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*Research Article*

# Special Cases of the Generalized Parabolic Cylinder Function Distribution

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This paper exploits the unique simplification in the computation of two special cases of the generalized parabolic cylinder function (or GPCF) distribution (or GPCFD) probability density function (pdf) and the cumulative distribution function (cdf) that comes from the reduction of the confluent hypergeometric function into an elementary function. These simplifications will significantly reduce the efficient computation of the GPCFD cdf because of the reduction of both the number of Kampé de Fériet functions and of the Srivastava functions (or hypergeometric series of three variables). Four original equations of the error functions that are not published anywhere else are obtained by means of expansion of the Kampé de Fériet functions. The computational complexity and numerical results of the special cases algorithm is compared and contrasted with the efficient algorithm derived earlier clearly demonstrate a *significant improvement in performance* in both speed, memory allocation, and accuracy.
*Index Terms*—Parabolic cylinder function, cumulative distribution function, special cases, exponential function, error function, hypergeometric series, confluent hypergeometric series, Kampé de Fériet function, pdf cdf optimization analysis.

## 1 Introduction

The derivation of the efficient and of the landmark generalized parabolic cylinder function (GPCF) [1] distribution (or GPCFD) probability density function (pdf) and of the cumulative distribution function (cdf) provided two unique computational tools that enabled the numerical computation of the GPCFD cdf in a record time in Progri (2021, [11]) and Progri (2021, [14]).

From 2016 when Dr. Progri began the investigation of the GPCFD cdf, Progri (2021, [11]), until the arrival of the efficient computation of the GPCFD cdf, Progri (2021, [13]), and the pinnacle or landmark computation of the GPCFD cdf, Progri (2021, [14]) a giant leap of accomplishments was achieved by means of producing several masterpieces computational tools or algorithms that complement each other from Progri et al. (2016, [4]) to Progri (2021, [12]).

In 2016, although Dr. Progri recognized that the Kampé de Fériet function might be a useful tool[i], and although he lacked the toolbox, i.e., he was able at the time to indirectly compute these functions in MATLAB for some special cases (see Progri (2016, [7])) based on a hypergeometric function partial derivatives technique developed in the same year (see Progri (2016, [8])).

In 2019 the investigation of the special cases of the generalized Bessel function distribution in (Progri (2019, [10])) produced the first masterpiece that showed the reduction of the Kampé de Fériet function into generalized hypergeometric functions [3]. This was a significant achievement in the development of the special cases because it is the first landmark publication that Dr. Progri exhausted every single avenue in the computation of the generalized Bessel function distribution pdf and cdf for the integer values of the parameter.

Since the reduction of the Kampé de Fériet function into either generalized hypergeometric functions [3] or other elementary functions is done on a case by case bases *the process of reduction is based on three fundamental principles*. The *first principle* is that you have to compute an integral or a function by means of the Kampé de Fériet function for a set of parameters. The *second principle* is that you have to differently compute the same function or integral for a particular subset of the set of the parameters that were used in the *first principle*. The *third principle* is that you have to equate the formula that was produced by means of the Kampé de Fériet function with the other formula that was produced differently for the *subset* of parameters, *only*.

This journal paper provides a unique and original derivation of the illustration of the process of reduction of the Kampé de Fériet function into an expansion of error functions by means of exactly the same three principles.

The *first* principle of reduction was accomplished in 2021 when Dr. Progri produced the first successful, original efficient computation of the GPCFD pdf and cdf (see Progri 2021, [13]) by means of the computation of eight Kampé de Fériet functions (or double hypergeometric series) [6]-[10] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [10]); i.e., it is computationally the most laborious function that Dr. Progri has ever come across with.

In the same publication in Appendix C (see Progri 2021, [13]) Dr. Progri showed that for a special case when $N = 1$ significant reduction of the computational complexity and computational time is achieved because four Kampé de Fériet functions (or double hypergeometric series) [6]-[10] are reduced into two error functions and four Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [10]) are reduced into four Kampé de Fériet functions (or double hypergeometric series) [6]-[10].

In Appendix B of Progri (2021, [14]) Dr. Progri recognized the same significant reductions that were discussed in Appendix C (see Progri 2021, [13]) of the efficient computational algorithm can also be obtained for $N = 2$.

In this paper Dr. Progri expands the usefulness of the special cases when $N > 2$. Up to this point the reader should have an understanding that the process of the *first principle of reduction* alone can be enormous; i.e., extremely laborious in nature, as illustrated by the efficient computation of the GPCFD cdf, as it required the derivations of three masterpiece journal papers.

However, the completion of the *first principle of reduction* alone is not sufficient to produce the reduction formulae. Hence, *the second and third principles of reduction* are produced in this journal paper.

Dr. Progri recognized that for a particular subset of parameters, in this case when $p = \{-1, -2\}$ and for any integer values of the parameter $N$ for a subset of the GPCFD cdf; i.e., two closed form expressions (cfe) are possible by means of the elementary and error function(s) and two other ones are possible by means of elementary, error function(s), and exponential functions. To derive these special cases' cfes Dr. Progri utilized material from three different unique references [15]-[17]. As expected, the results that we obtained from the three different references produced identical formulae. The completion of this process underlines the completion of *the second principle*. The completion of *the third principle* of the reduction is simply the equality that should exist between the two formulae one that was produced from the first principle and the other from the second principle.

It is exactly for these reasons presented in the paper that Dr. Progri has produced another masterpiece in the masterpiece marvels in analytical derivations and numerical computational series; namely the *illustration* of the three principles of reduction of for computing three original cfe that shows the relation of the error functions with Kampé de Fériet functions for particular values of the parameters.

The value of this process is illustrated in the numerical, theoretical results Sect. 4 where is it shown that significant reduction in computation time is achieved by means of the implementation of the special cases integration cfe that are produced in Sect. 3 of this superb journal paper.

This paper is organized as follows: in Sect. 2 the special cases of the partial GPCFD PDF are presented namely for $p = \{-1, -2\}$. The simplified computation of partial GPCFD CDF is discussed in Sect. 3. Section IV contains numerical results; Conclusion is provided in Section V along with a list of references.

## 2 Special Cases of the Partial GPCFD PDF

The beginning of the implementation of the *second principle* starts with the understanding of the special cases of the partial GPCFD pdf.

In Appendix C of Progri (2021, [13]) this special case was initially investigated for the first time for $N = 1$ when in fact the special case was aimed for the value of $p = -1$. For the second principle to be useful, this special case must be generalized to produce the same value of $p = -1$ for a broad range of positive integer values of the parameter $N$. Hence, the way to generalize this special case for all integer values of the parameter $N$ is by considering the following special case when:

$$p = -1 = k - N \Rightarrow k = N - 1; \begin{matrix} n = N - k - 1 = 0 \\ N = 1,2,3,\cdots \end{matrix} \tag{1}$$

Equation (1) generalizes all three indexes $p = -1$, $k$, and $n$ as a function of all possible, positive integer values of the of the input parameter $N$.

For this special; case from Progri (2021, [14]) (1) we have

$$f_{-1}(x) = \frac{\frac{H_{N-1}(N)}{a^{N-1} 2^{-\frac{N-1}{2}}}\left[e^{-\frac{1}{2}g_0^2(x)} V(-1,v_0) + e^{-\frac{1}{2}g_1^2(x)} V(-1,v_1)\right]}{2^{\frac{3(N+1)}{2}} 3^{-\frac{N}{2}} a e^{\frac{-a^2}{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N) n!}{a^k} \left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)}$$

$$= \frac{e^{\frac{a^2}{3c^4} - \frac{1}{2}g_0^2(x)}[1 - \Phi(v_0)] + e^{\frac{a^2}{3c^4} - \frac{1}{2}g_1^2(x)}[1 - \Phi(v_1)]}{2^{N+2} 3^{-\frac{N}{2}} a^N H_{N-1}^{-1}(N) \sum_{k=0}^{N-1} \frac{H_k(N) n!}{a^k} \left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)} \tag{2}$$

If we examine (2) we can clearly see that there are two components of the partial GPCFD pdf: one that is just a sum of the exponential functions and the other that is a weighted sum of a product of exponential functions with error functions. Therefore, if we were to integrate (2) one portion of the (2) will simply result is elementary functions the other portion has already been computed before in Progri (2021, [13]). The details of the integration of the portion of (2) are provided in Sect. 3.

