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Research Article

A Study of the Computation of the Exponential Function

To my High School math teacher, Gjergji Papanikolla, the best teacher I have ever known.

—Ilir F. Progri

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The computation of the exponential function provides a much-needed solution of the power series expansion of the exponential function for negative exponents. The issue is not that the numerical computation of the exponential function for negative exponents is wrong in various calculators, mathematical programing languages such as MATLAB, etc. The major issue is that there are several analytical computational integral problems in which the power series expansion of the exponential functions for negative exponents is improperly[i], mistakenly, wrongfully employed. A power series expansion of the exponential function is derived for negative exponents. The paper contains a wealth of analytical, graphical, and numerical justification for the newly power series expansion of the exponential function for negative exponents.

Index Terms—Exponential function, power series expansion, positive exponents, negative exponents, analytical derivations, binomial expansion, limit computation, derivative computation, integral computation.

1 Introduction

A study of the computation of the exponential function is the investigation of the power series expansion or computation of the exponential functions for negative exponents. The power series expansion of the exponential function is the centerpiece[ii] of the computation of the incomplete gamma function (IGF) that began in Progri (2016, [1]); and followed by a remarkable number of publications which required the computation of the IGF is some ways Progri (2016, [1]) until Progri (2022, [13]). However, while I was investigating the landmark computation of the generalized Bessel function cumulative distribution functions (cdf) of the first kind (GBFD1K) for integer values of the parameter p in Progri (2022, [13]-[15]) I understood for the first time that there was something not right (wrong, imperfect, erroneous, misleading) that was directly linked to the computation of the IGF based on the closed form expansions of the IGF from the literature [17][31].

It is well known that the computation of the IGF is based in part due to the power series expansion of the exponential function.

First, what are the major issues with the power series expansion of the exponential function?

The current power series expansion of the exponential function, $\exp(x)/\exp(-x)$, works well for positive exponent, values of $\forall x \in \mathbb{R}^{0+}$. However, there are very serious convergence problems for the exponential function $\exp(-x)$ for large values of $\forall x \in \mathbb{R}^{0+}$. In Sect. 2 I argue analytically that the current power series expansion of the exponential function, $\exp(-x)$ for large values of the exponent $\forall x \in \mathbb{R}^{0+}$ is either incorrect, erroneous, misleading, inaccurate, or suffers from serious convergence issues. With the help of the binomial expansion and the converge tests of the binomial expansion we were able to point out that the convergence issues of the power series expansion of the exponential function are due to the magnitude of the variable x.

Second, is it possible to produce a modified closed form expression of the power series expansion of the exponential function, $\exp(-x)$, for large values of the variable $\forall x \in \mathbb{R}^{0+}$? The modified power series expansion of the exponential function for negative exponents provides the much-needed correction of the computation of the exponential function for negative exponents, large values of the variable $\forall x \in \mathbb{R}^{0+}$.

This paper is organized as follows: in Sect. 2 the description of the expansion of the exponential function for negative exponents is discussed. The expansion of the exponential function for negative exponents: part 1 is presented in Sect. 3. The expansion of the exponential function for negative exponents: part 2 is derived in Sect. 4. Section 5 contains numerical, theoretical results; Conclusion is provided in Sect. 6 along with a list of references. In Appendix A we give a brief discussion on the use the absolute convergence tests of (2).

2 Description of the Expansion of the Exponential Function for Negative Exponents

The new integral closed form expression of the RIGF Progri (2022, [14]) points out that the power series expansion of the exponential function is the root-cause [iii] of computational issues associated with numerical evaluation of the RIGF, $\gamma(\alpha, x)$. Therefore, let is examine the exponential function, e^x, and computation of the power series of the exponential function.

For a discussion of the properties of the exponential function the reader may refer to several references, [19], [29], [32].

However, the exponential function, e^x is monotonically increasing with the exponent, x. The power series expansion of the exponential function, e^x, $\forall x \in \mathbb{R}^{0+}$ is given by

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{1}$$

Without too many details, the power series expansion of the exponential function satisfies all the absolute convergence criteria discussed in Gradshteyn, Ryzhik (2007, [29] pgs. 6, 7)) is also monotonically increasing and converges to the exact value of the e^x as the number of terms approaches to infinity when $\forall x \in \mathbb{R}^{0+}$.

