha+k}{\alpha}\right)\right]}{\alpha \Gamma(\alpha)}$$

$$= \frac{x^{\alpha-1} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k!}\right]}{\Gamma(\alpha)}$$

$$= \frac{x^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}}{\Gamma(\alpha)} = \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} \tag{29}$$

Next, let us take the derivative of the righthand side using (24) as follows:

$$\frac{\partial \gamma(\alpha, x)}{\partial x} = \frac{1}{\alpha \Gamma(\alpha)} \frac{\partial}{\partial x} \left\{ x^\alpha \Phi \left[\begin{matrix} 1 \\ \alpha+1 \end{matrix}; x \right] e^{-x} \right\}$$

$$= \frac{\frac{\partial}{\partial x}\left\{x^\alpha \Phi\left[\begin{matrix} 1 \\ \alpha+1 \end{matrix}; x\right]\right\} e^{-x} - x^\alpha \Phi\left[\begin{matrix} 1 \\ \alpha+1 \end{matrix}; x\right] e^{-x}}{\alpha \Gamma(\alpha)}$$

$$= \frac{x^{\alpha-1}\left\{\alpha \Phi\left[\begin{matrix} 1 \\ \alpha+1 \end{matrix}; x\right] + x \frac{\partial}{\partial x}\left\{\Phi\left[\begin{matrix} 1 \\ \alpha+1 \end{matrix}; x\right]\right\} - x \Phi\left[\begin{matrix} 1 \\ \alpha+1 \end{matrix}; x\right]\right\} e^{-x}}{\alpha \Gamma(\alpha)}$$

$$= \frac{\alpha x^{\alpha-1} e^{-x}}{\alpha \Gamma(\alpha)} \ (\text{see } (32))$$

$$= \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} \tag{30}$$

where we need to compute the derivative and the numerator separately before we can apply them in (30)

$$\frac{\partial}{\partial x}\left\{\Phi\left[\begin{matrix}1\\ \alpha+1\end{matrix};x\right]\right\} = \frac{\partial}{\partial x}\sum_{k=1}^{\infty}\frac{x^k}{(\alpha+1)_k}$$

$$= \sum_{k=1}^{\infty}\frac{k(1)_k x^{k-1}}{(\alpha+1)_k k!} \tag{31}$$

$$\alpha\Phi\left[\begin{matrix}1\\ \alpha+1\end{matrix};x\right] + x\frac{\partial}{\partial x}\left\{\Phi\left[\begin{matrix}1\\ \alpha+1\end{matrix};x\right]\right\} - x\Phi\left[\begin{matrix}1\\ \alpha+1\end{matrix};x\right] =$$

$$= \alpha\sum_{k=0}^{\infty}\frac{(1)_k x^k}{(\alpha+1)_k k!} + \sum_{k=0}^{\infty}\frac{k(1)_k x^k}{(\alpha+1)_k k!} - x\sum_{k=0}^{\infty}\frac{(1)_k x^k}{(\alpha+1)_k k!}$$

$$= \sum_{k=0}^{\infty}\frac{(\alpha+k-x)(1)_k x^k}{(\alpha+1)_k k!} \text{ (see (25), (31))}$$

$$= \alpha - x + \sum_{k=1}^{\infty}\frac{(\alpha+k-x)(1)_k x^k}{(\alpha+1)_k k!}$$

$$= \alpha - x + \alpha\sum_{k=1}^{\infty}\frac{(1)_k x^k}{(\alpha)_k k!} - x\sum_{k=1}^{\infty}\frac{(1)_k x^k}{(\alpha+1)_k k!}$$

$$= \alpha - x + \alpha\sum_{k=1}^{\infty}\frac{(1)_k x^k}{(\alpha)_k k!} - \sum_{k=1}^{\infty}\frac{x^{k+1}}{(\alpha+1)_k}$$

$$= \alpha - x + \alpha\sum_{k=1}^{\infty}\frac{(1)_k x^k}{(\alpha)_k k!} - \sum_{k=1}^{\infty}\frac{\alpha x^{k+1}}{(\alpha)_{k+1}}$$

$$= \alpha - x + \alpha\sum_{k=1}^{\infty}\frac{(1)_k x^k}{(\alpha)_k k!} - \sum_{k=2}^{\infty}\frac{\alpha x^{k}}{(\alpha)_{k}}$$

$$= \alpha - x + \alpha\sum_{k=1}^{\infty}\frac{(1)_k x^k}{(\alpha)_k k!} - \sum_{k=1}^{\infty}\frac{\alpha x^{k}}{(\alpha)_{k}} + \frac{\alpha x}{(\alpha)_1}$$

$$= \alpha - x + \alpha\sum_{k=1}^{\infty}\frac{(1)_k x^k}{(\alpha)_k k!} - \alpha\sum_{k=1}^{\infty}\frac{(1)_k x^k}{(\alpha)_k k!} + x$$

$$= \alpha \tag{32}$$

Equation (30) is important because it illustrated that Kummer's first transformation [8], [17], [18], [30] (see Appendix A for a detailed proof of the Kummer's Identity) does not change the answer of the derivative which is a product of elementary functions, but it improves the region of convergence in (24).

Next, if we apply the Leibniz integral rule [31], or differentiation under the integral sign rule [32], or Reynolds transport theorem [33] on the lefthand side of (24) we obtain the following:

$$\frac{\partial\gamma(\alpha,x)}{\partial x} = \frac{1}{\alpha\Gamma(\alpha)}\frac{\partial}{\partial x}\left(x^\alpha\int_0^1 e^{-xy^{\frac{1}{\alpha}}}dy\right)$$

$$= \frac{1}{\alpha\Gamma(\alpha)}\left[\alpha x^{\alpha-1}\int_0^1 e^{-xy^{\frac{1}{\alpha}}}dy + x^\alpha\frac{\partial}{\partial x}\left(\int_0^1 e^{-xy^{\frac{1}{\alpha}}}dy\right)\right]$$

$$= \frac{1}{\alpha\Gamma(\alpha)}\left[\alpha x^{\alpha-1}\int_0^1 e^{-xy^{\frac{1}{\alpha}}}dy + x^\alpha\int_0^1\frac{\partial}{\partial x}\left(e^{-xy^{\frac{1}{\alpha}}}\right)dy\right]$$

$$= \frac{1}{\alpha\Gamma(\alpha)}\left[\alpha x^{\alpha-1}\int_0^1 e^{-xy^{\frac{1}{\alpha}}}dy - x^\alpha\int_0^1 e^{-xy^{\frac{1}{\alpha}}}y^{\frac{1}{\alpha}}dy\right] \tag{33}$$

Since, we already know that the answer of (33) is then we equate (33) with (30) and we obtain:

$$x\int_0^1 e^{-xy^{\frac{1}{\alpha}}}y^{\frac{1}{\alpha}}dy = \alpha\left(\int_0^1 e^{-xy^{\frac{1}{\alpha}}}dy - e^{-x}\right),\ 0 \le x < \infty$$

$$= \alpha\left(\Phi\left[\begin{matrix}\alpha\\ \alpha+1\end{matrix};-x\right] - e^{-x}\right),\ (0 \le x < 1)$$

$$= \alpha e^{-x}\left(\Phi\left[\begin{matrix}1\\ \alpha+1\end{matrix};x\right] - 1\right),\ 0 \le x < 20 \tag{34}$$

Equation (34) might be a new integral that may not currently exist in the Table of Integral, Series, and Products similar to (see Gradshteyn, Ryzhik, 2007 [24])

This concludes the discussion on landmark computation of the derivatives of the IGF for non-integer values of a parameter.

5 Integration of the RIGF for Non-Integer Values of a Parameter

Having computed the new integrals of the RIGF, let us review them.