Equation (2) simplifies even more when $a = 1$. From Progri (2021, [13]), we already know that the solution for this special case is

$$f_{-1}(x) = \frac{e^{\frac{1}{3c^4} - \frac{1}{2}g_0^2(x)}[1 - \Phi(v_0)] + e^{\frac{1}{3c^4} - \frac{1}{2}g_1^2(x)}[1 - \Phi(v_1)]}{2^{N+2} 3^{-\frac{N}{2}} H_{N-1}^{-1}(N) \sum_{k=0}^{N-1} \frac{H_k(N) n!}{a^k} \left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2}{\sqrt{3}c^2}\right)} \tag{3}$$

In Appendix B of Progri (2021, [14]) the second special case was developed when $N = 2$ when in fact the special case was aimed for the value of $p = -2$. For exactly the same reasons that were stated earlier let us consider a broader special case when

$$p = -2 = k - N \Rightarrow k = N - 2; \begin{matrix} n = N - k - 1 = 1 \\ N = 2,3,4,\cdots \end{matrix} \tag{4}$$

Equation (4) generalizes all three indexes $p = -2$, $k$, and $n$ as a function of all possible, positive integer values of the of the input parameter $N$. For this special; case from Progri (2021, [14]) (1) we have

$$f_{-2}(x) = \frac{\frac{H_{N-2}(N) 1!}{a^{N-2} 2^{-\frac{N-2}{2}}}\left[e^{-\frac{1}{2}g_0^2(x)} V(-2,v_0) + e^{-\frac{1}{2}g_1^2(x)} V(-2,v_1)\right]}{2^{\frac{3(N+1)}{2}} 3^{-\frac{N}{2}} a e^{\frac{-a^2}{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N) n!}{a^k} \left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)}$$

$$= \frac{e^{\frac{a^2}{3c^4} - \frac{1}{2}g_0^2(x)}\left[\frac{{}_1F_1\left[\begin{matrix} -\frac{1}{2} \\ \frac{1}{2} \end{matrix}\middle| -v_0^2\right]}{\sqrt{\pi}} - v_0\right] + e^{\frac{a^2}{3c^4} - \frac{1}{2}g_1^2(x)}\left[\frac{{}_1F_1\left[\begin{matrix} -\frac{1}{2} \\ \frac{1}{2} \end{matrix}\middle| -v_1^2\right]}{\sqrt{\pi}} - v_1\right]}{2^{N+\frac{3}{2}} 3^{-\frac{N}{2}} a^{N-1} H_{N-2}^{-1}(N) \sum_{k=0}^{N-1} \frac{H_k(N) n!}{a^k} \left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)}$$

$$= \frac{e^{-\frac{1}{2}g_0^2(x)}\left[\frac{e^{-v_0^2}}{\sqrt{\pi}} + v_0[\Phi(v_0) - 1]\right] + e^{-\frac{1}{2}g_1^2(x)}\left[\frac{e^{-v_1^2}}{\sqrt{\pi}} + v_1[\Phi(v_1) - 1]\right]}{e^{-\frac{a^2}{3c^4}} 2^{N+\frac{3}{2}} 3^{-\frac{N}{2}} a^{N-1} H_{N-2}^{-1}(N) \sum_{k=0}^{N-1} \frac{H_k(N) n!}{a^k} \left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)} \tag{5}$$

Similarly, when we examine (5) we see that is made of two sums: one that contains only exponential functions and the other a weighted sum of exponential functions with confluent hypergeometric functions (see Appendix A of Progri (2021, [14]) for the expansion of a particular confluent hypergeometric function).

As a special case when $a = 1$, then (5) simplifies even further as

$$f_{-2}(x)=\frac{\frac{e^{\frac{c^{-4}}{3}-\frac{g_0^2(x)}{2}}\mathrm{M}\left[\begin{matrix}\frac{-1}{2}\\ \frac{1}{2}\end{matrix}\middle|-v_0^2\right]+e^{\frac{c^{-4}}{3}-\frac{g_1^2(x)}{2}}\mathrm{M}\left[\begin{matrix}\frac{-1}{2}\\ \frac{1}{2}\end{matrix}\middle|-v_1^2\right]}{\sqrt{\pi}}-e^{\frac{c^{-4}}{3}}\left[e^{-\frac{g_0^2(x)}{2}}v_0+e^{-\frac{g_1^2(x)}{2}}v_1\right]}{2^{N+\frac{3}{2}}3^{-\frac{N}{2}}H_{N-2}^{-1}(N)\sum_{k=0}^{N-1}\frac{H_k(N)n!}{a^k}\left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2}{\sqrt{3}c^2}\right)}$$

$$=\frac{\frac{e^{-\frac{g_0^2(x)}{2}}\left[e^{-v_0^2}+\sqrt{\pi}v_0\Phi(v_0)\right]+e^{-\frac{g_1^2(x)}{2}}\left[e^{-v_0^2}+\sqrt{\pi}v_0\Phi(v_0)\right]}{\sqrt{\pi}}-e^{-\frac{g_0^2(x)}{2}}v_0-e^{-\frac{g_1^2(x)}{2}}v_1}{2^{N+\frac{3}{2}}3^{-\frac{N}{2}}H_{N-2}^{-1}(N)e^{-\frac{c^{-4}}{3}}\sum_{k=0}^{N-1}\frac{H_k(N)n!}{a^k}\left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2}{\sqrt{3}c^2}\right)} \quad (6)$$

The first task in the process of *the second principle* is accomplished; namely, we have expanded both special cases to include for values of index $p=\{-1,-2\}$ we generalized all other indexes $k$ and $n$ as a function of all possible, positive integer values of the input parameter $N$. The second task of the process is also accomplished as we have produced the CFE of the partial GPCFD pdf and found out that it contains two main components: one component whose integration will result in an elementary function or a confluent hypergeometric function at worst and the other whose integration will be a much more complicated function whose solution is provided in Progri (2021, [13], [14]).

This concludes the discussion special cases of the partial GPCFD PDF. Next, we perform the integration of these formulae and produce the simplified computation of the partial GPCFD cdf.

## 3 Simplified Computation of Partial GPCFD CDF

In this section the simplified efficient computation of the landmark GPCFD cdf is performed. Even though in 2021 I was successful to develop for the first time the efficient computational algorithm of the GPCFD cdf the numerical results indicated that this algorithm was far from being optimized. This section contains two subsections: first special case of the partial GPCFD cdf and the second special case of the partial GPCFD cdf.

### 3.1 *First Special Case of the Partial GPCFD CDF*

The derivation of the simplified, partial GPCFD cdf is performed by taking advantage of the expansion of the series of Hermite polynomials and then of their recursive implementation [2].

