

Research Article

Landmark Computation of the Generalized Parabolic Cylinder Function Distribution

Ilir F. Progri¹

¹Giftet Inc., 5 Euclid Ave. #3, Worcester, MA 01610, USA

ORCID: 0000-0001-5197-1278

Correspondence should be addressed to Ilir Progri; iproгри@giftet.com

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This paper builds upon the originality, novelty, and innovative expansion of the efficient, recursive, and optimized computation of the generalized parabolic cylinder function (or GPCF) distributionⁱ (or GPCFD) probability density function (pdf) and the cumulative distribution function (cdf) in a manner that presents the pinnacle (or the landmark) in the *masterpiece marvels in analytical derivations and numerical computational series*. In this journal paper Dr. Progri has successfully utilized all the tools necessary to produce a landmark masterpiece of the original, novel, and innovative closed form expression (cfe) of the cdf of the GPCFD: the symmetry of the pdf and cdf, maneuvering, expansion by means of orthogonal polynomials such as Hermite polynomials, recursive algorithms, and computations of six pairs of generalized, modified Kampé de Fériet functions and generalized, modified confluent hypergeometric functions with Hermite polynomials in the numerator. The computational complexity and numerical results of the derived algorithm is compared and contrasted with the efficient algorithm derived earlier clearly demonstrate a *superior performance* in both speed, memory allocation, and accuracy.

Index Terms—Parabolic cylinder function, cumulative distribution function, Kampé de Fériet function, Hermite polynomials, landmark closed form expression, recursive algorithms, generalized functions, pdf cdf optimization analysis, hypergeometric series, confluent hypergeometric series.

1 Introduction

The derivation of the landmark generalized parabolic cylinder

function (PCF) [1] distribution probability density function (pdf) and of the cumulative distribution function (cdf) is performed by taking advantage of all the necessary tools in toolbox such as use of symmetry, maneuvering, expansion by means of the series of Hermite polynomialsⁱⁱ, efficient,

recursive, and optimized computation of six pairs of generalized, modified Kampé de Fériet functions and generalized, modified confluent hypergeometric functions with Hermite polynomials in the numerator [2].

In 2016, when Dr. Proгри began the investigation of the PCFDs in indoor geolocation applications [3], [4] recognized a shortage of the tools and of the toolbox for deriving the closed form expression of the GPCFD pdf and cdf and their efficient computation so he began a preliminary investigation GPCFD pdf and cdf (Proгри (2016, [5])) as he was investigating somewhat similar pdfs and cdfs (Proгри (2016, [7])) and their expansion by means of the Kampé de Fériet function.

In 2016, Dr. Proгри recognized that the Kampé de Fériet function might be a useful toolⁱⁱⁱ, but he lacked the toolbox, i.e., he was not able at the time to compute these functions in MATLAB (see Proгри (2016, [8])).

In 2018 Dr. Proгри produced the first masterpiece marvel for the efficient computation of the generalized Bessel function distribution for real values of the parameter by means of the Kampé de Fériet functions or of the double hypergeometric functions (Proгри (2018, [9])) because he was able to efficiently compute the Kampé de Fériet functions. This was a significant achievement in both the tools and the development of the toolbox.

In 2019 Dr. Proгри produced the second masterpiece while performing the investigation of the special cases of the generalized Bessel function distribution in (Proгри (2019, [10])). This was a significant achievement in the development of both the tools and of the computational toolbox because this is the first landmark publication that Dr. Proгри exhausted every single avenue in the computation of the generalized Bessel function distribution pdf and cdf for the integer value of the parameter. In this publication Dr. Proгри produced the first modified generalized Kampé de Fériet function that included a computation of the digamma function or the psi function in the numerator. He was also able to produce various recursive algorithms and compute the Srivastava or the triple hypergeometric series (see Appendix E in Proгри (2019, [10])).

In 2021, Dr. Proгри produced the first successful, original efficient computation of the GPCFD pdf and cdf (see Proгри 2021, [6]) and made significant advancement in the computation of the irregular singularities of the confluent hypergeometric function (see Proгри (2021, [11])). Although the efficient computation of the GPCFD pdf and cdf (see Proгри 2021, [6]) did not produce the results that Dr. Proгри expected

to achieve, it set the stage for this landmark publication. What was learned in Proгри (2021, [11])? First, it was the first successful attempt to eliminate the singularity in the integration process. Second, it was the first attempt that had a very good structure that will make use of the symmetry in comparison and contrast to the (Proгри (2016, [5])) which lacked both maneuvering and a good structure.