From the RIGF (19) and (23) we have

$$\gamma(\alpha,x) = \frac{\int_0^1 e^{-xy^{1/\alpha}}dy}{\int_0^\infty e^{-xy^{1/\alpha}}dy} \equiv \frac{\Phi\left[\begin{matrix}\alpha\\ \alpha+1\end{matrix};-x\right]}{\int_0^\infty e^{-xy^{1/\alpha}}dy} \tag{35}$$

From the RIGF (9) and Batman, ([21], pg. 133 (3) 1953) and (see Gradshteyn, Ryzhik, 2007 [24] pg. 899 ex. 8.351 2.)

$$\gamma(\alpha,x) = \frac{\int_0^x t^{\alpha-1}e^{-t}dt}{\Gamma(\alpha)} = \frac{x^\alpha\alpha^{-1}\Phi\left[\begin{matrix}\alpha\\ \alpha+1\end{matrix};-x\right]}{\Gamma(\alpha)} \tag{36}$$

Equating both sides of (35) and (36) yields

$$\int_0^\infty e^{-xy^{1/\alpha}}dy = \alpha x^{-\alpha}\Gamma(\alpha),\ x > 0 \tag{37}$$

Equation (37) exists in (see Gradshteyn, Ryzhik, 2007 [24] pg. 365 ex. 3.462 8.) which is a new entry, not found in any of the previous references

$$\int_0^\infty e^{-\beta x^n \pm a}dy = \frac{e^{\pm a}}{n\beta^{1/n}}\Gamma\left(\frac{1}{n}\right);\ \mathrm{Re}\beta > 0,\ \mathrm{Re}n > 0 \tag{38}$$

If we take

$$a = 0 \tag{39}$$

$$\beta = x \tag{40}$$

$$x = y \tag{41}$$

$$n = 1/\alpha \tag{42}$$

Substituting (39)-(42) into (38) yields:

$$\int_0^\infty e^{-\beta x^n \pm a}dy = \int_0^\infty e^{-xy^{1/\alpha}}dy = \frac{\alpha\Gamma(\alpha)}{x^\alpha} = \alpha x^{-\alpha}\Gamma(\alpha) \tag{43}$$

Let us define the new function as $\Gamma_x(\alpha)$

$$\Gamma_x(\alpha) = \frac{\alpha\Gamma(\alpha)}{x^\alpha} = \frac{\alpha}{x^\alpha}\frac{1}{\alpha}\prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^\alpha}{1+\frac{\alpha}{k}} = x^{-\alpha}\prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^\alpha}{1+\frac{\alpha}{k}} \tag{44}$$

Taking the logarithm of both side of (44) yields

$$\log[\Gamma_x(\alpha)] = -\alpha \log x + \sum_{k=1}^{\infty}\left[\alpha \log\left(1+\frac{1}{k}\right) - \log\left(1+\frac{\alpha}{k}\right)\right]$$

$$= \alpha\left[\sum_{k=1}^{\infty}\log\left(1+\frac{1}{k}\right) - \log x\right] - \sum_{k=1}^{\infty}\log\left(1+\frac{\alpha}{k}\right)$$

$$= \alpha \sum_{k=1}^{\infty}\left[\log\left(1+\frac{1}{k}\right) - \frac{\log\left(1+\frac{\alpha}{k}\right)}{\alpha} - \frac{1}{k}\left(1-\frac{1}{x}\right)^{k}\right] \quad (45)$$

Then we go back and utilizing the definition (44) into (36) we obtain a new definition of the RIGF as follows:

$$\gamma(\alpha, x) = \frac{\int_0^1 e^{-xy^{1/\alpha}}dy}{\Gamma_x(\alpha)} \equiv \frac{\Phi\left[\begin{matrix}\alpha\\ \alpha+1\end{matrix};-x\right]}{\Gamma_x(\alpha)}, \; 0 \le x < 1$$

$$\equiv \frac{e^{-x}\Phi\left[\begin{matrix}1\\ \alpha+1\end{matrix};x\right]}{\Gamma_x(\alpha)}, \; 0 \le x < 20 \quad (46)$$

Equation (46) is only suitable to identify that the issues with the region of convergence have to do with the computation of the numerator because it has simplified even further the regularization or the normalization of the RIGF. The only way to improve the computation of the RIGF is by means of transformations, which will be the subject of future research.

Moreover, in Appendix A I gave a proof and a profound new understanding of the Kummer's First Transformation as it relates to its uniqueness that is further explained in Appendix B.

At this point we have complete confidence that the new integrals (19), (20) are in fact valid integrals of the RIGF.

The integration of the RIGF occurs in many cumulative probability distribution functions (CDF) such as the generalized Bessel function distribution of the first kind (GBFD1K) [1], [5], [13] or second kind (GBFD2K) [1], [5], [6].

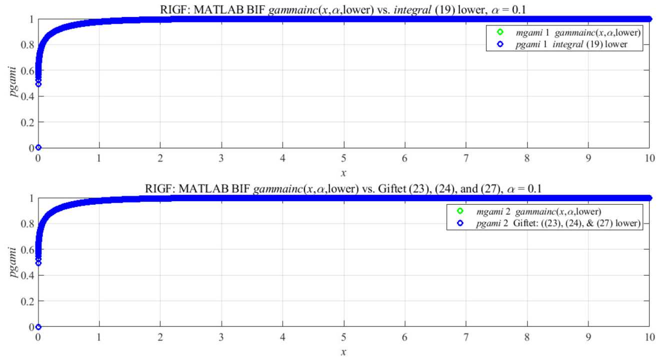
The landmark computation of the GBFD1K and GBFD2K will contain all the discussion on the landmark integration of the RIGF.

This concludes the discussion on the integration of the RIGF for non-integer values of a parameter.

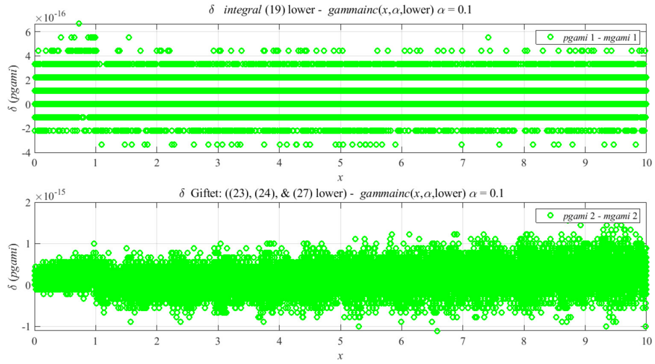
6 Numerical, Theoretical Results

Much has been said about the test setup that I began in Progri (2016, [1]) and I have significantly improved in Progri (2022, [14]).

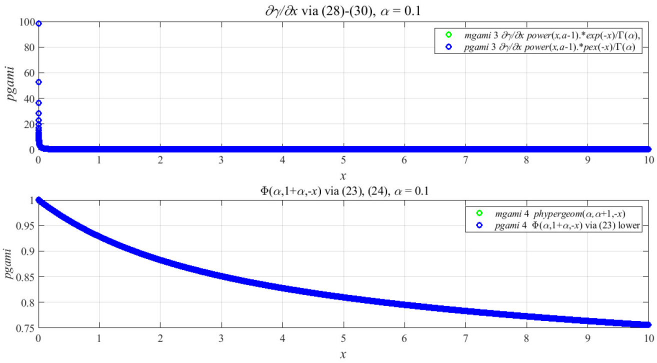
The numerical[xiii], computational, theoretical results consist of four major experiments. The first to the fourth experiments respectively are designed to compare and contrast the computation of:



(a) (top) *mgmi* the truth (see §6.1) and *pgmi* the estimated (see §6.1); (bottom) *mgmi* the truth (see §6.2) and *pgmi* the estimated (see §6.2).



(b) (top) absolute error of *pgmi* (a) minus *mgmi* (a) (see §6.1); (bottom) absolute error of *pgmi* (a) minus *mgmi* (a) (see §6.2).



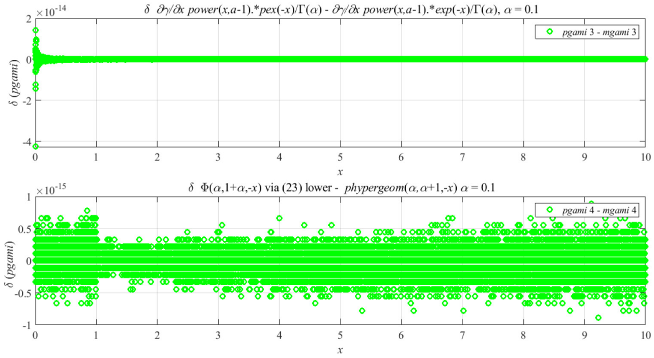
(c) (top) *mgmi* the truth (see §6.3) and *pgmi* the estimated (see §6.3); (bottom) *mgmi* the truth (see §6.4) and *pgmi* the estimated (see §6.4).

§6.1. *mgmi* (based on MATLAB built in function (BIF) *gammainc* $0 < \alpha < 1$) with the MATLAB BIF *integral* in (9) for $\alpha \ge 1$ in *pgmi* (based on MATLAB BIF *integral*) in (19) and (20).

§6.2. *mgmi* (same as in 1.) vs the MATLAB computation of (23), (24), and (27) in *pgmi*.