For the first special case we have already found that its partial pdf is given by

$$f_{-1}(x)=\frac{\frac{H_{N-1}(N)}{a^{N-1}2^{-\frac{N-1}{2}}}\left[e^{-\frac{g_0^2(x)}{2}}[1-\Phi(v_0)]+e^{-\frac{g_1^2(x)}{2}}[1-\Phi(v_1)]\right]}{2^{\frac{3(N+1)}{2}}3^{-\frac{N}{2}}ae^{\frac{-a^2}{3c^4}}\sum_{k=0}^{N-1}\frac{H_k(N)n!}{a^k}\left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)} \quad (7)$$

And if assume all possible positive integer values of the input parameter $N=1,2,\cdots$ we have

$$F'(x)=\frac{\frac{H_{N-1}(N)}{a^{N-1}2^{-\frac{N-1}{2}}}\int_x^\infty\left[e^{-\frac{g_0^2(t)}{2}}[1-\Phi(v_0)]+e^{-\frac{g_1^2(t)}{2}}[1-\Phi(v_1)]\right]dt}{2^{\frac{3(N+1)}{2}}3^{-\frac{N}{2}}ae^{\frac{-a^2}{3c^4}}\sum_{k=0}^{N-1}\frac{H_k(N)n!}{a^k}\left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)} \quad (8)$$

It remains to solve two integrals, (Gradshteyn, Ryzhik 2007, [15] pg. 1027 9.236 1.)

$$F_{-1i}{}'=\int_x^\infty e^{-\frac{g_i^2(t)}{2}}\left[1-\Phi\left(\frac{c_i(t)}{\sqrt{2}}\right)\right]dt;\ i=\{0,1\}$$

$$=\int_x^\infty e^{-\frac{g_i^2(t)}{2}}dt-\int_x^\infty e^{-\frac{g_i^2(t)}{2}}\Phi\left(\frac{c_i(t)}{\sqrt{2}}\right)dt \text{ ii} \quad (9)$$

which then produces four integrals,

$$F_{-10i}{}'(x)=\int_x^\infty e^{-\frac{g_i^2(t)}{2}}dt;\ i=\{0,1\} \quad (10)$$

$$F_{-11i}{}'(x)=-\int_x^\infty e^{-\frac{g_i^2(t)}{2}}\Phi\left(\frac{c_i(t)}{\sqrt{2}}\right)dt;\ i=\{0,1\} \quad (11)$$

where

$$c_i(t)=y=\frac{Nb}{a}-(-1)^i\frac{t-b}{2a};\ i=\{0,1\} \quad (12)$$

$$g_i(t)=\sqrt{3}\left[\frac{(t-b)}{2a}+\frac{(-1)^i a}{3c^2}\right]=\sqrt{3}(-1)^i\left[\frac{4Nb}{3a}-c_i(t)\right] \quad (13)$$

Substituting (12) into (10) we have

$$F_{-10i}{}'(x)=\int_x^\infty e^{-\frac{3\left[\frac{4Nb}{3a}-c_i(t)\right]^2}{2}}dt \quad (14)$$

Integral (14) can be solved at least three different ways using the three different tables of integrals [15], [15], and [17].

First, (14) can be written as

$$F_{-10i}{}'(x)=\int_x^\infty e^{-\frac{8N^2b^2}{3a^2}+4\frac{bN}{a}c_i(t)-\frac{3}{2}c_i^2(t)}dt$$

$$=e^{-\frac{8N^2b^2}{3a^2}}\int_x^\infty e^{-\frac{3}{2}c_i^2(t)+4\frac{bN}{a}c_i(t)}dt \quad (15)$$

Next, we make the substitution (12) into (15) produces

$$dy = (-1)^{i+1}\frac{dt}{2a};\ t \to |_x^{\infty} \Rightarrow\ y \to |_{y_i}^{(-1)^{i+1}\infty} \quad (16)$$

$$F_{-10i}{}'(x) = \frac{2ae^{-\frac{8N^2b^2}{3a^2}}}{(-1)^{i+1}}\int_{y_i}^{(-1)^{i+1}\infty} e^{-\frac{3}{2}y^2+4\frac{bN}{a}y}dy \quad (17)$$

Employing the integral from Ng, Geller (1969, [15] pg. 18 5.(A2)) we have

$$F_{-10i}{}'(x) = \frac{a}{(-1)^{i+1}}\frac{\sqrt{2\pi}}{\sqrt{3}}\Phi\left(\frac{\sqrt{3}}{\sqrt{2}}y - \frac{4Nb}{\sqrt{6}a}\right)|_{y_i}^{(-1)^{i+1}\infty} \quad (18)$$

$$F_{-100}{}'(x) = -\frac{\sqrt{2\pi}a}{\sqrt{3}}\Phi\left(\frac{\sqrt{3}}{\sqrt{2}}y - \frac{4Nb}{\sqrt{6}a}\right)|_{y_0}^{-\infty}$$

$$= 2a\sqrt{\frac{\pi}{6}}\left\{1 + \Phi\left[\frac{(3-2N)}{2\sqrt{6}}\frac{b}{a} - \frac{\sqrt{3}}{\sqrt{2}}\frac{x}{2a}\right]\right\}$$

$$= 2a\sqrt{\frac{\pi}{6}}\left\{1 - \Phi\left[\frac{3x}{2\sqrt{6}a} - \frac{(3-2N)}{2\sqrt{6}}\frac{b}{a}\right]\right\} \quad (19)$$

$$F_{-101}{}'(x) = 2a\sqrt{\frac{\pi}{6}}\left\{1 - \Phi\left[\frac{3x}{2a\sqrt{6}} - \frac{(3+2N)b}{2\sqrt{6}a}\right]\right\} \quad (20)$$

Second, if we make the substitution

$$z = \frac{4Nb}{3a} - c_i(t) = \frac{Nb}{3a} - (-1)^i\frac{b}{2a} + (-1)^i\frac{t}{2a} \quad (21)$$

Then we have

$$dz = (-1)^i\frac{dt}{2a};\ t \to |_x^{\infty} \Rightarrow\ z \to |_{z_i}^{(-1)^i\infty} \quad (22)$$

We substitute (21) and (22) into (14) we have

$$F_{-10i}{}'(x) = \frac{2a}{(-1)^i}\int_{z_i}^{(-1)^i\infty} e^{-\frac{3z^2}{2}}dz \quad (23)$$

Next, from (Gradshteyn, Ryzhik 2007, [15] pg. 336 3.322 1.)

$$F_{-100}{}'(x) = 2a\left[\int_{z_0}^{\infty} e^{-\frac{3z^2}{2}}dz = \int_0^{\infty} e^{-\frac{3z^2}{2}}dz - \int_0^{z_0} e^{-\frac{3z^2}{2}}dz\right]$$

$$= 2a\sqrt{\frac{\pi}{6}}\left[1 - \Phi\left(\frac{\sqrt{6}z_0}{2}\right)\right]$$

$$= 2a\sqrt{\frac{\pi}{6}}\left\{1 - \Phi\left[\frac{3x-(3-2N)b}{2\sqrt{6}a}\right]\right\} \quad (24)$$

Next, from (Gradshteyn, Ryzhik 2007, [15] pg. 336 3.321 1.)

$$F_{-101}{}'(x) = -2a\int_{z_1}^{-\infty} e^{-\frac{3z^2}{2}}dz = 2a\int_{-z_1}^{\infty} e^{-\frac{3z^2}{2}}dz$$

$$= 2a\sqrt{\frac{\pi}{6}}\left[1 - \Phi\left(\frac{-\sqrt{6}z_1}{2}\right)\right]$$

$$= 2a\sqrt{\frac{\pi}{6}}\left\{1 - \Phi\left[\frac{3x-(3+2N)b}{2a\sqrt{6}}\right]\right\} \quad (25)$$

Third, if we make the substitution (16) into (14) we have

$$F_{-10i}{}'(x) = \frac{2a}{(-1)^{i+1}}\int_{y_i}^{(-1)^{i+1}\infty} e^{-\left(\frac{\sqrt{3}}{\sqrt{2}}y - \frac{4N\sqrt{3}b}{3\sqrt{2}a}\right)^2}dy \quad (26)$$

Utilizing the (2019, [17] pg. 5 1.1 1.1.1 1.) the integral (26) is already solved

$$F_{-10i}{}'(x) = \frac{1}{(-1)^{i+1}}\frac{\sqrt{2}\sqrt{\pi}a}{\sqrt{3}}\Phi\left(\frac{\sqrt{3}}{\sqrt{2}}y - \frac{4Nb}{\sqrt{6}a}\right)|_{y_i}^{(-1)^{i+1}\infty} \quad (27)$$

$$F_{-100}{}'(x) = 2a\sqrt{\frac{\pi}{6}}\left\{1 - \Phi\left[\frac{3x-(3-2N)b}{2\sqrt{6}a}\right]\right\} \quad (28)$$

$$F_{-101}{}'(x) = 2a\sqrt{\frac{\pi}{6}}\left\{1 - \Phi\left[\frac{3x-(3+2N)b}{2a\sqrt{6}}\right]\right\} \quad (29)$$

Equations (19), (24), and (28) are identical as expected. Similarly, (20), (25), and (29) are also identical.