So then, where is the problem? The main problem is that (1) is mistakenly used for the series expansion of the e^{-x}, $\forall x \in \mathbb{R}^{+}$. We cannot use (1) to describe the series expansion of the e^{-x}. Why?

Suppose that we want to employ the series expansion of the exponential function, e^{-x}, $\forall x \in \mathbb{R}^{0+}$ as in (1), we have

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$$
$$= 1 + \frac{(-x)}{1!} + \frac{(-x)^2}{2!} + \cdots + \frac{(-x)^k}{k!} + \cdots \tag{2}$$

While the exponential function e^{-x} is monotonically decreasing, on the other hand, the series expansion given by the righthand side of (2) is not. Let us suppose that $x \gg 1$, we know that the exponential function e^{-x} is always positive; i.e., the following holds

$$0 < e^{-x} < 1 \tag{3}$$

Let us take the first two terms of the approximation in (2)

$$1 + \frac{(-x)}{1!} = 1 - x \ll 0 \tag{4}$$

The same argument can be said for (1) when $\forall x \in \mathbb{R}^{-}$.

Let us take the first three terms of the approximation in (2)

$$1 + \frac{(-x)}{1!} + \frac{(-x)^2}{2!} \equiv 1 + \frac{(1-2x+x^2)}{2!}$$
$$\equiv 1 + \frac{(1-x)^2}{2!} > 1 \tag{5}$$

Hence, the series expansion given by (2) is not monotonically decreasing. This is the strongest argument that we cannot employ the power series expansion of (2) for the exponential function for negative exponents.

In fact, as pointed out by Arfken and Weber, 1995, pg. 351, [19], it would be misleading to employ the absolute convergence tests in Gradshteyn, Ryzhik (2007, [29] pgs. 6, 7)

(2) because they will point out to a fact that (1) absolutely converges to the right value of e^x.

Moreover, I have also illustrated graphically as to why e^{-x} is a monotonically decreasing function (as shown in Fig. 1(a) and why (2) fails the absolute convergence test and the monotonically decreasing test (as shown in Fig. 1(b)).

Having identified that the current power series expansion of the exponential function is erroneous for large values of the variable x, in the following section we propose, study, and investigate the power series expansion of the exponential functions for negative exponents.

3 Expansion of the Exponential Function for Negative Exponents: Part 1

In the previous section we identified that the power series of the exponential function for negative exponents needs to be rewritten. In this section we propose, study, and investigate the power series expansion of the exponential functions for negative exponents for small values of the variable x.

The question then is, how do we compute the exponential function, e^{-x}, for $\forall x \in \mathbb{R}^+$? The correct computation of the exponential function is based on the properties of the exponential function [32]

$$e^{-x}e^{x} = e^{-x+x} = e^{0} = 1 \tag{6}$$

Since, $e^x > 0$ we can divide both sides of (6) then we apply the correct power series expansion of the exponential function e^x $\forall x \in \mathbb{R}^+$ as follows

$$e^{-x} = \frac{1}{e^{x}} = \frac{1}{\sum_{k=0}^{\infty}\frac{x^k}{k!}} = \frac{1}{1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^k}{k!}+\cdots} \tag{7}$$

Equation (7) is monotonically decreasing and provides a monotonically decreasing approximation of the exponential function e^{-x}, for $\forall x \in \mathbb{R}^{0+}$.

A short review of the power of the binomials (see Gradshteyn, Ryzhik (2007, 1.11 pg. 25 [29])) reveals the following:

$$(1+x)^q = 1 + \frac{qx}{1!} + \frac{q(q-1)x^2}{2!} + \cdots + \frac{q(q-1)\cdots(q-k+1)x^k}{k!} + \cdots$$ [iv]

$$= \sum_{k=0}^{\infty}\binom{q}{k}x^k \tag{8}$$

1. If q is neither a natural number nor zero, the series (8) converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.
2. For $x = 1$ the series (8) converges for $q > -1$ and diverges for $q \le -1$.
3. For $x = 1$ the series (8) converges absolutely for $q > 0$.
4. For $x = -1$, (8) converges absolutely for $q > 0$ and diverges for $q < 0$.
5. If $q = n$ is a natural number, the series (8) is reduced to a finite sum

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots + \frac{n(n-1)\cdots(n-n+1)x^n}{n!} + 0\cdots$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^k \tag{9}$$

Next, let us write the definition of the exponential function e^x as follows:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{n^k}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!} \tag{10}$$

Let us examine the absolute converge tests of (10). From (d'Alembert) convergence test we obtain:

$$\lim_{k\to\infty}\left|\frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}}\right| = x\lim_{k\to\infty}\frac{1}{k+1} = q \to 0 \tag{11}$$

The exponential function series expansion (10) converges absolutely, if $x < \infty$.