§6.3. the derivative based on MATLAB BIF *exp*, and ^ operator vs. the (28), (29), and (30) via the *pexp* Progri (2022, [14]) and the utilization of the power MATLAB BIF.

§6.4. the numerator on (35) based on *mgmi* (based on Giftet



(d) (top) absolute error of *pgmi* (c) minus *mgmi* (c); (bottom) absolute error of *pgmi* (c) minus *mgmi* (c).

FIGURE 1: (a)/(c) (top) *mgmi* the truth (see §6.1 (a) & §6.3 (c)) and *pgmi* the estimated (see §6.1 (a) & §6.3 (c)); (bottom) *mgmi* the truth (see §6.1 (a) & §6.3 (c)) and *pgmi* the estimated (see §6.2 (a) & §6.4 (c)) and (b)/(d) (top) absolute error of *pgmi* (a)/(c) minus *mgmi* (a) ((see §6.1 (a) & §6.3 (c))); (bottom) absolute error of *pgmi* (a)/(c) minus *mgmi* (a)/(c) (see §6.2) & §6.4).for $\alpha = 0.1,\ x = 0{:}\,\Delta x{:}\,10,\ \Delta x = 1e^{-2}$.

phypergeom) and then based on the MATLAB computation of (23), (24) in *pgmi*.

Detail explanation of the simulation results and the graphical illustration of these experiments are given below.

6.1 Landmark Integral Expression of the IGF

The first experiment is designed to show how capable is MATLAB to produce the truth, and how to integrate the integrals (9) vs. (19) and (20) by means of the MATLAB BIF *integral*.

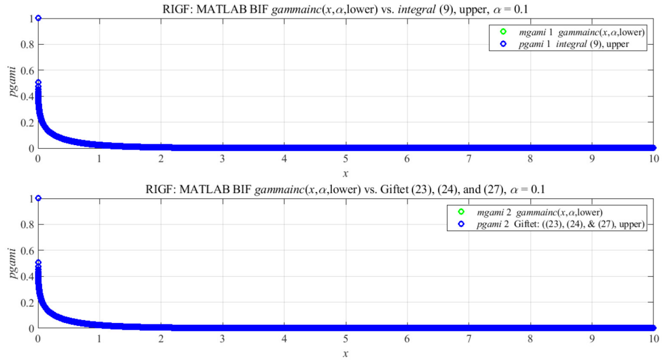
The *mgmi* (based on MATLAB built in function (BIF) *gammainc* $0 < \alpha < 1$) (see Figs. 1-3 (a)(top)) or the MATLAB BIF *integral* in (9) for $\alpha \geq 1$ (see Figs. 4-7(a)(top))) is considered the truth; *pgmi* (based on MATLAB BIF *integral*) is the function we would like to compare and contrast to *mgmi*.

The MATLAB function *mgami* performs comparison of the above and outputs the results in *mgmi* and *pgmi*.

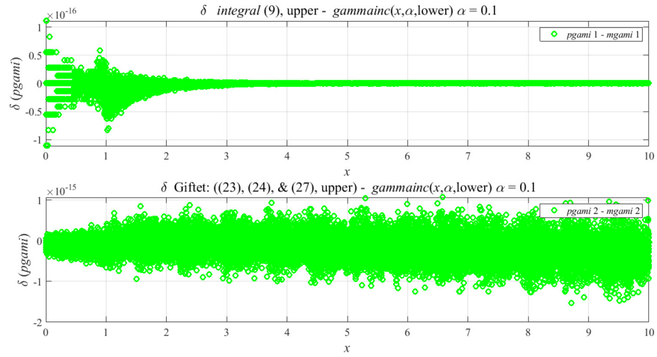
The simulation rest setup has three options:

1. Option 1 is used to compute the *lower* RIGF, and it is the default option.
2. Option 2 is used to compute the *upper* RIGF.
3. Option 3 is used to compute the *natural logarithm* of the *upper* RIGF as is the case of §6.1, §6.2 or the *lower* RIGF is the case of §6.3, §6.4.

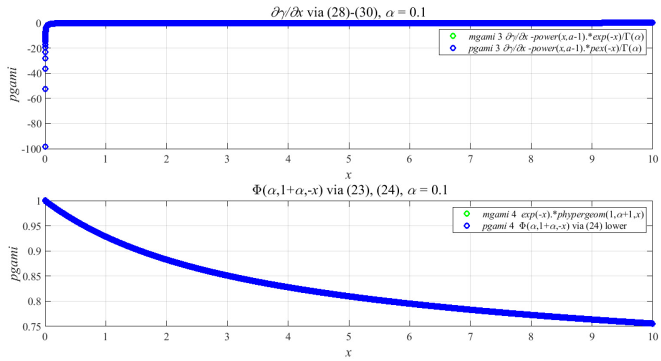
As we shall see from the simulation results is that the computation of the (9), (19), (20) is successful, somewhat



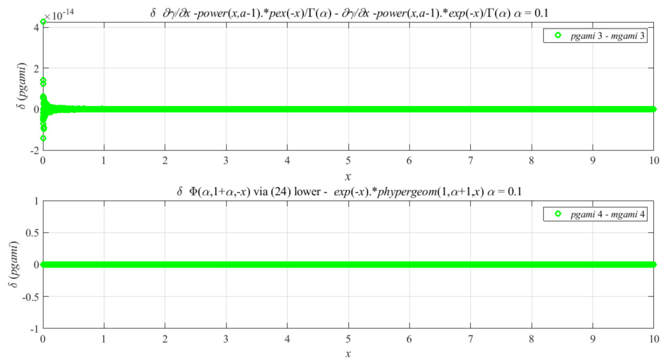
(a) Ditto Fig.1(a).



(b) Ditto Fig.1(b).



(c) Ditto Fig.1(c).



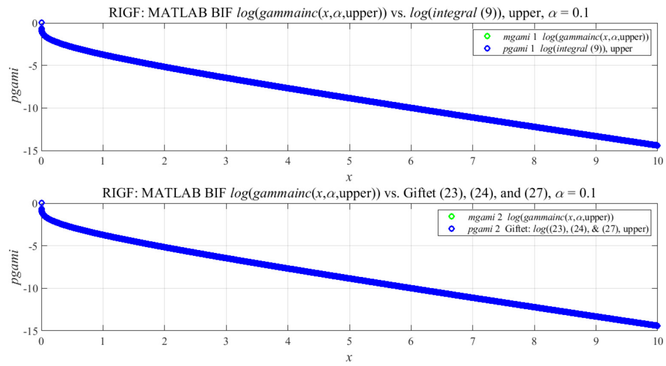
(d) Ditto Fig.1(d).

FIGURE 2: Same as Fig. 1 but, ‘upper’.

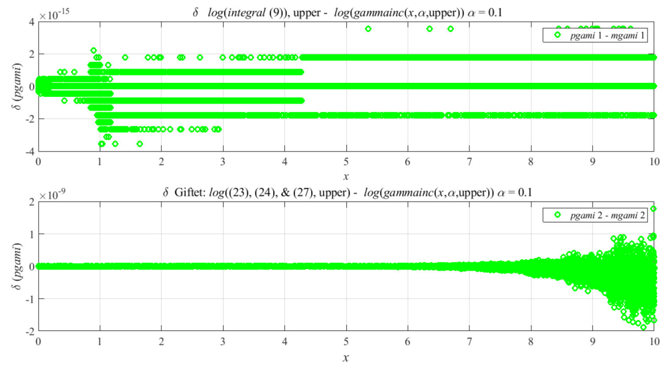
successful, unsuccessful respectively.

The first experiment showed three major weaknesses of the current MTLAB BIFs *gammainc* and *integral*.

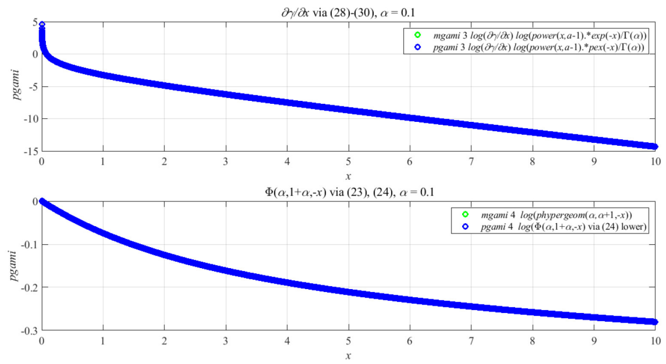
1. The MTLAB BIF *gammainc* is a very good approximation



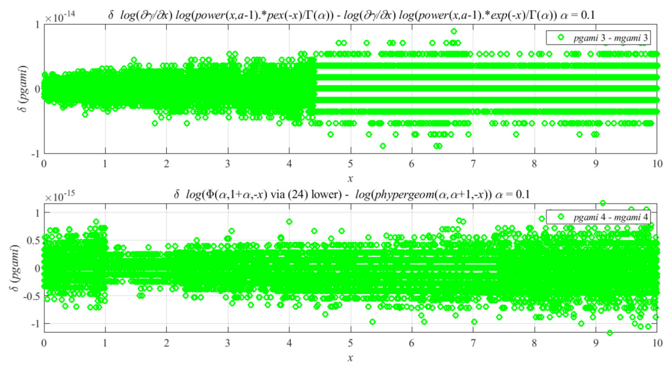
(a) Ditto Fig.1(a).