The question then is as follows: Are we able to produce exactly the same result from (149) in Progri (2021, [13])?

$$F_{-10i}{}'' = \frac{2a\sqrt{\pi}}{\sqrt{6}} - (-1)^i\frac{2a\sqrt{\pi}}{\sqrt{6}}\Phi\left(\frac{4Nb}{\sqrt{6}a}\right) \quad (30)$$

Hence, we have,

$$F_{-100}{}'' = \frac{2a\sqrt{\pi}}{\sqrt{6}} - \frac{2a\sqrt{\pi}}{\sqrt{6}}\Phi\left(\frac{4Nb}{\sqrt{6}a}\right) \quad (31)$$

$$F_{-101}{}'' = \frac{2a\sqrt{\pi}}{\sqrt{6}} + \frac{2a\sqrt{\pi}}{\sqrt{6}}\Phi\left(\frac{4Nb}{\sqrt{6}a}\right) \quad (32)$$

From, Progri (2021, [14]) (102) and (103) we have,

$$S_{-1001}(x) = \sum_{j,n=0}^{\infty}\frac{(1)_n}{\left(\frac{3}{2}\right)_{j+n}}\frac{x_2^j}{j!}\frac{x_3^n}{n!} = F_{1:0;0}^{0:0;1}\left[\begin{matrix}-:-;1; \\ \frac{3}{2}:-;-;\end{matrix}x_2, x_3\right] \quad (33)$$

$$S_{-1002}(x) = \sum_{j,n=0}^{\infty}\frac{(1)_j(1)_n\frac{x_2^j x_3^n}{j!\ n!}}{(2)_{j+n}\left(\frac{3}{2}\right)_j} = F_{1:1;0}^{0:1;1}\left[\begin{matrix}-:1;1; \\ 2:\frac{3}{2};-;\end{matrix}x_2, x_3\right] \quad (34)$$

From, Progri (2021, [14]) (102) and (103) we have,

$$S_{-1011}(x) = F_{1:0;0}^{0:0;1}\left[\begin{matrix}-:-;1; \\ \frac{3}{2}:-;-;\end{matrix}x_5, x_6\right] \quad (35)$$

$$S_{-1012}(x) = F_{1:1;0}^{0:1;1}\left[\begin{matrix}-:-;1;1; \\ 2:\frac{3}{2};-;\end{matrix}x_5, x_6\right] \quad (36)$$

Substituting, (31)/(32), (33)/(35), and (34)/(36) into (78) of Progri (2021, [14]) we have,

$$F_{-100}{}'(x) = F_{-100}{}'' + \mathcal{F}_{-100}(x)$$

$$= \frac{2a\sqrt{\pi}\left[1-\Phi\left(\frac{4Nb}{\sqrt{6}a}\right)\right]}{\sqrt{6}} + \frac{2ay_0S_{-1001}(x)+x_2c^2S_{-1002}(x)}{e^{a_2+x_3}} \quad (37)$$

$$F_{-101}{}'(x) = F_{-101}{}'' + \mathcal{F}_{-101}(x)$$

$$= \frac{2a\sqrt{\pi}\left[1+\Phi\left(\frac{4Nb}{\sqrt{6}a}\right)\right]}{\sqrt{6}} + \frac{2ay_1 S_{-1011}(x)+x_5c^2S_{-1012}(x)}{e^{a_2+x_6}} \quad (38)$$

Where

$$y_0 = \left(\frac{2a^2+bc^2}{2ac^2} \equiv \frac{b'}{2a}\right) - \frac{x}{2a} = \frac{2b\left(N+\frac{1}{2}\right)-x}{2a} \quad (39)$$

$$y_1 = \frac{2a^2-bc^2}{2ac^2} + \frac{x}{2a} = \frac{2b\left(N-\frac{1}{2}\right)+x}{2a} \quad (40)$$

$$x_2 = \frac{4a^2y_0^2}{c^4} = \frac{4N^2b^2y_0^2}{a^2} = \frac{N^2b^2\left[2b\left(N+\frac{1}{2}\right)-x\right]^2}{a^4} \quad (41)$$

$$x_5 = \frac{4a^2y_1^2}{c^4} = \frac{4N^2b^2y_1^2}{a^2} = \frac{N^2b^2\left[2b\left(N-\frac{1}{2}\right)+x\right]^2}{a^4} \quad (42)$$

$$x_3 = \frac{3y_0^2}{2} = \frac{3}{2}\frac{\left[2b\left(N+\frac{1}{2}\right)-x\right]^2}{4a^2} \quad (43)$$

$$x_6 = \frac{3y_1^2}{2} = \frac{3}{2}\frac{\left[2b\left(N-\frac{1}{2}\right)+x\right]^2}{4a^2} \quad (44)$$

$$2ay_0 = 2b\left(N+\frac{1}{2}\right) - x \quad (45)$$

$$2ay_1 = 2b\left(N-\frac{1}{2}\right) + x \quad (46)$$

$$x_2c^2 = \frac{4a^2y_0^2}{c^2} = 4Nby_0^2 = \frac{Nb\left[2b\left(N+\frac{1}{2}\right)-x\right]^2}{a^2} \quad (47)$$

$$x_5c^2 = \frac{4a^2y_1^2}{c^2} = 4by_1^2 = \frac{Nb\left[2b\left(N-\frac{1}{2}\right)+x\right]^2}{a^2} \quad (48)$$

If we compare and contrast (19), (24), and (28) with (31) or (20), (25), and (29) with (32) we have the following

$$F_{-100}{}'(x \equiv b') = \frac{\sqrt{2\pi}a}{\sqrt{3}}\left[1-\Phi\left(\frac{3x}{2\sqrt{6}a} - \frac{(3-2N)}{2\sqrt{6}}\frac{b}{a}\right)\right]$$

$$= \frac{2a\sqrt{\pi}}{\sqrt{6}}\left[1-\Phi\left(\frac{4Nb}{a\sqrt{6}}\right)\right] \quad (49)$$

$$F_{-101}{}'(x \equiv -b'') = \frac{2a\sqrt{\pi}}{\sqrt{6}}\left[1-\Phi\left(\frac{3x}{2a\sqrt{6}} - \frac{(3+2N)b}{2\sqrt{6}a}\right)\right]$$

$$= \frac{2a\sqrt{\pi}}{\sqrt{3}}\left[1-\Phi\left(\frac{-4Nb}{a\sqrt{6}}\right)\right]$$

$$= \frac{2a\sqrt{\pi}}{\sqrt{3}}\left[1+\Phi\left(\frac{4Nb}{a\sqrt{6}}\right)\right] \quad (50)$$