Next, let us write the definition of the exponential function e^{-x} as follows:

$$\lim_{n\to\infty}\left(1+\frac{-x}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^k}{n^k}$$

$$= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots + \frac{(-x)^n}{n!} + \cdots$$

$$= \sum_{k=0}^{\infty}\frac{(-x)^k}{k!} \tag{12}$$

The series expansion of the e^{-x} (12) converges to the exact value of the e^{-x} only if $0 \le x < 1$. If suffers major computational inaccuracies and diverges if $x \ge 1$[v].

Another analog definition of the exponential function e^{-x} is as follows:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-n} = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{-n}{k}\frac{x^k}{n^k}$$

$$= \lim_{n\to\infty}\left[\begin{array}{l} 1 + \frac{-n\frac{x}{n}}{1!} + \frac{-n(-n-1)\frac{x^2}{n^2}}{2!} + \cdots \\ + \frac{-n(-n-1)\cdots(-n-k+1)\frac{x^k}{n^k}}{k!}\cdots \end{array}\right]$$

$$= 1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{(-1)^k x^k}{k!}\cdots$$

$$= \sum_{k=0}^{\infty}\frac{(-x)^k}{k!} \tag{13}$$

Since, $n \to \infty$ then based on the convergence properties of the power of the binomials the series expansion (13) converges only if and only if $0 \le x < 1$.

This concludes the discussion on the power series expansion of the exponential function for negative exponents. Next, we discuss the modification of the power series expansion of the exponential functions for large values of the variable x.

4 Expansion of the Exponential Function for Negative Exponents: Part 2

While the power series expansion of the exponential functions for negative exponents (12) and (13) was designed for small values of the variable, x; it suffers from significant computational errors for large values of the variable x.

The question then is, how do we compute the exponential function, e^{-x}, for $\forall x \gg 1 \in \mathbb{R}^+$?

Suppose that $\forall x \gg 1 \in \mathbb{R}^+$ we can rewrite (8) as follows:

$$\frac{\left(1+\frac{1}{x}\right)^q}{x^{-q}} = \frac{1+\frac{qx^{-1}}{1!}+\frac{q(q-1)x^{-2}}{2!}+\cdots+\frac{q(q-1)\cdots(q-k+1)x^{-k}}{k!}+\cdots}{x^{-q}}$$

$$= x^q \sum_{k=0}^{\infty} \binom{q}{k} x^{-k} \quad (14)$$

The series (14) absolutely converges for $\forall x \gg 1 \in \mathbb{R}^+$.

Unfortunately, we will not be able to employ this technique for the computation of the e^{-x}, for $\forall x \gg 1 \in \mathbb{R}^+$ because

$$e^{-x} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-n} = \lim_{n\to\infty} x^{-n}\left(\frac{1}{x}+\frac{1}{n}\right)^{-n} = 0 \times \infty \quad (15)$$

In this case we must employ a different technique. The correct computation of the exponential function for large values of the variable, x, is based on the properties of the exponential function, e^{-x}, [32]. Let us suppose that $x_{\max} = q$ is the largest value of the variable, x, that is defined as

$$q = x_{\max} = \max(x) \quad (16)$$

I.e., the variable $0 \le x \le q$. In this case we can write the computation of the exponential function, e^{-x}, [32] shown in (7) as follows:

$$e^{-x} = \left(e^{-\frac{x}{q}}\right)^q$$

$$= \left[\sum_{m=0}^{\infty} \frac{\left(-\frac{x}{q}\right)^m}{m!}\right]^q$$

$$\cong \left[\sum_{m=0}^{\infty} \frac{\left(-\frac{x}{q}\right)^m}{m!}\right]^q$$

$$= \left[1 - \frac{\frac{x}{q}}{1!} + \frac{\frac{x^2}{q^2}}{2!} - \frac{\frac{x^3}{q^3}}{3!} + \cdots + \frac{(-1)^k \frac{x^k}{q^k}}{k!}\right]^q \quad (17)$$