(b) Ditto Fig.1(b).



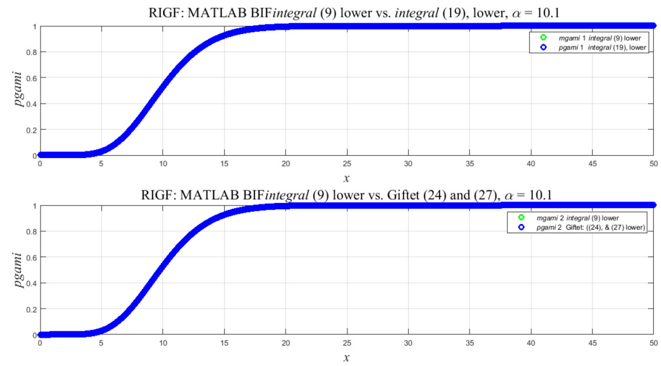
(c) Ditto Fig.1(c).



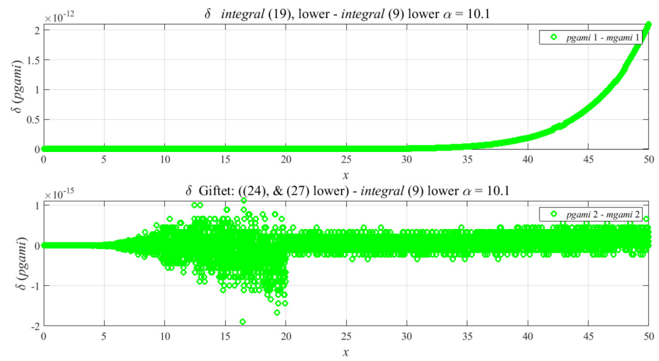
(d) Ditto Fig.1(d).

FIGURE 3: Same as Fig. 1 but log('upper') (a) and (b) and log('lower') (c) and (d).

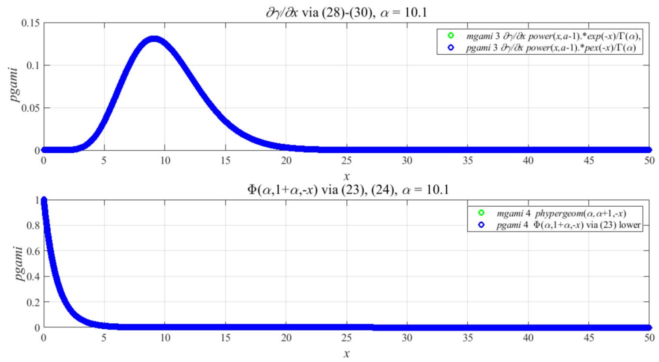
of the truth $0 < \alpha < 1$, (see Figs. 1-3 (a)(top)). For values of the parameter $\alpha \geq 1$ the MATLAB BIF *integral* (9) is a much better approximation of the truth (see Figs. 4-7(a)(top))). The only problem with using the MATLAB BIF *integral* (9) vs. *gammainc* is that *integral* (9) will be a lot slower than *gammainc*.



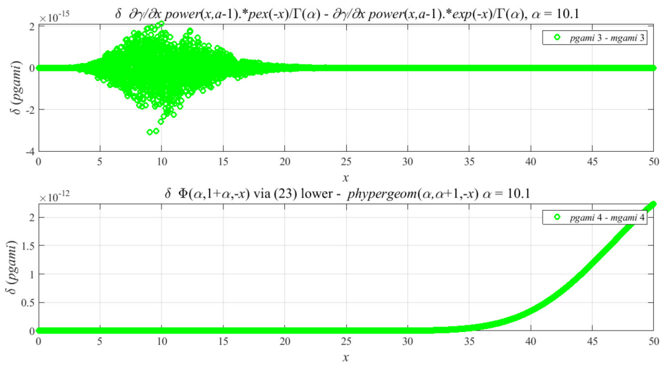
(a) Ditto Fig.1(a).



(b) Ditto Fig.1(b).

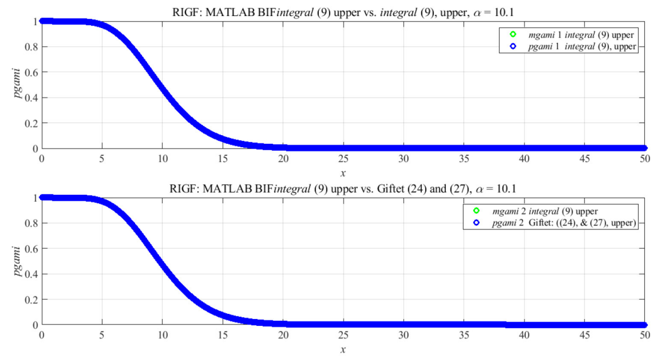


(c) Ditto Fig.1(c).

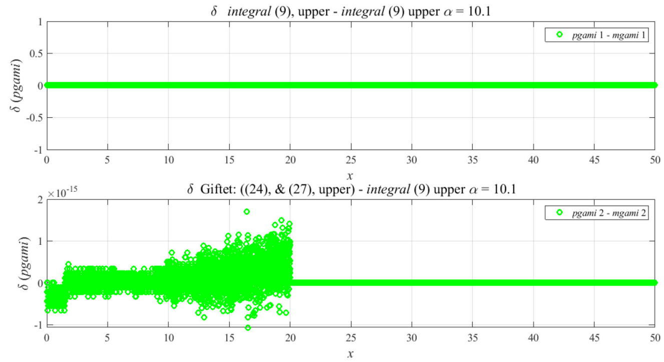


(d) Ditto Fig.1(c).

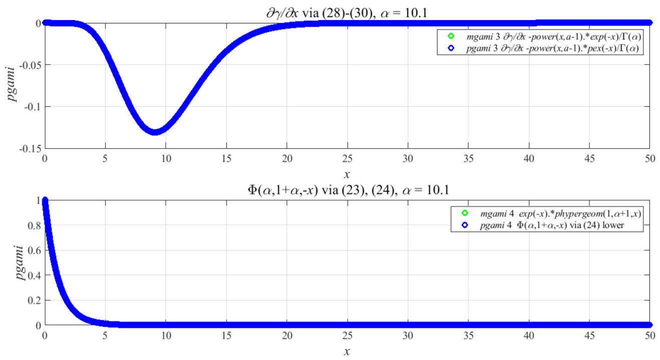
FIGURE 4: Same as Fig. 1 but for $\alpha = 10.1$, $x = 0{:}\,\Delta x{:}\,50$, $\Delta x = 1e^{-2}$, 'lower'.



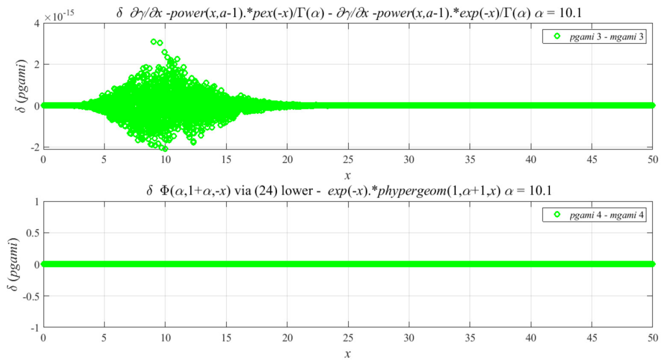
(a) Ditto Fig.4(a).



(b) Ditto Fig.4(b).

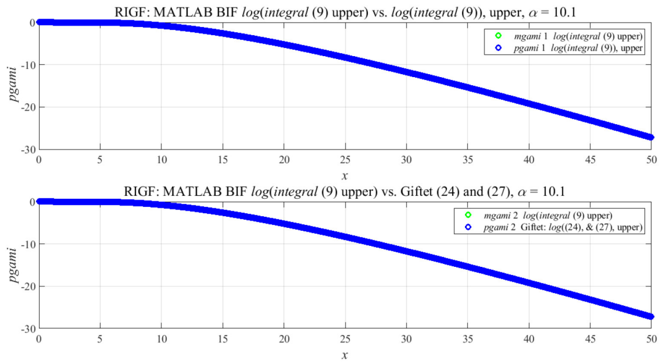


(c) Ditto Fig.4(c).