Up to this point, Dr. Proгри has made significant advancement in both the analytical and computational tools and of the toolbox. Up to this point, the stage was set perfectly for a major breakthrough, the creation of a landmark publication. This was accomplished by adding the use of the expansion via the orthogonal polynomials such as the Hermite polynomials [2], [12] and the use of the recursive algorithms in the computation of the GPCFD pdf and cdf and the improvement of the overall structure of the integration process and of the tools and of the toolbox^{iv}.

This paper is organized as follows: in Sect. 2 the efficient computation of the landmark GPCFD pdf is presented. The efficient computation of the landmark GPCFD cdf is discussed in Sect. 3. Section IV contains numerical results; Conclusion is provided in Section V along with a list of references and a series of Appendices.

2 The Landmark GPCFD PDF

The main purpose of the efficient computation of a special case of the landmark GPCFD pdf is not only to validate the computational work that was performed in the previous section but also to provide great insight into the nature of the landmark GPCFD pdf and then later of the landmark GPCFD cdf.

2.1 Optimization of the Landmark GPCFD PDF

The main purpose of the landmark GPCF pdf was to enable a fast computation of the integration using the MALAB integral function. From Tab. I (column 1) in Proгри (2021, [6]) the integral function computational performance was on the order of from 2.73 sec for $N = 1$ to 6.75 sec for $N = 1$. The main purpose of this section is to reduce this computation by at least an order of magnitude. How is this accomplished? This is accomplished by eliminating redundancy in computation or performing the order of the computations that eliminates redundancy in computation.

From Proгри (2021, [6]), (53) we have

$$\begin{aligned}
f_{PCD}(x) &= \frac{\sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2} \frac{k}{2}} \left[e^{-\frac{1}{2}g_0^2(x)} V(p, v_0) + e^{-\frac{1}{2}g_1^2(x)} V(p, v_1) \right]}{\frac{3(N+1)}{2} \frac{N}{2} \frac{-a^2}{ae^{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^k} \left(\frac{\sqrt{3}}{2} \right)^k D_{k-N} \left(\frac{2a}{\sqrt{3}c^2} \right)} \\
&= \frac{e^{-\frac{1}{2}g_0^2(x)} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2} \frac{k}{2}} V(p, v_0) + e^{-\frac{1}{2}g_1^2(x)} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2} \frac{k}{2}} V(p, v_1)}{\frac{3(N+1)}{2} \frac{N}{2} \frac{-a^2}{ae^{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^k} \left(\frac{\sqrt{3}}{2} \right)^k D_{k-N} \left(\frac{2a}{\sqrt{3}c^2} \right)} \\
&= \frac{e^{-\frac{1}{2}g_0^2(x)} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2} \frac{k}{2}} \left| \frac{{}_1F_1 \left[\frac{p+1}{2}, \frac{1}{2} \right] - v_0^2}{\Gamma \left(\frac{1-p}{2} \right)} - \frac{v_0 {}_1F_1 \left[\frac{p+1}{2}, \frac{3}{2} \right] - v_0^2}{2^{-1} \Gamma \left(\frac{-p}{2} \right)} \right|}{\frac{3(N+1)}{2} \frac{N}{2} \frac{-a^2}{ae^{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^k} \left(\frac{\sqrt{3}}{2} \right)^k D_{k-N} \left(\frac{2a}{\sqrt{3}c^2} \right)} \\
&\quad + \frac{e^{-\frac{1}{2}g_1^2(x)} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2} \frac{k}{2}} \left| \frac{{}_1F_1 \left[\frac{p+1}{2}, \frac{1}{2} \right] - v_1^2}{\Gamma \left(\frac{1-p}{2} \right)} - \frac{v_1 {}_1F_1 \left[\frac{p+1}{2}, \frac{3}{2} \right] - v_1^2}{2^{-1} \Gamma \left(\frac{-p}{2} \right)} \right|}{\frac{3(N+1)}{2} \frac{N}{2} \frac{-a^2}{ae^{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^k} \left(\frac{\sqrt{3}}{2} \right)^k D_{k-N} \left(\frac{2a}{\sqrt{3}c^2} \right)} \quad (1)
\end{aligned}$$