Let us define

$$z = -\frac{\frac{x}{q}}{1!} + \frac{\frac{x^2}{q^2}}{2!} - \frac{\frac{x^3}{q^3}}{3!} + \cdots + \frac{(-1)^k \frac{x^k}{q^k}}{k!} \quad (18)$$

Hence, (15) can be written as

$$e^{-x} = (1+z)^q$$

$$= 1 + \frac{qz}{1!} + \frac{q(q-1)z^2}{2!} + \cdots + \frac{q(q-1)\cdots(q-k+1)z^k}{k!} + \cdots \quad (19)$$

The computation of the exponential function for negative exponents can be summarized as follows:

$$e^{-x} = \begin{cases} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} & 0 \le x < 1 \\ \left[\sum_{k=0}^{\infty} \frac{\left(-\frac{x}{q}\right)^k}{k!}\right]^q & 1 \le x \le q < \infty \end{cases} \quad (20)$$

Now that we established a way to compute the exponential functions for negative exponents, let us examine another way for computing the exponential function for negative exponents.

Let us write the variable x as follows:

$$x = (\lfloor x \rfloor \equiv n) + \Delta x \quad (21)$$

where $\lfloor x \rfloor \equiv n = \{1,2,\cdots\}$ is the integer portion of the number, and $0 \le \Delta x < 1$ is the fractional part of the variable x. Substituting (21) into the expansion of the exponential function e^{-x}, for $\forall x \gg 1 \in \mathbb{R}^+$ we obtain:

$$e^{-x} = e^{-n-\Delta x} = e^{-n} e^{-\Delta x} \quad (22)$$

Since, the computation of the integer part is done in a way that we discussed earlier as

$$e^{-n} = (e^{-1})^n = (1/e)^n \quad (23)$$

However, for the computation of the fractional part we can use the series expansion because $0 \le \Delta x < 1$.

$$e^{-\Delta x} = \sum_{k=0}^{\infty} \frac{(-\Delta x)^k}{k!} \quad (24)$$

Therefore, the computation of the exponential function for negative exponents for large values of the variable x can be written as

$$e^{-x} = (e^{-1})^n \sum_{k=0}^{\infty} \frac{(-\Delta x)^k}{k!} \quad (25)$$

Next, if we take the derivative of the exponential function e^{-x} we obtain:

$$\frac{de^{-x}}{dx} = -e^{-x} \quad (26)$$

We can compute the derivative of the exponential function e^{-x} based on the calculations (20).

Next, if we take the integral of the exponential function e^{-x}

we obtain:

$$\int_0^x e^{-t}dt = -\int_0^x e^{-t}d(-t)$$
$$= -e^{-t}|_0^x = 1 - e^{-x} \tag{27}$$

Similarly, we can compute the integral of the exponential function e^{-x} based on the calculations of (20).

This concludes the discussion on the expansion of the exponential function e^{-x} for large values of the exponent the variable, x.

5 Numerical, Theoretical Results

The numerical theoretical results present the computation of the exponential function e^{-x} via either MATLAB identities or via the computation presented in Sects. 2-4.

The main purpose of this section is to illustrate numerically that what was discussed in Sects. 2-4 is accomplished via MATLAB simulations via the use of the MATLAB built in function (BIF) and identities.

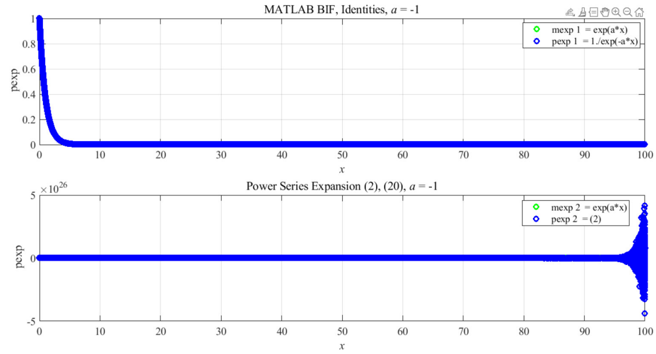
There are four main exponential functions that we are comparing and contrasting one vs. another. The first function is via either MATLAB BIF or identities called mexp. The second function is via the series expansion of the exponential function via (1), (2), (7), and (20) called pexp. The third function is the computation of the derivative of the exponential function via either MATLAB identities or via (26) called dexp. And the fourth and final function is the computation of the integral via either MATLAB identities or via (27) called iexp.