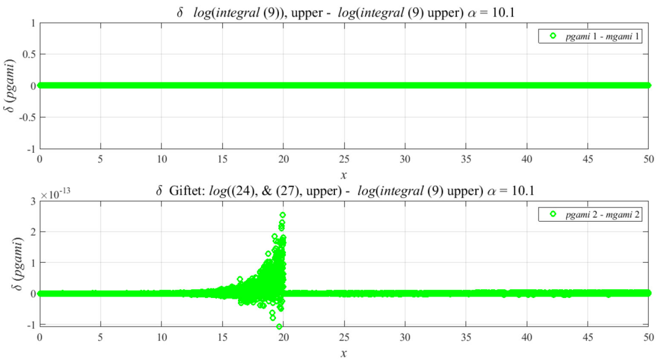


(d) Ditto Fig.4(c).

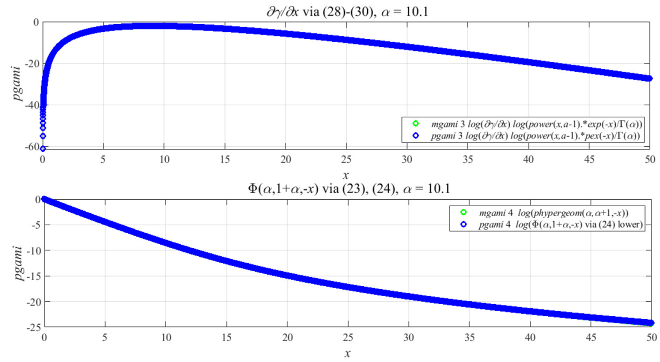
FIGURE 5: Same as Fig. 4 but, ‘upper’.



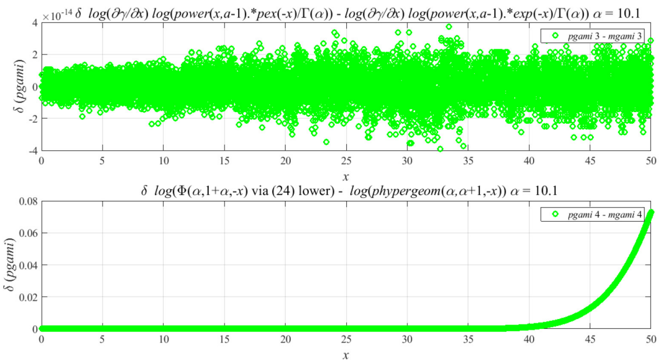
(a) Ditto Fig.4(a).



(b) Ditto Fig.4(b).



(c) Ditto Fig.4(c).

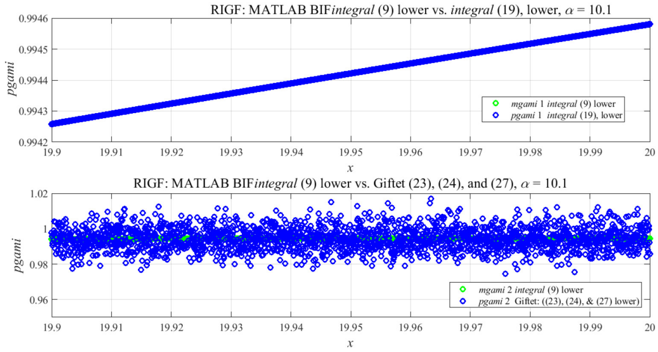


(d) Ditto Fig.4(c).

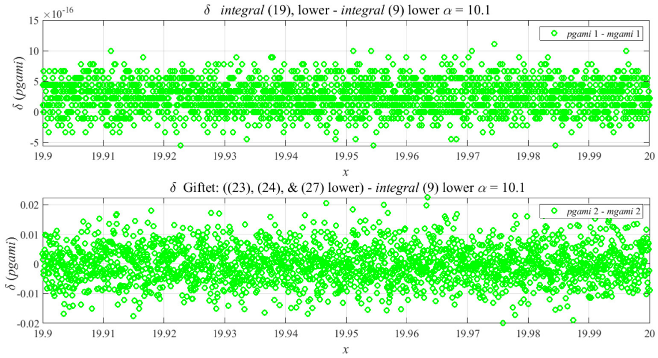
FIGURE 6: Same as Fig. 4 but log(‘upper’).

2. The MATLAB BIF *integral* (19) appears to fade in precision for large values of x (see Fig. 4(b)(top)).
3. The first experiment exposed a major weakness of MATLAB in its inability to compute the integral of the form of (20). Integral (20) is the computation of the integral from one to infinity of an exponential function of a complicated negative exponent.

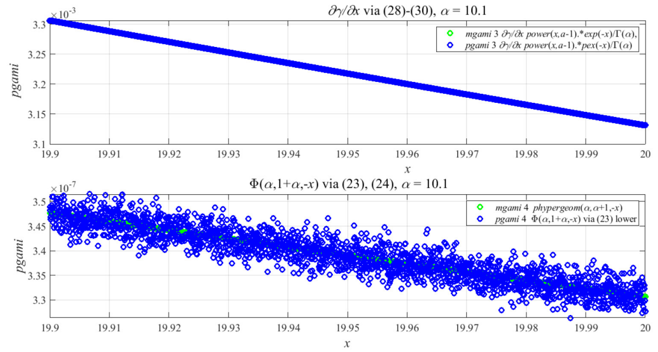
The first experiment is designed to give MATLAB an



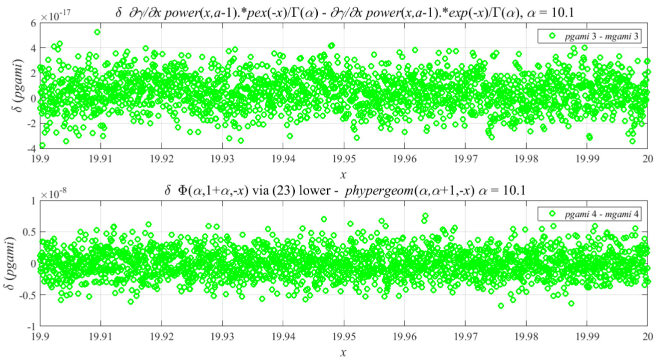
(a) Ditto Fig.4(a).



(b) Ditto Fig.4(b).



(c) Ditto Fig.4(c).



(d) Ditto Fig.4(c).

FIGURE 7: Same as Fig. 4 but for $x = 19.9{:}\Delta x{:}20,\ \Delta x = 1e^{-5}$, 'lower'.

understanding on how good its BIF for computing the RIGF (9), (19), and (20) are.

6.2 Landmark Computation of the IGF for Non-Integer Values of a Parameter

The second experiment is designed to show how realistic are the computation of RIGF by means of (23), (24), and (27) using the MATLAB function *pgami* in *pgmi* (see Figs. 1-7(a)(bottom)).

Ditto *mgmi*. I.e., the MATLAB function *pgami* compares and contrasts *pgmi* with *mgmi* which is considered the truth.

In addition to the three options used in the first experiment, in the second experiment there are two options within each of the options of the first experiment; i.e., the second experiment contains a total of six options. Sub options are designed to show the split of the interval of time into

1. three successive (sequential, or consecutive, or one after another) intervals. For each interval we apply (23), (24), and (27) respectively. This sub option is only suitable for $0 < \alpha \ll 1$ (see Figs. 1-3(a)(bottom)).
2. two successive intervals. For each interval we apply (24) and (27) respectively. This sub option is only suitable for $0 < \alpha \sim 1$ or $\alpha \gg 1$ (see Figs. 4-7(a)(bottom)).

The results of this experiment were quite successful.

6.3 Computation of the Derivative of the RIGF for Non-Integer Values of a Parameter

The third experiment is designed to compare and contrast the computation of the RIGF in the *dgami* where the *mgmi* contains the truth via MATLAB (BIF *gammainc*, *exp*, and MATLAB identities via MATLAB operator ^) and the computation of (28), (29), and (30) via the *pexp* Progri (2022, [14]) and the utilization of the power MATLAB BIF (see Figs. 1-7(c)(top)).

In the third experiment we can make use of the first three options that were utilized in the first two experiments.