On the other hand, we have from (37) and (38) we have,

$$F_{-100}{}'(b') = \frac{2a\sqrt{\pi}\left[1-\Phi\left(\frac{4Nb}{\sqrt{6}a}\right)\right]}{\sqrt{6}} + \frac{2ay_0S_{-1001}(b')+x_2c^2S_{-1002}(b')}{e^{a_2+x_3}}$$

$$= \frac{2a\sqrt{\pi}\left[1-\Phi\left(\frac{4Nb}{\sqrt{6}a}\right)\right]}{\sqrt{6}} \quad (51)$$

$$F_{-101}{}'(-b) = \frac{2a\sqrt{\pi}\left[1+\Phi\left(\frac{4Nb}{\sqrt{6}a}\right)\right]}{\sqrt{6}} + \frac{2ay_1S_{-1011}(-b)+x_5c^2S_{-1012}(-b)}{e^{a_2+x_6}}$$

$$= \frac{2a\sqrt{\pi}\left[1+\Phi\left(\frac{4Nb}{\sqrt{6}a}\right)\right]}{\sqrt{6}} \quad (52)$$

As expected, (49)/(50) is identical to (51)/(52). Since, they have to be identical for all values of x; hence, we have

$$\Phi\left[\frac{3x-(3-2N)b}{2\sqrt{6}a}\right] = \Phi\left(\frac{4Nb}{\sqrt{6}a}\right) - \frac{2ay_0S_{-1001}(x)+x_2c^2S_{-1002}(x)}{e^{a_2+x_3}} \quad (53)$$

$$\Phi\left[\frac{3x-(3+2N)b}{2a\sqrt{6}}\right] = -\Phi\left(\frac{4Nb}{\sqrt{6}a}\right) - \frac{2ay_1S_{-1011}(x)+x_5c^2S_{-1012}(x)}{e^{a_2+x_6}} \quad (54)$$

Dr. Progri produced two original formulas of the error function that are not found in any of the references [15]-[17] on may not be in any other references or that matter.

It remains to solve the other integral

$$F_{p1i}{}'(x) = -\int_x^{\infty} e^{-\frac{g_i^2(t)}{2}}\Phi\left(\frac{c_i(t)}{\sqrt{2}}\right)dt \quad (55)$$

Integral (55) is extremely difficult, hence, aside from the unique solutions already provided in Progri (2021, [13], [14]) I will not provide any other alternative computations unless they become available and if they do then I will provide that solution is another journal paper.

This concludes the derivations of the first special case of the partial GPCFD CDF. Next, we consider the second special case of the partial GPCFD cdf.

### *3.2 Second Special Case of the Partial GPCFD CDF*

The second special case also offers some savings in computation of the efficient GPCFD cdf.

For the second special case its partial pdf is given by

$$f_{-2}(x) = \frac{\frac{H_{N-2}(N)1!}{a^{N-2}2^{-\frac{N-2}{2}}}\left[e^{-\frac{1}{2}g_0^2(x)}V(-2,v_0)+e^{-\frac{1}{2}g_1^2(x)}V(-2,v_1)\right]}{2^{\frac{3(N+1)}{2}}3^{-\frac{N}{2}}ae^{\frac{-a^2}{3c^4}}\sum_{k=0}^{N-1}\frac{H_k(N)n!}{a^k}\left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)} \quad (56)$$

and if assume all possible positive integer values of the input parameter $N = 2,3,\cdots$ we have

$$F'(x) = \frac{\frac{H_{N-2}(N)1!}{a^{N-2}2^{-\frac{N-2}{2}}}F_{-2}{}'(x)}{2^{\frac{3(N+1)}{2}}3^{-\frac{N}{2}}ae^{\frac{-a^2}{3c^4}}\sum_{k=0}^{N-1}\frac{H_k(N)n!}{a^k}\left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)}$$

$$= \frac{F_{-2}{}'(x)+\frac{4\sqrt{2}}{a}F_{-1}{}'(x)}{2^{4.5}3^{-1}ae^{\frac{-a^2}{3c^4}}\left[D_{-2}\left(\frac{2a}{\sqrt{3}c^2}\right)+\frac{2\sqrt{3}}{a}D_{-1}\left(\frac{2a}{\sqrt{3}c^2}\right)\right]} \quad (57)$$

It remains to solve two partial integrals:

$$F_{-2i}{}' = \int_x^\infty e^{-\frac{g_i^2(t)}{2}} V\left(-2, \frac{c_i(t)}{\sqrt{2}}\right) dt^{\text{ iii}};\ i = \{0,1\}$$

$$= \int_x^\infty e^{-\frac{g_i^2(t)}{2}} \frac{{}_1F_1\left[\begin{matrix}-\frac{1}{2}\\ \frac{1}{2}\end{matrix}\middle| -\frac{c_i^2(t)}{2}\right]}{\Gamma\left(\frac{3}{2}\right)} dt - 2\int_x^\infty e^{-\frac{g_i^2(t)}{2}} \frac{c_i(t)}{\sqrt{2}} dt$$

$$= \frac{2\int_x^\infty e^{-\frac{g_i^2(t)}{2}}\, {}_1F_1\left[\begin{matrix}-\frac{1}{2}\\ \frac{1}{2}\end{matrix}\middle| -\frac{c_i^2(t)}{2}\right] dt}{\sqrt{\pi}} - 2\int_x^\infty e^{-\frac{g_i^2(t)}{2}} \frac{c_i(t)}{\sqrt{2}} dt \tag{58}$$

which then produces four integrals,

$$F_{-20i}{}'(x) = \frac{2\int_x^\infty e^{-\frac{g_i^2(t)}{2}}\, {}_1F_1\left[\begin{matrix}-\frac{1}{2}\\ \frac{1}{2}\end{matrix}\middle| -\frac{c_i^2(t)}{2}\right] dt}{\sqrt{\pi}};\ i = \{0,1\} \tag{59}$$

$$F_{-21i}{}'(x) = -2\int_x^\infty e^{-\frac{g_i^2(t)}{2}} \frac{c_i(t)}{\sqrt{2}} dt;\ i = \{0,1\} \tag{60}$$

The first integral $F_{-20i}{}'(x)$ is very difficult; hence, aside from the solutions provided in Progri (2021, [13], [14]) I will not provide any other alternative computations; however, should they become available I will provide them in a separate journal paper.

For the second integral; however, we can obtain a much faster alternative solutions than the one found in Progri (2021, [13]). If we make the substitution (13) into (60) we have

$$F_{-21i}{}'(x) = -2\int_x^\infty e^{-\frac{3\left(\frac{4Nb}{3a}-y\right)^2}{2}} \frac{y}{\sqrt{2}} dt;\ i = \{0,1\} \tag{61}$$

Next, we make the substitution (12) and (16) into (61) produces

$$F_{-21i}{}'(x) = (-1)^i 4a \int_{y_i}^{(-1)^{i+1}\infty} e^{-\frac{3\left(\frac{4Nb}{3a}-y\right)^2}{2}} \frac{y}{\sqrt{2}} dy \tag{62}$$

Next, we make the following substitution

$$z = \frac{\sqrt{3}\left(\frac{4Nb}{3a}-y\right)}{\sqrt{2}};\ dz = \frac{-\sqrt{3}dy}{\sqrt{2}};\ y \to |_{y_i}^{(-1)^{i+1}\infty};\ z \to |_{z_i}^{(-1)^i\infty} \tag{63}$$

we have

$$F_{-21i}{}'(x) = \frac{(-1)^{i+1}4a}{\sqrt{3}} \int_{z_i}^{(-1)^i\infty} e^{-z^2}\left(\frac{4Nb}{3a} - \frac{\sqrt{2}}{\sqrt{3}} z\right) dz \tag{64}$$

Integral (64) can be split into two integrals:

$$F_{-21i}{}'(x) = \frac{(-1)^i 4}{3}\left[\frac{\sqrt{2}\int_{z_i}^{(-1)^i\infty} e^{-z^2} z dz}{a^{-1}} - \frac{4Nb\int_{z_i}^{(-1)^i\infty} e^{-z^2} dz}{\sqrt{3}}\right] \tag{65}$$