where

$$a^2 = Nbc^2 \quad (\text{Progr 2016, [3] 6, 7}) \quad (2)$$

$$v_i^2 = \frac{c_i^2(x)}{2} \quad (3)$$

$$c_i(x) = \frac{a}{c^2} - (-1)^i \frac{x-b}{2a} \quad (4)$$

$$\begin{aligned}
g_i(x) &= \frac{\sqrt{3}(x-b)}{2a} + \frac{(-1)^i a}{\sqrt{3}c^2} \\
&= \frac{(-1)^i 4a}{\sqrt{3}c^2} - \sqrt{3}(-1)^i c_i(x) \\
&= \frac{(-1)^i 4a}{\sqrt{3}c^2} - \sqrt{6}(-1)^i v_i(x) \quad (5)
\end{aligned}$$

$$\frac{g_i^2(x)}{2} = \frac{8a^2}{3c^4} - \frac{4\sqrt{2}a}{c^2} v_i(x) + 6v_i^2(x) \quad (6)$$

First, in (1) we have taken the exponential function outside of the summation; hence, this eliminates the redundancy in computation. Second, we have performed a recursive computation of the term inside the summation in both the numerator and the denominator that depends on the index k . These two changes will significantly improve the computational performance of the integral function that is discussed in the numerical section.

This concludes the discussion on the optimization of the landmark GPCFD pdf. Next, we consider a few special cases.

2.2 Special Cases of the Landmark GPCFD PDF

Further improvements of the computational performance can only be achieved if we know exactly certain values of the

parameters. For these special cases the computation of the confluent hypergeometric function can be reduced to the computation of well-known functions. The main purpose here is not to consider all the possible special cases because there is a significant large number of special cases; but it is to consider a small number of special cases and provide enough knowledge, guidance, insights that all the remaining special cases, if need be, can be derived or considered in like manner as the special cases presented in this subsection. From the Tab. I in Progr (2021, [6]) we notice that the computational performance is roughly a linear function of the parameter N ; i.e., the computational performance corresponding to $N = 2$ is roughly twice as high as the one corresponding to $N = 1$. The only way to reduce the computational performance is to consider these two cases one at a time and see how we can improve the computational performance.

First, let us consider a special case when:

$$N = 1 \Rightarrow p = -1; n = 0; k = 0 \quad (7)$$

Substituting (7) into (1) and from (Gradshteyn, Ryzhik 2007, [11] pg. 1027 9.236 1.) we obtain the solution for this special case as

$$\begin{aligned}
f_{PCD}(x) &= \frac{\frac{1 - \frac{2v_0 {}_1F_1 \left[\frac{1}{2}, \frac{1}{2} \right] - v_0^2}{\sqrt{\pi}}}{e^{\frac{g_0^2(x)}{2}}} + \frac{1 - \frac{2v_1 {}_1F_1 \left[\frac{1}{2}, \frac{1}{2} \right] - v_1^2}{\sqrt{\pi}}}{e^{\frac{g_1^2(x)}{2}}}}{2^3 3^{-\frac{1}{2}} a e^{\frac{-a^2}{3c^4}} D_{-1} \left(\frac{2a}{\sqrt{3}c^2} \right)} \\
&= \frac{\sqrt{3} e^{-0.5g_0^2(x)} [1 - \Phi(v_0)] + e^{-0.5g_1^2(x)} [1 - \Phi(v_1)]}{4a \sqrt{2\pi} \left[1 - \Phi \left(\frac{2a}{\sqrt{6}c^2} \right) \right]} \quad (8)
\end{aligned}$$

Equation (8) simplifies even more when $a = 1$. From Progr 2021, [6], we already know that the solution for this special case is

$$f_{PCD}(x) = \frac{e^{-\frac{(\sqrt{3}x - \frac{b}{\sqrt{3}})^2}{8}} [1 - \Phi(\frac{3b-x}{2\sqrt{2}})] + e^{-\frac{(\sqrt{3}x - \frac{5b}{\sqrt{3}})^2}{8}} [1 - \Phi(\frac{b+x}{2\sqrt{2}})]}{4 \times 3^{-0.5} \times \sqrt{2\pi} [1 - \Phi(\frac{2b}{\sqrt{6}})]} \quad (9)$$

From (8) and (9) we obtain all the insight into the nature of the special case of the GPCFD pdf. For this special case the GPCFD pdf is reduced to a difference of a sum of two normally distributed pdfs with a weighted sum of two normally distributed pdfs weighted by error functions.