The results of this experiment were quite successful.

6.4 Integration of the RIGF for Non-Integer Values of a Parameter

The fourth and the final experiment is designed to show how important is the region of convergence of the numerator on (35) based on *mgmi* (based on Giftet *phypergeom* function) and then based on the MATLAB computation of (23), (24) in *pgmi* (see Figs. 1-7(c)(bottom)).

If the RIGF will be utilized in the integration problems then we must explain really well that the issues of the numerical accuracy of the integration of the RIGF have to do with the ability to precisely compute the numerator on (35) or (23), (24) in *pgmi.*

The graphical simulation results of this experiment were aided with the help of six options that were discussed in the second experiment.

We were quite successful in pointing out that understanding the regions of convergence makes a significant on how the integration of RIGF will be performed in Progri (2022, [15]).

Figs. 1-7(b), (d) show the error which is defined as the difference of *pgmi* the estimated with *mgmi* the truth from Figs. 1-7(a), (c) respectively.

We have illustrated the simulation results with a total of fifty-six plots: thirty-eight of them belong to the computation of the truth and thirty-eight of the estimated and thirty-eight of the error which is the difference between the estimated with the truth.

7 Conclusions

This journal paper provides significantly added value in the computation of the RIGF because:

1. Three new integrals have been added that demonstrate that the RIGF is in fact the integration of an exponential function with a more complicated exponent.
2. The power series expansion of the exponential function is in fact the root cause of all the computational limitations to obtain full precision in the computation of the RIGF.
3. The Kummer's first transformation is unique and it expands the region of convergence of the RIGF.
4. The inverse of the CHF is not easily obtainable.
5. The Derivative of RIGF is the same regardless of the transformations.
6. A practical implementation of all the CFEs of the RIGF are beautifully assessed in the journal paper.
7. It is recommended that MATLAB should improve both *gammainc* and *integral* BIFs to meet the expectations that are required in the computation of the RIGF.

Therefore, because of 1-7 this journal paper is considered a *landmark* computation of the RIGF.

Giftet MATLAB functions which are part of the Giftet Indoor Geolocation Systems—Theory and Applications library produce a remarkably accurate computation of the RIGF based on the CFEs discussed in this journal paper.

8 Acknowledgement

This work was supported by Giftet Inc. executive office.

I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [37] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications*. *Vol. I* (Not yet available in print) [34].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say, I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [35] reminds me of my grandfather who I loved him very much.

Finally, I am also dedicating this journal paper to our most beloved President Ronald Reagan, who reminds me of my grandfather, [36] who spoke from the heart, who spoke the truth, who was a staunch supporter of personal freedom, the greatest leader of the free world, and perhaps the greatest critic of the wasteful government control and spending[xiv].

"*Government is not the answer, government is the problem*."—Ronald Reagan

I would like to express my deepest gratitude to my undergraduate University Physics professor, Niko Thomo. When I was studying at the Tirana Polytechnic University, Tirana, Albania, from 1989-1994, I used to visit him at his home frequently and show him my progress in my physics exercises. He inspired me unlike any other professor. As a result of my persistent work under his direction, I was accepted to study at Worcester Polytechnic University, Worcester, Massachusetts in August of 1995.

9 References

[1] I.F. Progri, "Generalized Bessel function distributions," *J. Geol. Geoinfo. Geointel.,* vol. 2016, article ID 2016071602, 15 pg., Nov. 2016. DOI: https://doi.org/10.18610/JG3.2016.071602

[2] I.F. Progri, "Exponential generalized Beta distribution," *J. Geol. Geoinfo. Geointel.,* vol. 2016, article ID

2016071603, 18 pg., Nov. 2016. DOI: https://doi.org/10.18610/JG3.2016.071603

[3] I.F. Progri, "Hypergeometric function partial derivatives," vol. 2016, article ID 2016071604, 21 pg., Nov. 2016. DOI: https://doi.org/10.18610/JG3.2016.071604

[4] I.F. Progri, "Generalized parabolic cylinder function distribution," vol. 2016, article ID 2016071605, 14 pg., Nov. 2016. DOI: https://doi.org/10.18610/JG3.2016.071605

[5] I.F. Progri, "Efficient computation of generalized Bessel function distributions," *J. Geol. Geoinfo. Geointel.*, vol. 2018, article ID 2018071604, 12 pg., Nov. 2018. DOI: https://doi.org/10.18610/JG3.2018.071605

[6] I.F. Progri, "Efficient computation of the special cases of the generalized Bessel function distributions," *J. Geol. Geoinfo. Geointel.,* vol. 2019, article ID 2019071601, 31 pg., Nov. 2019. DOI: https://doi.org/10.18610/JG3.2019.071601

[7] I.F. Progri, "The computation of a 2F2 hypergeometric function," *J. Geol. Geoinfo. Geointel.,* vol. 2020, article ID 2020071601, 12 pg., Nov. 2020. DOI: https://doi.org/10.18610/JG3.2020.071601

[8] I.F. Progri, "Confluent hypergeometric function irregular singularities," *J. Geol. Geoinfo. Geointel.,* vol. 2021, article ID 2021071601,12 pg., Nov. 2021. DOI: https://doi.org/10.18610/JG3.2021.071601

[9] I.F. Progri, "Efficient computation of the generalized parabolic cylinder function distribution," *J. Geol. Geoinfo. Geointel.,* vol. 2021, article ID 2021071602, 17 pg., Nov. 2021. DOI: https://doi.org/10.18610/JG3.2021.071602

[10] I.F. Progri, "Landmark computation of the generalized parabolic cylinder function distribution," *J. Geol. Geoinfo. Geointel.,* vol. 2021, article ID 2021071603, 15 pg., Nov. 2021. DOI: https://doi.org/10.18610/JG3.2021.071603

[11] I.F. Progri, "Special cases of the generalized parabolic cylinder function distribution," *J. Geol. Geoinfo. Geointel.,* vol. 2021, article ID 2021071604, 12 pg., Nov. 2021. DOI: https://doi.org/10.18610/JG3.2021.071604

[12] I.F. Progri, "The significance of Kampé de Fériet functions in the computation of certain generalized distributions," *J. Geol. Geoinfo. Geointel.,* vol. 2021, article ID 2021071605, 7 pg., Nov. 2021. DOI: https://doi.org/10.18610/JG3.2021.071605

[13] I.F. Progri, "Landmark computation of generalized Bessel function distributions of the first kind: part 1," *J. Geol. Geoinfo. Geointel.*, vol. 2022, article ID 2022071601, 13 pg., Nov. 2022. DOI: https://doi.org/10.18610/JG3.2022.071601

[14] I.F. Progri, "A study of the computation of the exponential function," *J. Geol. Geoinfo. Geointel.*, vol. 2022, article ID 2022071602, 10 pg., Nov. 2022. DOI: https://doi.org/10.18610/JG3.2022.071602

[15] I.F. Progri, "Landmark computation of generalized Bessel function distributions of the first kind: part 2," *J. Geol. Geoinfo. Geointel.*, vol. 2022, article ID 2022071604, 10 pg., Nov. 2022. DOI: https://doi.org/10.18610/JG3.2022.071604

[16] A.M. Legendre, *Traité des Fonctions Elliptiques et des Intégrales Eulériennes, avec des Tables pour en Faciliter le Calcul Numérique*. Vol. 2. Paris: Huzard-Courcier, 1826.

[17] E.E. Kummer, "De integralibus quibusdam definitis et seriebus infinitis," *Journal für die reine und angewandte Mathematik (in Latin)*, vol. 17, pp. 228-242, 1837. DOI: https://doi.org/10.1515/crll.1837.17.228

[18] N. Nielsen, *Handbuch der Theorie der Gammafunktion*. Leipzig: Teubner, 326 pg., 1906.

[19] K. Pearson, *Tables of the Incomplete Gamma-Function*, London: Dept. Sci. & Indus. Res., 167 pg., March. 1922.

[20] H. Bateman, *Higher Transcendental Functions*. Vol. 1, New York, NY: McGraw-Hill, 302 pg., 1953.

[21] H. Bateman, *Higher Transcendental Functions*. Vol. 2, New York, NY: McGraw-Hill, 414 pg. 1953.

[22] H. Bateman, *Higher Transcendental Functions*. Vol. 3, New York, NY: McGraw-Hill, 292 pg. 1953.

[23] G.B. Arfken, H.J. Weber, *Mathematical Methods for Physicists*, San Diego, CA: Academic Press, 1995.