From (Gradshteyn, Ryzhik 2007, [15] pg. 336 3.322 1.) or (2019, [17] pg. 5 1.1 1.1.1 1.), or Ng, Geller (1969, [15] pg. 18 5. (A2)) we solve the second of the two integrals first as follows

$$\int_{z_0}^\infty e^{-z^2} dz = \frac{\sqrt{\pi}}{2}[1 - \Phi(z_0)] \tag{66}$$

$$-\int_{z_1}^{-\infty} e^{-z^2} dz = \int_{-z_1}^\infty e^{-z^2} dz = \frac{\sqrt{\pi}}{2}[1 - \Phi(-z_1)] \tag{67}$$

The first integral is easily solved as follows

$$\int_{z_i}^{(-1)^i\infty} \frac{z}{e^{z^2}} dz = \frac{-1}{2}\left[\int_{z_i}^{(-1)^i\infty} \frac{d(-z^2)}{e^{z^2}} = \frac{1}{e^{z^2}}|_{z_i}^{(-1)^i\infty}\right] = \frac{e^{-z_i^2}}{2} \tag{68}$$

Finally, substituting (66) through (68) into (65) yields

$$F_{-210}{}'(x) = \frac{4}{3}\left\{\frac{ae^{-z_0^2}}{\sqrt{2}} + \frac{2N\sqrt{\pi}b\Phi(z_0)}{\sqrt{3}} - \frac{2N\sqrt{\pi}b}{\sqrt{3}}\right\} \tag{69}$$

$$F_{-211}{}'(x) = \frac{4}{3}\left\{\frac{-ae^{-z_1^2}}{\sqrt{2}} - \frac{2N\sqrt{\pi}b\Phi(z_1)}{\sqrt{3}} - \frac{2N\sqrt{\pi}b}{\sqrt{3}}\right\} \tag{70}$$

Where

$$z_i = \frac{\sqrt{3}\left(\frac{4Nb}{3a}-y_i\right)}{\sqrt{2}} = \frac{\sqrt{3}\left[\frac{2Nb}{3a}+(-1)^i\frac{x-b}{2a}\right]}{\sqrt{2}};\ i = \{0,1\} \tag{71}$$

$$z_0 = \frac{\sqrt{3}\left(\frac{4Nb}{3a}-y_0\right)}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}}\left[\frac{3x-(3-2N)b}{6a}\right]; \tag{72}$$

$$z_1 = \frac{\sqrt{3}\left(\frac{4Nb}{3a}-y_1\right)}{\sqrt{2}} = \frac{\sqrt{3}\left[\frac{(3+2N)b-3x}{6a}\right]}{\sqrt{2}}. \tag{73}$$

Let us examine the solution from (138), (145), and (150) in Progri (2021, [13]).

$$F_{-21i}{}'(x) = F_{-21i}{}'' + \mathcal{F}_{-21i}(x) \tag{74}$$

Where

$$F_{-210}{}'' = \frac{4a\left[e^{-a_2}+\sqrt{\pi}a_1\Phi(a_1)\right]}{3\sqrt{2}} - \frac{a_2c^2\sqrt{\pi}}{\sqrt{3}}$$

$$= \frac{4a\left[e^{-a_2}+\sqrt{\pi}a_1\Phi(a_1)\right]}{3\sqrt{2}} - \frac{4Nb\sqrt{\pi}}{3\sqrt{3}} \tag{75}$$

$$F_{-211}{}'' = -\frac{4a\left[e^{-a_2}+\sqrt{\pi}a_1\Phi(a_1)\right]}{3\sqrt{2}} - \frac{a_2c^2\sqrt{\pi}}{\sqrt{3}}$$

$$= -\frac{4a\left[e^{-a_2}+\sqrt{\pi}a_1\Phi(a_1)\right]}{3\sqrt{2}} - \frac{4Nb\sqrt{\pi}}{3\sqrt{3}} \tag{76}$$

$$\mathcal{F}_{-210}(x) = \frac{\sqrt{2}ay_0^2S_{-2101}(x)+\frac{2\sqrt{2}x_2c^2y_0}{3}S_{-2102}(x)}{e^{a_2+x_3}} \tag{77}$$

$$\mathcal{F}_{-211}(x) = \frac{-\sqrt{2}ay_1^2S_{-2111}(x)-\frac{2\sqrt{2}x_5c^2y_5}{3}S_{-2112}(x)}{e^{a_2+x_6}} \tag{78}$$

$$S_{-2101}(x) = F_{1:1;0}^{0:1;1}\left[\begin{matrix}-:1;1;\\ 2:\frac{1}{2};-;\end{matrix} x_2, x_3\right] \tag{79}$$

$$S_{-2102}(x) = F^{0:0;1}_{1:0;0}\left[\begin{matrix} -:-;1; \\ \frac{5}{2}:-;-; \end{matrix} x_2, x_3\right] \tag{80}$$

$$S_{-2111}(x) = F^{0:1;1}_{1:1;0}\left[\begin{matrix} -:1;1; \\ 2:\frac{1}{2};-; \end{matrix} x_5, x_6\right] \tag{81}$$

$$S_{-2112}(x) = F^{0:0;1}_{1:0;0}\left[\begin{matrix} -:-;1; \\ \frac{5}{2}:-;-; \end{matrix} x_5, x_6\right] \tag{82}$$

If we compare and contrast (69) with (75) or (70) with (76) we have the following

$$F_{-100}{}'(x \equiv b') = \frac{4}{3}\left\{\frac{ae^{-z_0^2}}{\sqrt{2}} - \frac{\sqrt{\pi}Nb[1-\Phi(z_0)]}{\sqrt{3}}\right\}$$

$$= \frac{4}{3}\left[\frac{ae^{-a_2}}{\sqrt{2}} + \frac{\sqrt{\pi}Nb\Phi(a_1)}{\sqrt{3}}\right] - \frac{4\sqrt{\pi}Nb}{3\sqrt{3}} \tag{83}$$

$$F_{-211}{}'(x = -b) = \frac{4}{3}\left\{\frac{-ae^{-z_1^2}}{\sqrt{2}} - \frac{\sqrt{\pi}Nb[1+\Phi(z_1)]}{\sqrt{3}}\right\}$$

$$= -\frac{4}{3}\left\{\frac{ae^{-a_2}}{\sqrt{2}} + \frac{\sqrt{\pi}Nb[1+\Phi(a_1)]}{\sqrt{3}}\right\} \tag{84}$$

As expected, (83)/(84) is identical to (75)/(76). Since, they have to be identical for all values of x; hence, we have

$$\frac{e^{-z_0^2}-e^{-a_2}}{\sqrt{2}a^{-1}} + \frac{\Phi(z_0)-\Phi(a_1)}{N^{-1}\pi^{-\frac{1}{2}}\sqrt{3}b^{-1}} = \frac{\frac{3ay_0^2 S_{-2101}(x)}{2\sqrt{2}} + \frac{3x_2c^2y_0 S_{-2102}(x)}{\sqrt{2}}}{e^{a_2+x_3}} \text{iv} \tag{85}$$

$$\frac{e^{-z_1^2}-e^{-a_2}}{\sqrt{2}a^{-1}} + \frac{\Phi(z_1)-\Phi(a_1)}{N^{-1}\pi^{-\frac{1}{2}}\sqrt{3}b^{-1}} = \frac{\frac{3ay_1^2}{2\sqrt{2}} S_{-2111}(x) + \frac{3x_5c^2y_5}{\sqrt{2}} S_{-2112}(x)}{e^{a_2+x_6}} \text{v} \tag{86}$$

Dr. Progri again produced two original formulas of the difference of the error functions and the exponential functions that are not found in any of the references [15]-[17] on may not be in any other references or that matter.

Here we conclude the *third principle of the reduction* of the formulae that for a subspace of the index parameter $p = \{-1,-2\}$ and for all possible positive integer values of the input parameter $N = \{2,3,\cdots\}$.