For the computation of the second function, pexp, we illustrate three cases:

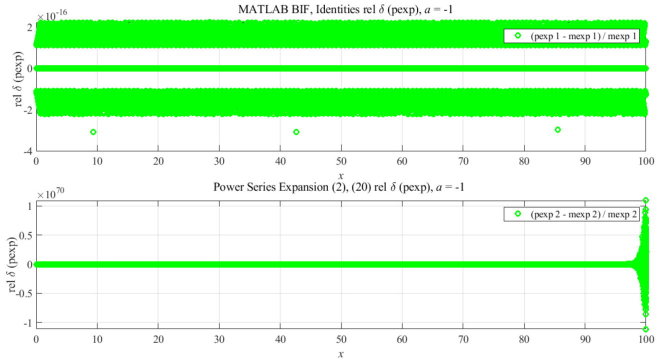
1. the power series expansion without any modifications (or corrections) (1) and (2).
2. the modified power series expansion via (20).
3. when appropriate we take the logarithm to illustrate the computation of the modified power series expansion via (20).

The mexp is considered the truth. This function results either from MATLAB BIF or identities of BIFs. Each function has two outputs: mex and pex. The function mex is computed wither via MATLAB BIF or identities of BIFs. The pex ie computed via identities computed via (1), (2), (7), (20), (26), oe (27).

The main question we would like to answer from this work is as follows:



(a) the computation of the exponential function (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Power Series Expansion.



(b) The relative error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) Power Series Expansion.

FIGURE 1: Numerical calculations of (a) exponential function and (b) relative error (top) MATLAB BIF vs MATLAB identities; (bottom) Power Series Expansion.

How does the computation of the exponential function e^{-x} via pex compares to the ones obtained from MATLAB identities of BIFs mex?

As a performance measure of the comparisons, we define the relative error, δ, via the following:

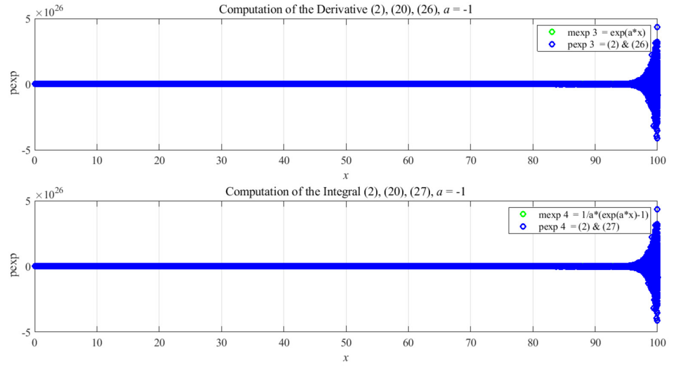
$$\delta = \frac{\text{pex}-\text{mex}}{\text{mex}} \tag{28}$$

The numerical results in the following Subsects. will illustrate the degree of similarity of the pex from mex.

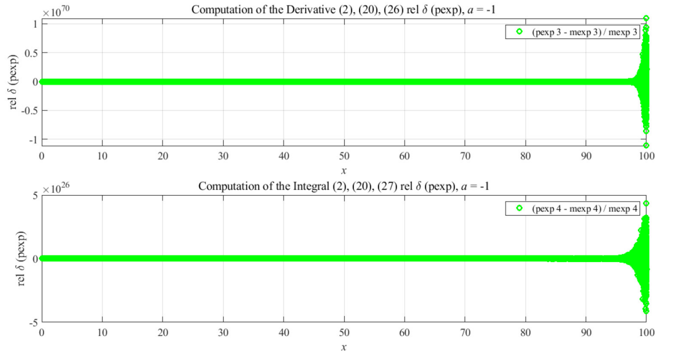
Let us describe the main simulation parameters: the vector of the variable x starts from zero until one hundred with a step size of $dx = 5 \times 10^{-4}$. The total number of points from vector x is equal to 2×10^5. Other simulation parameters are similar to the ones discussed in Progri (2021, [8])-Progri (2022, [13]),

5.1 *Description of the Expansion of the Exponential Function for Negative Exponents*

First, we illustrate why the expansion of the exponential function for negative exponents needs modification.