[24] I.S. Gradshteyn, I.M. Ryzhik, (A. Jeffrey, D. Zwillinger, editors) *Table of Integrals, Series, and Products*, 7th ed., Burlington, MA: Academic Press, 1171 pp., 2007.

[25] Anon, "Gamma function," *Wikipedia, the free encyclopedia*, June 2022. https://en.wikipedia.org/wiki/Gamma_function

[26] Anon, "Incomplete Gamma function," *Wikipedia, the free encyclopedia*, June 2022. https://en.wikipedia.org/wiki/Incomplete_gamma_function

[27] Anon, "Regularization (mathematics)," *Wikipedia, the free encyclopedia*, June 2022.

https://en.wikipedia.org/wiki/Regularization_(mathematics)

[28] Anon, “Hypergeometric function,” *Wikipedia, the free encyclopedia*, June 2022. https://en.wikipedia.org/wiki/Hypergeometric_function

[29] Anon, “Pochhammer symbol,” *Wikipedia, the free encyclopedia*, June 2022. https://en.wikipedia.org/wiki/Pochhammer_symbol

[30] Anon., “Confluent hypergeometric function,” *From Wikipedia, the free encyclopedia*, June 2022. https://en.wikipedia.org/wiki/Confluent_hypergeometric_function

[31] Anon., “Leibniz integral rule,” *From Wikipedia, the free encyclopedia*, Mar. 2022. https://en.wikipedia.org/wiki/Leibniz_integral_rule

[32] Anon., “Differentiation under the integral sign,” *From Wikipedia, the free encyclopedia*, Mar. 2022. https://en.wikipedia.org/wiki/Differentiation_under_the_integral_sign

[33] Anon., “Reynolds transport theorem,” *From Wikipedia, the free encyclopedia*, Mar. 2022. https://en.wikipedia.org/wiki/Reynolds_transport_theorem

[34] I. Progri, *Indoor Geolocation Systems—Theory and Applications. Vol. I*, 1st ed., Worcester, MA: Giftet Inc., ~800 pp., ~ 2022. (not yet available in print)

[35] Anon., “Jack Nicklaus,” *Wikipedia, the free encyclopedia*, Oct. 2021. https://en.wikipedia.org/wiki/Jack_Nicklaus

[36] Anon., “Ronald Reagan,” *Wikipedia, the free encyclopedia*, June 2022. https://en.wikipedia.org/wiki/Ronald_Reagan

[37] Anon, “MATLAB 2022b,” *The MathWorks, Inc.*, Natick, MA, Copyright © 1994-2022, The MathWorks, Inc., http://mathworks.com/products/new_products/release2022b.html?requestedDomain=mathworks.com

values of the parameter α.

10 Appendix A: Identity of Kummer’s First Transformation

In this Appendix we attempt to search for a transformation similar to the Kummer’s first transformation. The question that we would like to ask is the following: Is Kummer’s first transformation or identity unique? We will assume the opposite, which is that we will assume that there exists more than one solution in the form of the product of two functions similar to the Kummer’s first transformation. Then we will prove that there can only be one solution which is identical to the the Kummer’s first transformation which is known as the Kummer’s identity.

Lemma 1: Let us assume that the solution of the numerator of the RIGF is equal to

$$\int_0^1 e^{-xy^{\frac{1}{\alpha}}}dy = e^{-\beta x}\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]^{\text{xv}} = e^{-x}\Phi\left[\begin{matrix}1\\ \alpha+1\end{matrix};x\right] \quad (47)$$

such that $0<\beta\leq 1$. $\rho\in\mathbb{R}$.

Objective: If the Kummer’s first transformation is unique in the form of (47) then there should be only one solution for the parameters β, ρ. If the Kummer’s first transformation is not unique then there should be more than one solution.

Proof: From (47) we have two unknowns β, ρ but only one equation.

If we take the derivative from both sides, we can obtain another relation as given below:

$$\begin{aligned}
\frac{\partial\gamma(\alpha,x)}{\partial x} &= \frac{1}{\alpha\Gamma(\alpha)}\frac{\partial}{\partial x}\left\{x^{\alpha}\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]e^{-\beta x}\right\}\\
&= \frac{\frac{\partial}{\partial x}\left\{x^{\alpha}\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]\right\}e^{-\beta x}-\beta x^{\alpha}\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]e^{-\beta x}}{\alpha\Gamma(\alpha)}\\
&= \frac{x^{\alpha-1}\left\{\alpha\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]+x\frac{\partial}{\partial x}\left\{\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]\right\}-\beta x\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]\right\}e^{-\beta x}}{\alpha\Gamma(\alpha)}\\
&= \frac{x^{\alpha-1}\alpha\Phi\left[\begin{matrix}\rho-1\\ \alpha\end{matrix};x\right]e^{-\beta x}}{\alpha\Gamma(\alpha)}\\
&= \frac{x^{\alpha-1}\Phi\left[\begin{matrix}\rho-1\\ \alpha\end{matrix};x\right]e^{-\beta x}}{\Gamma(\alpha)} = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} \qquad (48)
\end{aligned}$$

where we need to compute the derivative and the numerator separately before we can apply them in (48)

$$\begin{aligned}
\frac{\partial}{\partial x}\left\{\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]\right\} &= \frac{\partial}{\partial x}\sum_{k=1}^{\infty}\frac{(\rho)_k x^k}{(\alpha+1)_k k!}\\
&= \sum_{k=1}^{\infty}\frac{k(\rho)_k x^{k-1}}{(\alpha+1)_k k!} \qquad (49)
\end{aligned}$$

$$\begin{aligned}
&\alpha\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]+x\frac{\partial}{\partial x}\left\{\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]\right\}-\beta x\Phi\left[\begin{matrix}\rho\\ \alpha+1\end{matrix};x\right]=\\
&= \alpha\sum_{k=0}^{\infty}\frac{(\rho)_k x^k}{(\alpha+1)_k k!}+\sum_{k=0}^{\infty}\frac{k(\rho)_k x^k}{(\alpha+1)_k k!}-x\beta\sum_{k=0}^{\infty}\frac{(\rho)_k x^k}{(\alpha+1)_k k!}\\
&= \sum_{k=0}^{\infty}\frac{(\alpha+k-\beta x)(\rho)_k x^k}{(\alpha+1)_k k!} \text{ (see (24), (49))}\\
&= \alpha-\beta x+\sum_{k=1}^{\infty}\frac{(\alpha+k-\beta x)(\rho)_k x^k}{(\alpha+1)_k k!}\\
&= \alpha-\beta x+\alpha\sum_{k=1}^{\infty}\frac{(\rho)_k x^k}{(\alpha)_k k!}-\beta x\sum_{k=1}^{\infty}\frac{(\rho)_k x^k}{(\alpha+1)_k k!}\\
&= \alpha-\beta x+\alpha\sum_{k=1}^{\infty}\frac{(\rho)_k x^k}{(\alpha)_k k!}-\beta\sum_{k=1}^{\infty}\frac{(\rho)_k x^{k+1}}{(\alpha+1)_k k!}
\end{aligned}$$

$$= \alpha - \beta x + \alpha \sum_{k=1}^{\infty} \frac{(\rho)_k x^k}{(\alpha)_k k!} - \beta \sum_{k=2}^{\infty} \frac{\alpha(\rho)_{k-1} x^k}{(\alpha)_k (k-1)!}$$

$$= \alpha - \beta x + \alpha \sum_{k=1}^{\infty} \frac{(\rho)_k x^k}{(\alpha)_k k!} - \beta \sum_{k=1}^{\infty} \frac{\alpha(\rho)_{k-1} x^k}{(\alpha)_k (k-1)!} + \frac{\beta \alpha x}{(\alpha)_1}$$

$$= \alpha + \alpha \sum_{k=1}^{\infty} \frac{(\rho)_k x^k}{(\alpha)_k k!} - \alpha \sum_{k=1}^{\infty} \frac{k(\rho)_k x^k}{(\rho+k-1)(\alpha)_k k!}$$

$$= \alpha + \alpha \sum_{k=1}^{\infty} \left(1 - \frac{k}{\rho+k-1}\right) \frac{(\rho)_k x^k}{(\alpha)_k k!}$$

$$= \alpha + \alpha(\rho - 1) \sum_{k=1}^{\infty} \frac{(\beta)_{k-1} x^k}{(\alpha)_k k!}$$

$$= \alpha \left[1 + (\rho - 1) \sum_{k=1}^{\infty} \frac{(\rho)_{k-1} x^k}{(\alpha)_k k!}\right]$$

$$= \alpha \left[1 + \sum_{k=1}^{\infty} \frac{(\rho-1)_k x^k}{(\alpha)_k k!}\right]$$

$$= \alpha \left[\sum_{k=0}^{\infty} \frac{(\rho-1)_k x^k}{(\alpha)_k k!}\right] = \alpha \Phi \left[\begin{matrix}\rho - 1\\ \alpha\end{matrix}; x\right] \tag{50}$$