This concludes the discussion on the special cases of the partial GPCDF cdf. Next, we discuss in great detail a few numerical examples.

## 4 Numerical, Theoretical Results

In my first publication in Progri (2016, [11]) I made a modest effort in the computation of the GPCFD cdf. At the time I was only able to compute the linear approximation.

In my second publication in Progri (2021, [13]) the numerical, theoretical results Sect. 4 provides significantly improved capability in computation of the GPCFD cdf. I was able to compute the efficient GPCFD cdf in cfe by means of eight Kampé de Fériet function (or double hypergeometric series) [6]-[10] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x,y,z]$ (see Appendix E of Progri (2019, [10]); i.e., it is computationally the most laborious function that I have ever come across with.

TABLE I: THE QUANTITATIVE COMPUTATIONAL PERFORMANCE

| GPCDF cdf: CFE 1 Efficient (151) vs. CFE 2 Special Cases | | | |
|---|---|---|---|
| $a = 2,\ b = 1,\ N = 1$, Op 4, $\sum(\cdot) = 30$ terms | | | |
| Integral (ms) | L. Ap. (ms) | CFE 1 (sec) | CFE 2 (sec) |
| 43.1 | 1.5 | 0.98 | 0.71 |
| $a = 2,\ b = 1,\ N = 2$, Op 4, $\sum(\cdot) = 35$ terms | | | |
| Integral (ms) | L. Ap. (ms) | CFE 1 (sec) | CFE 2 (sec) |
| 66.6 | 1.4 | 2.58 | 1.74 |
| $a = 2,\ b = 1,\ N = 3$, Op 4, $\sum(\cdot) = 40$ terms | | | |
| Integral (sec) | L. Ap. (ms) | CFE 1 (sec) | CFE 2 (sec) |
| 5.18 | 64.9 | 5.54 | 4.53 |
| $a = 2,\ b = 1,\ N = 4$, Op. 4, $\sum(\cdot) = 40$ terms | | | |
| Integral (sec) | L. Ap. (ms) | CFE 1 (sec) | CFE 2 (sec) |
| 9.1 | 115 | 7.78 | 6.73 |

For a special case when $a = 1;\ b = 1;\ N = 1 \Rightarrow c = 1$ the computation of the closed form expression of the GPCFD cdf is reduced because four Kampé de Fériet function (or double hypergeometric series) [6]-[10] are reduced into two error functions and four Srivastava's Triple Hypergeometric Series $F^{(3)}[x,y,z]$ (see Appendix E of Progri (2019, [10]) are reduced into four Kampé de Fériet function (or double hypergeometric series) [6]-[10] as discussed in Appendix C.

However, the computational algorithms were grossly (extremely, or excessively) unoptimized as indicated by the red color of the computational times in Tab. I of Progri (2021, [13]).

The majority of the computational, quantitative, optimization issues of the efficient GPCFD cdf in addition to the landmark GPCFD cdf were addressed in Progri (2021, [14]) as indicated by the green color of the computational times in Tab. I of Progri (2021, [14]).

In this Sect. further refinements of the optimization algorithm are performed in addition to the inclusion of the cfe of the partial GPCFD cdf from Sect. 3 of this publication. Therefore, we would expect greener computational times in Tab. I of the numerical results.

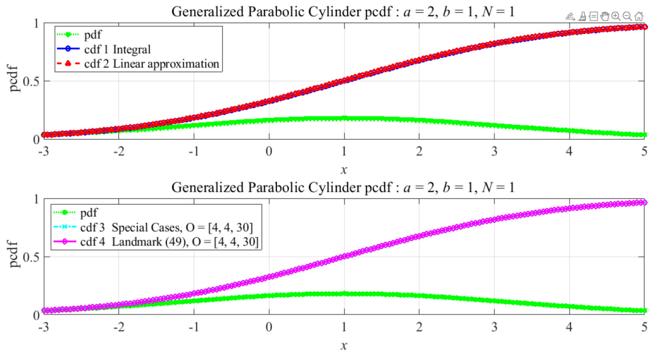


Fig. 1. The computation of the GPCFD pdf and cdf for $-3 \leq x \leq 5,\ a = 2,\ b = 1,\ N = 1$.

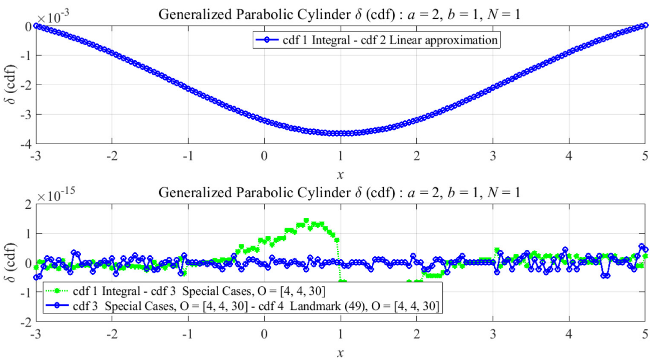


Fig. 2. The absolute error the GPCFD cdf for $-3 \leq x \leq 5$, $a = 2,\ b = 1,\ N = 1$.

### *4.1 Examples*

For the Num. Ex. Sect., we employ a slightly different the same setup from the one in Num. Ex. Sect. Progri (2021, [13], [14]).

We ran the simulations on a Dell computer Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz using MATLAB 2020b [20] using a vector of $-3 \leq x \leq 5$ with a sample increment of $\Delta x = 0.05$ that leads to 160 points. Here we have reduced the number of points so as to reduce the computation time but have improved the accuracy due to further refinement of the algorithm. How have we done that? By comparing and contrasting the computation times shown in Tab. I of this paper with the same results shown in Tab. I of Progri (2021, [14]). As shown in Tab. I for exactly the same set of the input parameters $a$, $b$, $N$, and Op. same accuracy and faster computation times are obtained for fewer number of terms or $\sum(\cdot)$.

Table 1 presents the computational duration times in either seconds or milliseconds of the cdf 1 or the integral, cdf 2 or linear approximation, cdf 3 via Efficient (151) of Progri (2021, [13]) without Special Cases simplification of this paper, and cdf 4 of the same as cdf 3 but with the Special Cases (19), (20), (69), and (70). The first row in Tab. 1 correspond to the

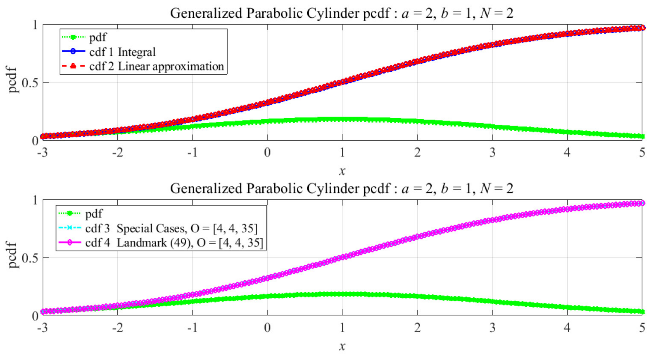


Fig. 3. The computation of the GPCFD pdf and cdf for $-3 \leq x \leq 5,\ a = 2,\ b = 1,\ N = 2$.

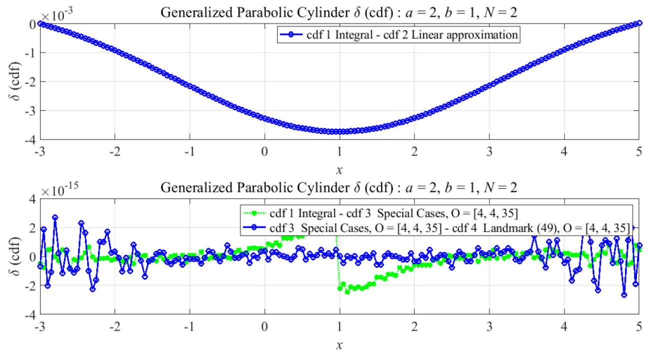


Fig. 4. The absolute error the GPCFD cdf for $-3 \leq x \leq 5$, $a = 2,\ b = 1,\ N = 2$.

computation times for values of $a = 1;\ b = 1;\ N = 1$ for thirty terms for option four that is used for the computation of the four Kampé de Fériet functions (or double hypergeometric series) [6]-[10] and four Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [10])). Again, by comparing and contrasting the computation times in the first row with the computation times in fourth row of Progri (2021, [14]) we see a reduction of the computation times by almost a factor of 3.70/5.11 for the efficient algorithm without special cases/efficient algorithm with special cases.