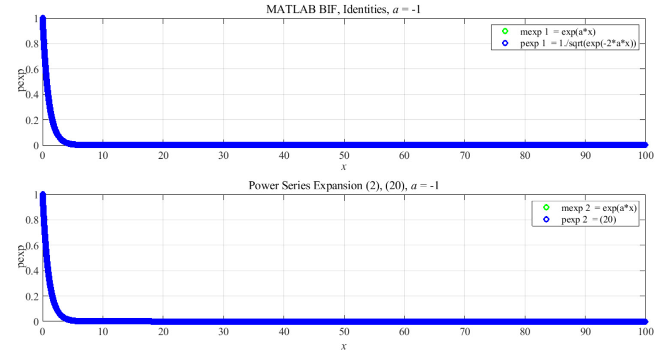


(a) the computation of the exponential function (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Power Series Expansion.

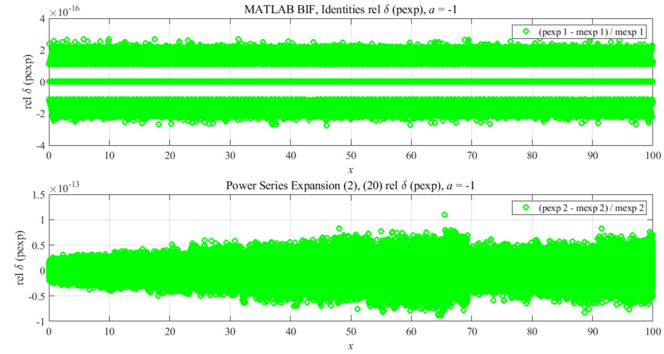


(b) The relative error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) Power Series Expansion.

FIGURE 2: Numerical calculations of (a) exponential function and (b) relative error (top) MATLAB BIF vs MATLAB identities; (bottom) Power Series Expansion.



(a) the computation of the exponential function (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Modified Power Series Expansion.



(b) The relative error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) Modified Power Series Expansion.

FIGURE 3: Numerical calculations of (a) exponential function and (b) relative error (top) MATLAB BIF vs MATLAB identities; (bottom) Modified Power Series Expansion.

Figure 1(a) illustrates the plots of (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Power Series Expansion.

Figure 1(b) depicts the computation of the relative error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) Power Series Expansion.

As indicated from the simulation results of Fig. 1(a) and (b), the power series expansion of the exponential function, e^{-x}, fails; therefore, it needs modification for large values of the exponent or variable x.

Figure 2(a) displays the plots of computation of the derivative and integral of the exponential function (top) derivative via MATLAB identities vs Power Series Expansion; (bottom) integral MATLAB identities vs. Power Series Expansion.

Figure 2(b) illustrates the computation of the relative error; δ, for computing the derivative and integral of the exponential function (top) MATLAB identities of MATLAB BIFs, (bottom) Power Series Expansion.

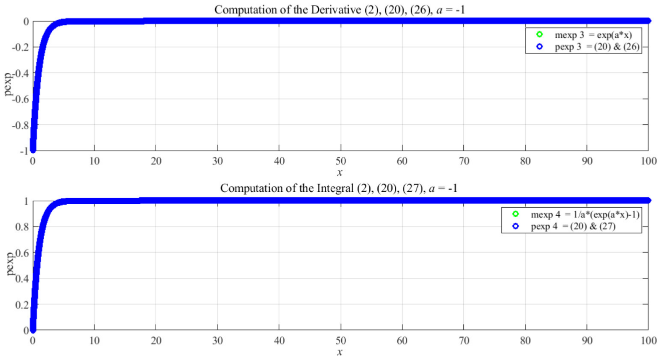
If the computation of the exponential function fails, then the computation of the derivative and the integral of the exponential function will fail also. Hence, the root cause of the failure of the computation of the IGF, and of the GBF1K is as a result of the lack of modification or clarification from the power series expansion of the exponential function, e^{-x}.