Therefore, not only is (60) very significant as an original expression not published anywhere else, but also because it provides great insight into the nature and the efficient computation of the GPCFD cdf.

Significant savings in the computation of the integral function are obtained from (8) because the GPCFD pdf is

reduced to the computation of elementary functions.

Next, let us consider another special case when $N = 2 \Rightarrow k = \{0,1\}; p = \{-2, -1\}; n = \{1,0\}$ (10)

then (1) becomes

$$\begin{aligned}
 f_{PCD}(x) = & \frac{e^{-\frac{1}{2}g_0^2(x)} \sum_{k=0}^1 \frac{H_k(2)n!}{a^{k_2} \frac{-k}{2}} \left| \frac{{}_1F_1\left[\frac{p+1}{2}, \frac{1}{2}, -v_0^2\right]}{\Gamma\left(\frac{1-p}{2}\right)} - \frac{v_{01} {}_1F_1\left[\frac{p+1}{2}, \frac{3}{2}, -v_0^2\right]}{2^{-1}\Gamma\left(\frac{-p}{2}\right)} \right|}{\frac{9}{2^2 3^{-1} a} e^{-\frac{a^2}{3c^4}} \sum_{k=0}^1 \frac{H_k(2)n!}{a^{k_2} \frac{-k}{2}} \left| \frac{{}_1F_1\left[\frac{p+1}{2}, \frac{1}{2}, -v_1^2\right]}{\Gamma\left(\frac{1-p}{2}\right)} - \frac{v_{11} {}_1F_1\left[\frac{p+1}{2}, \frac{3}{2}, -v_1^2\right]}{2^{-1}\Gamma\left(\frac{-p}{2}\right)} \right|} \\
 & \frac{3e^{\frac{a^2}{3c^4}}}{\frac{1}{e^{\frac{1}{2}g_0^2(x)}} + \frac{1}{e^{\frac{1}{2}g_1^2(x)}}} \\
 & = \frac{\frac{7}{2^2 a} \left[D_{-2}\left(\frac{2a}{\sqrt{3c^2}}\right) + \frac{2\sqrt{3}}{a} D_{-1}\left(\frac{2a}{\sqrt{3c^2}}\right) \right]}{\frac{1}{e^{\frac{1}{2}g_0^2(x)}} + \frac{1}{e^{\frac{1}{2}g_1^2(x)}}} \\
 & = \frac{\frac{2\sqrt{2}}{a} - v_0 \left[1 - \Phi(v_0) \right] + \frac{e^{-v_0^2}}{\sqrt{\pi}}}{\frac{1}{e^{\frac{1}{2}g_0^2(x)}}} + \frac{\frac{2\sqrt{2}}{a} - v_1 \left[1 - \Phi(v_1) \right] + \frac{e^{-v_1^2}}{\sqrt{\pi}}}{\frac{1}{e^{\frac{1}{2}g_1^2(x)}}} \\
 & = \frac{3^{-1} 2^3 a \sqrt{\pi} \left[\frac{2}{\sqrt{\pi}} e^{-\frac{2a^2}{3c^4}} + 2 \left(\frac{\sqrt{3}}{a} \frac{a}{\sqrt{3c^2}} \right) \left[1 - \Phi\left(\frac{\sqrt{2}a}{\sqrt{3c^2}}\right) \right] \right]}{1} \quad (11)
 \end{aligned}$$

where in Appendix A we give the proof of the following identity

$${}_1F_1\left[\frac{-1}{2}, \frac{1}{2}, -z^2\right] = e^{-z^2} + \sqrt{\pi} z \Phi(z) \quad (12)$$