The rest of the data in Tab. I is self-explanatory as we ran and recorded the computation times for various values of $a = \{2\}$, $b = \{1\}$, $N = \{1,2,3,4\}$, and $Op. = 4$.

Figure 1 presents the GPCFD pdf and four GPCFD cdfs cdf 1 corresponds to the integral, cdf 2 to linear approximation (top), cdf 3 to a closed form expression via (151) Progri (2021, [13]) with the Special Cases (19), (20), (69), and (70) op. 4 using thirty terms and cdf 4 Landmark (49) using the same number of terms (bottom). In Fig. 2 the absolute error of the cdf 1 with cdf 2 is shown (top); and cdf 1 with cdf 3 and cdf 3 with cdf 4 (bottom).

Figures 3, 5, 7 and 4, 6, 8 present the same result as Figs. 1

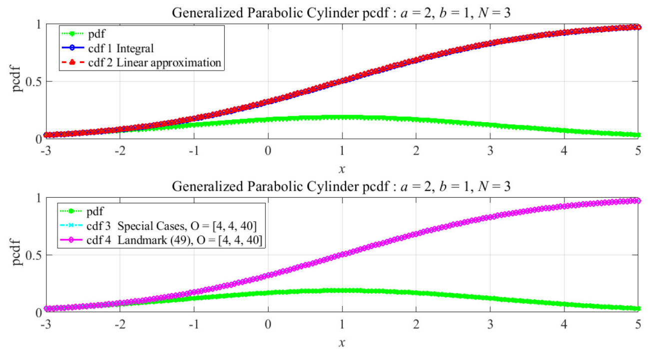


Fig. 5. The computation of the GPCFD pdf and cdf for $-3 \leq x \leq 5,\ a = 2,\ b = 1,\ N = 3$.

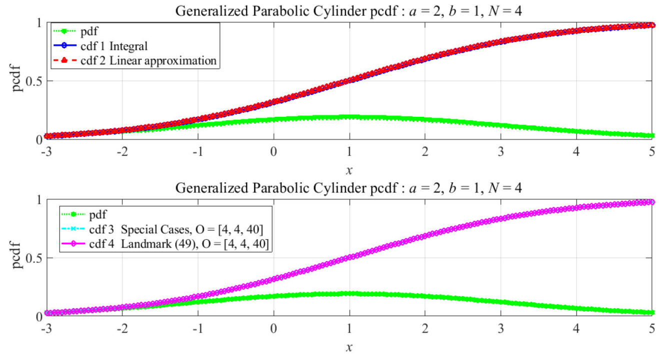


Fig. 7. The computation of the GPCFD pdf and cdf for $-3 \leq x \leq 5,\ a = 2,\ b = 1,\ N = 4$.

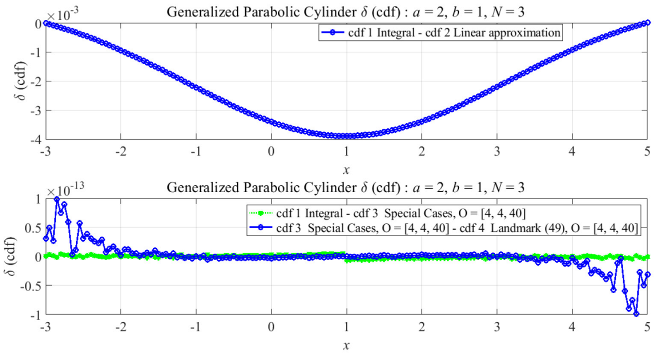


Fig. 6. The absolute error the GPCFD cdf for $-3 \leq x \leq 5$, $a = 2,\ b = 1,\ N = 3$.

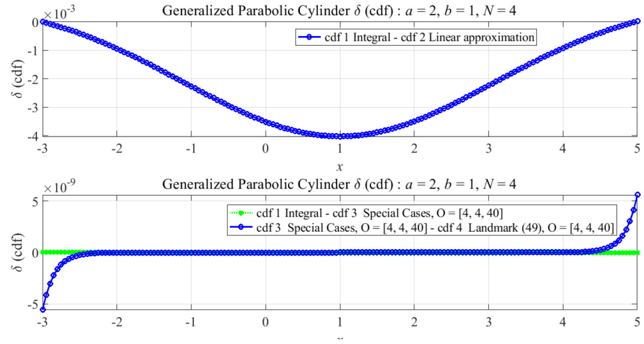


Fig. 8. The absolute error the GPCFD cdf for $-3 \leq x \leq 5$, $a = 2,\ b = 1,\ N = 4$.

and 2 but for values of $a = 2;\ b = 1;\ N = \{2,3,4\}$.

As shown from Figs. 4, 6, 8 the performance accuracy of the landmark algorithm begins to deteriorate as $N$ increases from two to four.

The computation times for the third and fourth rows increase significantly as $N$ increases from two to four; however, we notice that the computation times of the Efficient algorithm (151) Progri (2021, [13]) with the Special Cases (19), (20), (69), and (70) is faster than the Integral cdf 1 for $N = \{3,4\}$. This is very significant because for the first time the computation of the Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [10])) is faster than the Integral routine from MATLAB 2020b.

This concluded the discussion on examples and numerical, theoretical results.

## 5 Conclusions

This journal paper provides exceptional discussions on the importance of special cases and a complete illustration of the three principles of the reduction of the formulae of the error functions and their connection to the Kampé de Fériet functions [6]-[10]. As a result of the applications of the three principles of reduction four original formulae are obtained as illustrated by (53), (54), (85), and (86).

The introduction, description, expansion, and derivation of the two special cases in this outstanding journal paper is one of the most significant highlights of the importance of special cases in the computation of the efficient GPCFD cdf in comparison and contrast to the special cases discussed in Progri (2019, [10]). The analytical, computational work is not done until all possible special cases are discussed in great detail. This work is not meant to exhaust every single possible special case; but to provide enough insights that should a reader, scholar, practitioner in the field require the development of a particular special case then this body of work provides more than enough material for the person to do so.

Based on the computations times and the performance accuracy shown in this journal paper the efficient algorithm is still a good and useful algorithm even though it still requires the computations of Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [10])).

Since, a lot more has already been said it makes no sense to reiterate and therefore this concludes the discussion on the conclusions Sect.

## 6 Acknowledgement

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I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [20] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications*. *Vol. I* (Not yet available in print) [19].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless, to say I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [21] reminds me of my grandfather who I loved him very much.

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[i] In 2016 Dr. Progri stated: "We can only compute Kampé de Fériet functions indirectly as part of the integration or integral. The direct computation of the Kampé de Fériet function will be considered in a future publication," (Progri (2016, [7])).

[ii] Because in general

$$F_{-1}{}'(x, N > 2) \neq F_{-1}{}'(x, N = 2) \neq F_{-1}{}'(x, N = 1) \tag{1}$$

we have to develop a general solution for

$$N > 2 \Rightarrow k = N - 1 \Rightarrow \begin{array}{l} p = k - N = -1 \\ n = N - 1 - k = 0 \end{array} \tag{2}$$

[iii] Ditto ii.

[iv] Please note that the solution is only valid for integer values of $N = \{2, 3, \cdots\}$.

[v] Ditto iv.