5.2 *Modified Expansion of the Exponential Function for Negative Exponents*

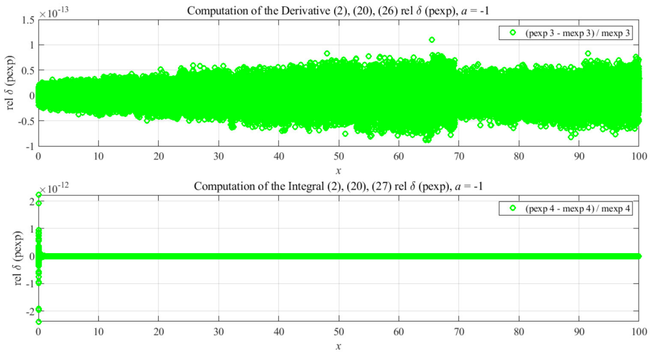
In this Subsect. we repeat exactly the same simulation scenario that we ran in Subsect 5.1 with the only difference we employ the modified power series expansion of the exponential function as given by (20) instead of the standard power series expansion of the exponential function in (2).

Figure 3(a) pictures the plots of (top) MATLAB BIFs, Identities; (bottom) Power Series Expansion (20).

Figure 3(b) depicts the computation of the relative error; δ, (top) MATLAB BIFs, Identities, (bottom) Power Series Expansion.



(a) the computation of the exponential function (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Power Series Expansion.



(b) The relative error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) Power Series Expansion.

FIGURE 4: Numerical calculations of (a) exponential function and (b) relative error (top) MATLAB BIF vs MATLAB identities; (bottom) Power Series Expansion.

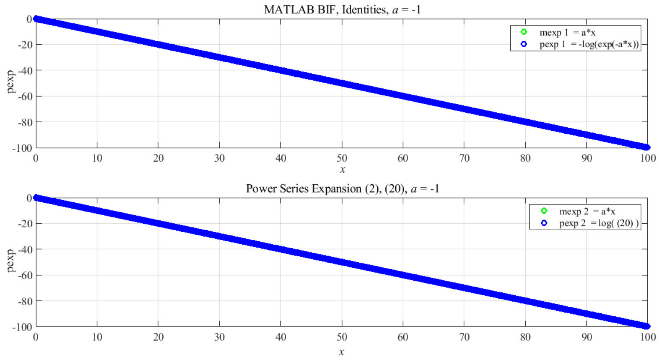
Figure 4(a) displays the plots of computation of the derivative and integral of the exponential function (top) derivative via MATLAB identities vs Modified Power Series Expansion; (bottom) integral MATLAB identities vs. Modified Power Series Expansion.

Figure 4(b) illustrates the computation of the relative error; δ, for computing the derivative and integral of the exponential function (top) MATLAB identities of MATLAB BIFs, (bottom) Modified Power Series Expansion.

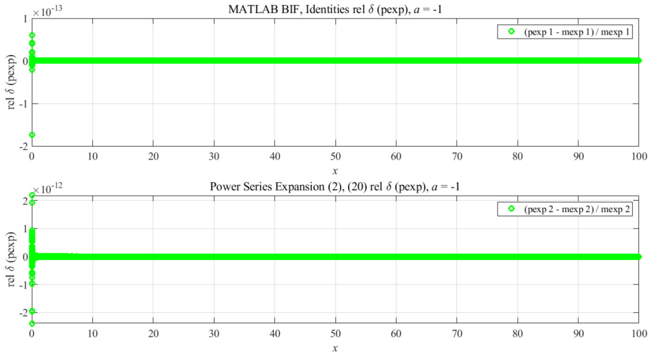
Figure 5(a) displays the plots of (top) MATLAB BIFs, Identities; (bottom) Power Series Expansion in log.

Figure 5(b) depicts the computation of the relative error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) pexp 2 Power Series Expansion in log (2), (20).

As illustrated from Figs. 4 and 5, the modified power series expansion of the exponential functions does not fail, for large values of the variable x.[vi]



(a) the computation of the logarithm of the exponential function (top) MATLAB BIF vs MATLAB identities; (bottom) MATLAB BIF vs. Modified Power Series Expansion.



(b) The relative error; δ, (top) MATLAB identities of MATLAB BIFs, (bottom) modified Power Series Expansion.

FIGURE 5: Numerical calculations of (a) the logarithm of the exponential function and (b) relative error (top) MATLAB BIF vs MATLAB identities; (bottom) Power Series Expansion.