As a special case when $a = 1$, then (11) simplifies even further as

$$\begin{aligned}
 f_{PCD}(x) = & 3 \frac{e^{-\frac{1}{2}g_0^2(x)} \left| \frac{e^{-v_0^2}}{\sqrt{\pi}} + (2\sqrt{2} - v_0)[1 - \Phi(v_0)] \right|}{\frac{1}{e^{\frac{1}{2}g_0^2(x)}} + \frac{1}{e^{\frac{1}{2}g_1^2(x)}}} \\
 & + \frac{e^{-\frac{1}{2}g_1^2(x)} \left| \frac{e^{-v_1^2}}{\sqrt{\pi}} + (2\sqrt{2} - v_1)[1 - \Phi(v_1)] \right|}{\frac{1}{e^{\frac{1}{2}g_0^2(x)}} + \frac{1}{e^{\frac{1}{2}g_1^2(x)}}} \\
 & = \frac{8\sqrt{\pi} \left[\frac{2}{\sqrt{\pi}} e^{-\frac{2}{3c^4}} + 2 \left(\sqrt{3} - \frac{2b}{\sqrt{3}} \right) \left[1 - \Phi\left(\frac{2}{\sqrt{6c^2}}\right) \right] \right]}{1} \quad (13)
 \end{aligned}$$

Even for this special case we have reduced the computation of the landmark GPCFD pdf to a computation of twice as many elementary functions if we compare and contrast (11) or (13) with (8) or (9). Presumably, the same process can be followed for other values of $N = 3, 4, \dots$ if there is a need to compute the landmark GPCFD pdf and there is a tighter requirement for the integration performance than the one that is obtained from employing only the reduction from the savings that we

discussed in the optimization of the landmark GPCFD pdf.

This concludes the discussion of the computation of the landmark GPCFD pdf. Next, we investigate the discussion on the optimized computation of the landmark GPCFD CDF.

3 The Landmark GPCFD CDF

In this section the optimized efficient computation of the landmark GPCFD cdf is performed. Even though in 2021 I was successful to develop for the first time the efficient computational algorithm of the GPCFD cdf the numerical results indicated that this algorithm was far from being optimized. This section contains two subsections: the derivation of the landmark GPCFD cdf and special cases of the landmark cdf.

3.1 Derivation of the Landmark GPCFD CDF

The derivation of the landmark GPCFD cdf is performed by maneuvering then taking advantage of the expansion of the series of Hermite polynomials and then of their recursive implementation [2].

The derivation of the landmark GPCFD cdf is based on the direct integration method which was discussed in great detail in Appendix B of Proгри (2021, [6]). However, for the landmark GPCFD cdf to work a slight modification of the direct integration method is required.

We recall that the direct integration is given by

$$F_{PCD}(x) \equiv F(x) = \int_{-\infty}^x [f_{PCD}(t) \equiv f(t)] dt \quad (14)$$

where $f(x)$ is given by (1).

Since the function is symmetric around $x = b$ then we have to take advantage of the symmetry and not compute the entire integral.

$$F(x) = \int_{-\infty}^x f(t) dt \equiv \begin{cases} \frac{1}{2} - \int_b^{2b-x} f(t) dt & x \leq b \\ \frac{1}{2} & x = b \\ \frac{1}{2} + \int_b^x f(t) dt & x \geq b \end{cases} \quad (15)$$

The computation of (15) is slightly different from the computation of (81) in Appendix B of Proгри (2021, [6]). The same comment can be applied for all the other formulas.

Hence, it only makes sense to compute the following integral

$$F'(x) = \int_b^x f(t) dt; \quad x \geq b > 0 \quad (16)$$

Substituting (16) into (1) and after some elementary algebra like changing the order of summation and integration we have

$$\begin{aligned}
F'(x) &= \frac{\sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2-2}} \int_b^x \left[e^{-\frac{g_0^2(t)}{2}} V(p, v_0) + e^{-\frac{g_1^2(t)}{2}} V(p, v_1) \right] dt}{\frac{3(N+1)}{2} \frac{N}{2} \frac{-a^2}{ae^{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2k_3-2}} D_{k-N} \left(\frac{2a}{\sqrt{3c^2}} \right)} \\
&= \frac{\sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2-2}} F_p'(x)}{\frac{3(N+1)}{2} \frac{N}{2} \frac{-a^2}{ae^{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2k_3-2}} D_p \left(\frac{2a}{\sqrt{3c^2}} \right)} \quad (17)
\end{aligned}$$