Then we want the solution for ρ, β to be equal to

$$\Phi \left[\begin{matrix}\rho - 1\\ \alpha\end{matrix}; x\right] e^{-\beta x} = e^{-x} \tag{51}$$

$$\Phi \left[\begin{matrix}\rho - 1\\ \alpha\end{matrix}; x\right] = e^{(\beta-1)x} \tag{52}$$

$$e^{-x} \Phi \left[\begin{matrix}1\\ \alpha + 1\end{matrix}; x\right] = e^{-\beta x} \Phi \left[\begin{matrix}\rho\\ \alpha + 1\end{matrix}; x\right] \tag{53}$$

$$\Phi \left[\begin{matrix}\rho - 1\\ \alpha\end{matrix}; x\right] \Phi \left[\begin{matrix}1\\ \alpha + 1\end{matrix}; x\right] = \Phi \left[\begin{matrix}\rho\\ \alpha + 1\end{matrix}; x\right] \tag{54}$$

Since (54) must be true for all values of x then let us take the first two terms approximation as follows:

$$\left(1 + \frac{\rho-1}{\alpha} x\right)\left(1 + \frac{1}{\alpha+1} x\right) \equiv 1 + \frac{\rho}{\alpha+1} x \tag{55}$$

Equating only the first order terms we have

$$\frac{\rho-1}{\alpha} + \frac{1}{\alpha+1} \equiv \frac{\rho}{\alpha+1} \Leftrightarrow (\rho - 1)(\alpha + 1) + \alpha \equiv \alpha\rho \tag{56}$$

Which is equivalent with

$$\rho\alpha - \alpha + \rho - 1 + \alpha \equiv \alpha\rho \Leftrightarrow \rho - 1 \equiv 0 \tag{57}$$

The solution then for the parameter ρ is

$$\rho \equiv 1 \tag{58}$$

If we substitute (58) into (51) we also obtain

$$\beta \equiv 1 \tag{59}$$

Equating only the second order terms we have

$$\left(1 + \frac{\rho-1}{\alpha} x + \frac{\rho(\rho-1)}{\alpha(\alpha+1)} \frac{x^2}{2}\right)\left(1 + \frac{x}{\alpha+1} + \frac{2}{(\alpha+1)(\alpha+2)} \frac{x^2}{2}\right) \equiv$$

$$1 + \frac{\rho}{\alpha+1} x + \frac{\rho(\rho+1)}{(\alpha+1)(\alpha+2)} \frac{x^2}{2} \tag{60}$$

$$\frac{\rho^2+\rho-2}{\alpha} + \frac{2}{(\alpha+2)} \equiv \frac{\rho(\rho+1)}{(\alpha+2)} \tag{61}$$

$$\rho^2\alpha + \rho\alpha - 2\alpha + 2(\rho^2 + \rho - 2) + 2\alpha \equiv \rho^2\alpha + \rho\alpha \tag{62}$$

$$\rho^2 + \rho - 2 \equiv 0 \Rightarrow \rho = 1 \text{ or } \rho = -2 \tag{63}$$

For $\rho = 1$ we obtain $\beta \equiv 1$; hence. we obtain the Kummer's first transformation which is independent of the variable x. The second solution for $\rho = -2$ is not really a valid solution because it will not work for the first order approximation.

We can employ the total mathematic induction to prove that in fact Kummer's first transformation is the only valid solution for every single term.

This Lemma 1 which is in fact Kummer's first identity was a significant contribution because it eliminated the search for any other transformation in the form of the Kummer's first transformation.

Therefore, we cannot do any better than the Kummer's first transformation. What other transformation we can use and how we can compute them will be the subject of a future journal paper.

11 Appendix B: The Problem of the CHF Inverse

In this Appendix from Kummer's first transformation we provide a new expansion of the exponential functions for negative exponents.

Let us rewrite the equation that relates the Kummer's first transformation:

$$\Phi \left[\begin{matrix}\alpha\\ \alpha + 1\end{matrix}; -x\right] = e^{-x} \Phi \left[\begin{matrix}1\\ \alpha + 1\end{matrix}; x\right] = \tag{64}$$

Let us further assume that the inverse transformation of $\Phi[1, \alpha + 1; x]$ and it is called $\Phi^{-1}[1, \alpha + 1; x]$ and it is

$$\Phi \left[\begin{matrix}\alpha\\ \alpha + 1\end{matrix}; -x\right] \Phi^{-1} \left[\begin{matrix}1\\ \alpha + 1\end{matrix}; x\right] = e^{-x} \tag{65}$$

The question is what is $\Phi^{-1}[1, \alpha + 1; x]$ equal to? This problem will be investigated in the future.

Nevertheless, the following is still true so we can write:

$$e^{-x} = \Phi \left[\begin{matrix}\alpha\\ \alpha + 1\end{matrix}; -x\right] / \Phi \left[\begin{matrix}1\\ \alpha + 1\end{matrix}; x\right] \tag{66}$$

So, since, this is true for every α.

We can also rewrite the Kummer's first transformation in a slightly different form:

$$\Phi \left[\begin{matrix}1\\ \alpha + 1\end{matrix}; x\right] = e^{x} \Phi \left[\begin{matrix}\alpha\\ \alpha + 1\end{matrix}; -x\right] \tag{67}$$

In the same manner we can obtain:

$$e^{x} = \Phi\begin{bmatrix} 1 \\ \alpha+1 \end{bmatrix}; x\Big] \Phi^{-1}\begin{bmatrix} \alpha \\ \alpha+1 \end{bmatrix}; -x\Big] \quad (68)$$

The only way to further improve the understand of the computation of the exponential functions and then of the RIGF would be by means of finding the inverse of CHF.

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i Landmark, it means a significantly improved understanding, a turning point, or a quantum leap in the computation of the IGF.

ii In mathematics, statistics, finance, computer science, particularly in machine learning and inverse problems, regularization is a process that changes the result answer to be "simpler"; it is often used to obtain results for ill-posed problems or to prevent overfitting. It is an attempt in this journal paper to make a second order regularization.

iii Advantages, soundness, firmness, robustness due to lack of the singularities that is expected from the limit values.

iv Disadvantages, drawbacks, stumbling blocks, weaknesses, defects, flaws, imperfections, etc. due to the complexity of the computation of wo integrals.

v Twisted or folded, with many hidden details or pitfalls that needs to be taken in to consideration.

vi Equation (19) is perhaps a new integral expression of the RIGF not found in any textbooks or table of integrals, series, and products similar to Gradshteyn, Ryzhik, 2007 [24] 1171 pg.).

vii Equation (23) is preferred as opposed to Batman, ([21], pg. 133 (3) 1953) and (see Gradshteyn, Ryzhik, 2007 [24] pg. 899 ex. 8.351 2.) because it is the regularized version of the IGF; as it is scaled by x^{α}/α.

viii For $\alpha \ll 1$ the region of convergence expands to $0 \le x < 5$ if the power series expansion is utilized.

ix Ditto vi.

x $N = 200,\ 0 \le x < 20$ provide a very good approximation up to sixteen decimal points. Although (24) converges for values of $20 \le x < \infty$ but it's precision will not improve in fact its precession is expected to deteriorate and N is expected to increase, which is the reason why the Nielsen ([18], 1906). Transformation for descending powers is so important because it will require fewer terms for convergence.

xi $M = 20$ and $20 \le x < \infty$ are very important computational details that I could have found in neither Batman, ([21], pg. 133 (3) 1953) nor Nielsen ([18], 1906). For large values of x the computation of the RIGF is expected to be a lot faster than for smaller values of x.

xii Ditto ix.

xiii What is another word for numeric or numerical? It is quantitative, quantifiable, computable, calculable, and or measurable.

xiv There is a false impression that only the research that is funded by the government bureaucrats is worthy of financial support, awards, & recognition. This is not to say that President Ronald Reagan wanted to demonize the role of the government as an enterprise, or resource, or originator of many inventions. But, when the government spending is abused for wasteful spending of the taxpayers trillions of $ dollars, and moreover, when it controls and suppresses innovation and ignores or completely denies funding to innovators and incubators based on political beliefs, then said President Ronald Reagan this type of government is the problem.

xv We can use a different parameter instead of α and then apply Lemma I and we would still obtain the Kummer's first transformation.