6 Conclusions

We have successfully investigated the power series expansion of the exponential function, e^{-x}, for both small and large values of the variable x.

For large values of the variable x the power series expansion fails, as indicated in Fig.1(a), (b); however, for the modified power series expansion (2) works well as indicated from Figs. 3-5.

Further investigation on a more convenient way for expanding the exponential function for negative exponents for large values of the exponents is required.

7 Acknowledgement

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I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [36] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications*. *Vol. I* (Not yet available in print) [16].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say, I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [34] reminds me of my grandfather who I loved him very much.

Finally, I am also dedicating this journal paper to our most beloved President Ronald Reagan, who reminds me of my grandfather, [35] who spoke from the heart, who spoke the truth, who was a staunch supporter of personal freedom, the greatest leader of the free world, and perhaps the greatest critic of the wasteful government control and spending[vii].

"*Government is not the answer, government is the problem*."—Ronald Reagan

I would like to express my deepest gratitude to my high school math teacher, Gjergji Papanikolla. When I was in high school, Themistokli Germenji, Korcë, Albania, from 1985-1989, I used to visit him at his home once every week and show him my progress in my math exercises. He inspired me unlike any other teacher. As a result of my persistent work under his direction, I won prizes in three consecutive Math National Competitions in Albania from 1986-1989.

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values of the parameter α.

9 Appendix A: Absolute Convergence Tests of (2)

We cannot use the absolute convergence tests of (2) to prove that the power series expansion of (2) converges to the exact value of e^{-x}.

The absolute convergence tests of (2) will show that in fact the power series expansion of (2) e^{-x} absolutely converges to the power series expansion of (1) or e^{x} which is misleading and erroneous.

Let us start with the Cauchy absolute convergence test (see Gradshteyn, Ryzhik (2007, 0.22 pg. 6 [29]))

$$\lim_{k\to\infty} |u_k|^{1/k} = \lim_{k\to\infty} \frac{x^{k^{1/k}}}{k!^{1/k}} = q = x \lim_{k\to\infty} \frac{1}{k!^{1/k}} \to 0 < 1 \qquad (29)$$

The series expansion of e^{x} given by (1) is said to be converging absolutely; however, it would be misleading to use this test to show that (2) converges to e^{-x} because of the discussion given in Sect. 2.

Next, let us consider the d'Almbert absolute convergence test (see Gradshteyn, Ryzhik (2007, 0.222 pg. 6 [29]))

$$\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}} = q = x \lim_{k\to\infty} \frac{1}{k+1} \to 0 < 1 \qquad (30)$$

The series expansion of e^{x} given by (1) is said to be converging absolutely; however, it would be misleading to use this test to show that (2) converges to e^{-x} because of the discussion given in Sect. 2.

The use of the absolute convergence tests of (2) based on (see Gradshteyn, Ryzhik (2007, 0.22 pgs. 6, 7 [29])) would be misleading, erroneous, inaccurate, wrong. Instead, we will use the properties of the monotonically increasing and decreasing properties to qualify or disqualify the power series expansion of the exponential function e^{x}/e^{-x} $\forall x \in \mathbb{R}^{0+}$.

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[i] Amiss, erroneously, faultily, inaccurately, inappropriately, inaptly, incorrectly, mistakenly employed without clear understanding, with tremendous lack of analytical and numerical justification and evaluation.

[ii] The focal point, the best part, the center of attention, the nucleus, the core, the kernel, the climax of the computation of the incomplete gamma function.

[iii] Ditto ii.

[iv] The proof of the binomial expansion is given via mathematical induction, [33].

[v] In reality it diverges for values of $x \geq 15$.

[vi] The only issue is that is not given in any convenient way and it may not always be practical to employ in its current form.

[vii] There is a false impression that only the research that is funded by the government bureaucrats is worthy of financial support, awards, & recognition. This is not to say that President Ronald Reagan wanted to demonize the role of the government as an enterprise, or resource, or originator of many inventions. But, when the government spending is abused for wasteful spending of the taxpayers trillions of $ dollars, and moreover, when it controls and suppresses innovation and ignores or completely denies funding to innovators and incubators based on political beliefs, then said President Ronald Reagan this type of government is the problem.