It remains to solve two integrals,

$$\begin{aligned}
F_{pi}' &= \int_b^x e^{-\frac{g_i^2(t)}{2}} V \left(p, \frac{c_i(t)}{\sqrt{2}} \right) dt; \quad i = \{0,1\} \\
&= \int_b^x e^{-\frac{g_i^2(t)}{2}} \frac{{}_1F_1 \left[\frac{p+1}{2} \middle| \frac{p+1}{2} \right] - \frac{c_i^2(t)}{2}}{\Gamma \left(\frac{1-p}{2} \right)} dt + \\
&\quad - \int_b^x e^{-\frac{g_i^2(t)}{2}} \frac{\frac{c_i(t)}{\sqrt{2}} {}_1F_1 \left[\frac{p+1}{2} \middle| \frac{p+1}{2} \right] - \frac{c_i^2(t)}{2}}{2^{-1} \Gamma \left(-\frac{p}{2} \right)} dt; \quad i = \{0,1\} \quad (18)
\end{aligned}$$

which then produces four integrals,

$$F_{p0i}'(x) = \frac{\int_b^x e^{-\frac{g_i^2(t)}{2}} {}_1F_1 \left[\frac{p+1}{2} \middle| \frac{p+1}{2} \right] - \frac{c_i^2(t)}{2} dt}{\Gamma \left(\frac{1-p}{2} \right)}; \quad i = \{0,1\} \quad (19)$$

$$F_{p1i}'(x) = \frac{-\sqrt{2} \int_b^x c_i(t) e^{-\frac{g_i^2(t)}{2}} {}_1F_1 \left[\frac{p+1}{2} \middle| \frac{p+1}{2} \right] - \frac{c_i^2(t)}{2} dt}{\Gamma \left(-\frac{p}{2} \right)}; \quad i = \{0,1\} \quad (20)$$

Next, before we discuss the expansion of the series of Hermite polynomials, let us perform the maneuvering of the exponential function

$$\begin{aligned}
e^{-\frac{1}{2}g_i^2(x)} &= e^{-\frac{8a^2}{3c^4} + 2\frac{4a}{\sqrt{6c^2}} \sqrt{\frac{3}{2}} c_i(x) - \frac{3}{2} c_i^2(x)} \\
&= e^{-a_2} e^{-\frac{3}{2} c_i^2(x) + 2\frac{2\sqrt{2}a}{\sqrt{3c^2}} \sqrt{\frac{3}{2}} c_i(x)} \quad (21)
\end{aligned}$$

Where

$$a_2 = 8 \times 3^{-1} a^2 c^{-4} = a_1^2 \quad (22)$$

$$a_1 = 2 \times 2^{0.5} \times 3^{-0.5} a c^{-2} \quad (23)$$

Next, we make the substitution

$$z_i(x) = \sqrt{\frac{3}{2}} c_i(x) \quad (24)$$

Hence, we have

$$e^{-\frac{1}{2}g_i^2(x)} = e^{-a_2} e^{-\frac{3}{2} c_i^2(x) + 2\frac{2\sqrt{2}a}{\sqrt{3c^2}} \sqrt{\frac{3}{2}} c_i(x)}$$

$$\begin{aligned}
&= e^{-a_2} e^{-z_i^2 + 2\frac{2\sqrt{2}a}{\sqrt{3c^2}} z_i} \\
&= e^{-a_2} e^{-z_i^2 + 2a_1 z_i} \quad (25)
\end{aligned}$$

Next, we employ the series of Hermite polynomials (Gradshteyn, Ryzhik (2007, [11]) pg. 997 8.957 1.) for the above function then we have

$$\begin{aligned}
e^{-\frac{1}{2}g_i^2(x)} &= e^{-a_2} \sum_{m=0}^{\infty} \frac{z_i^m}{m!} H_m(a_1) \\
&= e^{-a_2} \sum_{m=0}^{\infty} \frac{3^{0.5m} c_i^m(x)}{2^{0.5m} m!} H_m(a_1) \quad (26)
\end{aligned}$$

If we make the substitution

$$y_i(t) = c_i(t) = \frac{a}{c^2} - (-1)^i \frac{t-b}{2a} \quad (27)$$

$$dy_i(t) = \frac{(-1)^{i+1} dt}{2a}; \quad t \rightarrow \left|_b^x \Rightarrow y_i(t) \rightarrow \left|_{y_i(b)}^{y_i(x)} \right. \quad (28)$$

then the direct integral, $F_{p0i}'(x)$, from (19) can be solved as

$$\begin{aligned}
F_{p0i}'(x) &= \frac{\int_b^x e^{-\frac{g_i^2(t)}{2}} {}_1F_1 \left[\frac{p+1}{2} \middle| \frac{p+1}{2} \right] - v_i^2(t) dt}{\Gamma \left(\frac{1-p}{2} \right)}; \quad i = \{0,1\} \\
&= \frac{2a \sum_{m,r=0}^{\infty} \frac{H_m(a_1) \left(\frac{p+1}{2} \right)_r 3^{0.5m} (-1)^r \int_{y_i(b)}^{y_i(x)} y_i^{m+2r}(t) dy_i(t)}{\left(\frac{1}{2} \right)_r 2^{0.5m} m_2 r! m!}}{(-1)^{i+1} e^{a_2} \Gamma \left(\frac{1-p}{2} \right)} \\
&= \frac{2a \sum_{m,r=0}^{\infty} \frac{H_m(a_1) \left(\frac{p+1}{2} \right)_r (-1)^r 3^{0.5m} y_i^{m+2r+1}(t) \Big|_{y_i(b)}^{y_i(x)}}{\left(\frac{1}{2} \right)_r 2^{0.5m} m_2 r! m! (m+2r+1)}}{(-1)^{i+1} e^{a_2} \Gamma \left(\frac{1-p}{2} \right)} \\
&= \frac{2ae^{-a_2} \left| \sum_{k,r=0}^{\infty} \frac{H_{2k}(a_1) \left(\frac{p+1}{2} \right)_r 3^k (-1)^r}{\left(\frac{1}{2} \right)_r 2^k 2^r r! (2k)!} \frac{y_i^{2k+2r+1}(x)}{2(k+r+\frac{1}{2})} - \frac{y_i^{2k+2r+1}(b)}{2(k+r+\frac{1}{2})} \right.}{(-1)^{i+1} \Gamma \left(\frac{1-p}{2} \right)} \\
&\quad \left. + \sum_{k,r=0}^{\infty} \frac{H_{2k+1}(a_1) \left(\frac{p+1}{2} \right)_r 3^{k+0.5} (-1)^r}{\left(\frac{1}{2} \right)_r 2^{k+0.5} 2^r r! (2k+1)!} \frac{y_i^{2k+2r+2}(x)}{2(k+r+1)} - \frac{y_i^{2k+2r+2}(b)}{2(k+r+1)} \right|}{(-1)^{i+1} \Gamma \left(\frac{1-p}{2} \right)} \\
&= \frac{2a \left| y_i(x) K_{0ie} - y_i(b) K_{0ieb} + \frac{1}{2} \sqrt{\frac{3}{2}} \left[y_i^2(x) K_{0io} - y_i^2(b) K_{0ieb} \right] \right|}{(-1)^{i+1} e^{a_2} \Gamma \left(\frac{1-p}{2} \right)} \quad (29)
\end{aligned}$$

Where

$$z_{1i} = \frac{3y_i^2(x)}{8}; \quad z_{2i} = -\frac{y_i^2(x)}{2}; \quad (30)$$

$$z_{1b} = \frac{3y_i^2(b)}{8}; \quad z_{2b} = -\frac{y_i^2(b)}{2}; \quad (31)$$

$$K_{0ie} = \sum_{k,r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{k+r} H_{2k}(a_1) \left(\frac{p+1}{2}\right)_r z_{1i}^k z_{2i}^r}{\left(\frac{3}{2}\right)_{k+r} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_r k! r!}$$

$$= F_{1:1;1}^{1:H_e;1} \left[\begin{matrix} 1: (a_1), \frac{p+1}{2}; \\ 2: \frac{3}{2}, \frac{1}{2}; \end{matrix} ; z_{1i}, z_{2i} \right] \quad (32)$$

$$K_{0io} = \sum_{k,r=0}^{\infty} \frac{(1)_{k+r} H_{2k+1}(a_1) \left(\frac{p+1}{2}\right)_r z_{1i}^k z_{2i}^r}{(2)_{k+r} \left(\frac{3}{2}\right)_k \left(\frac{1}{2}\right)_r k! r!}$$

$$= F_{1:1;1}^{1:H_o;1} \left[\begin{matrix} 1: (a_1), \frac{p+1}{2}; \\ 2: \frac{3}{2}, \frac{1}{2}; \end{matrix} ; z_{1i}, z_{2i} \right] \quad (33)$$

$$K_{0ieb} = \sum_{k,r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{k+r} H_{2k}(a_1) \left(\frac{p+1}{2}\right)_r z_{1b}^k z_{2b}^r}{\left(\frac{3}{2}\right)_{k+r} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_r k! r!}$$

$$= F_{1:1;1}^{1:H_e;1} \left[\begin{matrix} 1: (a_1), \frac{p+1}{2}; \\ 2: \frac{3}{2}, \frac{1}{2}; \end{matrix} ; z_{1b}, z_{2b} \right] \quad (34)$$

$$K_{0iob} = \sum_{k,r=0}^{\infty} \frac{(1)_{k+r} H_{2k+1}(a_1) \left(\frac{p+1}{2}\right)_r z_{1b}^k z_{2b}^r}{(2)_{k+r} \left(\frac{3}{2}\right)_k \left(\frac{1}{2}\right)_r k! r!}$$

$$= F_{1:1;1}^{1:H_o;1} \left[\begin{matrix} 1: (a_1), \frac{p+1}{2}; \\ 2: \frac{3}{2}, \frac{1}{2}; \end{matrix} ; z_{1b}, z_{2b} \right] \quad (35)$$

For $i = 0$ we obtain $F_{p00}'(x)^v$ from (29) as follows

$$F_{p00}'(x) = \frac{2[y_0(b)K_{00eb} - y_0(x)K_{00e}] + \sqrt{\frac{3}{2}}[y_0^2(b)K_{00eb} - y_0^2(x)K_{00e}]}{a^{-1}e^{a_2}\Gamma\left(\frac{1-p}{2}\right)} \quad (36)$$

From (36) the landmark computation of $F_{p00}'(x)$ requires the optimized computation of four modified generalized Kampé de Fériet functions (or double hypergeometric series) (32)-(35) [7]-[10]. Now, since these functions are simultaneously computed in pairs, (30) requires the optimized computation of two pairs of the modified generalized Kampé de Fériet functions; one pair is (32) with (33) and the other pair is (34) with (35).

Likewise, for $i = 1$ we obtain $F_{p01}'(x)^{vi}$ from (29) as follows

$$F_{p01}'(x) = \frac{2[y_1(x)K_{01e} - y_1(b)K_{01eb}] + \sqrt{\frac{3}{2}}[y_1^2(x)K_{01e} - y_1^2(b)K_{01eb}]}{a^{-1}e^{a_2}\Gamma\left(\frac{1-p}{2}\right)} \quad (37)$$

However, since from (27) $y_0(b)$ is identical to $y_1(b)$

$$y_0(b) = c_0(t) = \frac{a}{c^2} \equiv c_1(t) = \frac{a}{c^2} = y_1(b) \quad (38)$$

then the landmark computation of $F_{01}'(x)$ requires the optimized computation of just one pair modified generalized Kampé de Fériet functions (or double hypergeometric series)

[7]-[10] (32) with (33).

Similarly, we solve the function, $F_{p1i}'(x)$, from (20) as follows

$$F_{p1i}'(x) = \frac{-2 \int_b^x v_i(t) e^{-\frac{g_i^2(t)}{2}} {}_1F_1\left[\begin{matrix} \frac{p+1}{2} \\ \frac{3}{2} \end{matrix}; -v_i^2(t) \right] dt}{\Gamma\left(-\frac{p}{2}\right)}$$

$$= \frac{-\sqrt{2} \sum_{m,r=0}^{\infty} \frac{H_m(x_1) \left(\frac{p+1}{2}\right)_r {}_3F_2\left[\begin{matrix} 0.5m, (-1)r \\ 2.05m, 2rm!r! \end{matrix}; \int_b^x y_i^{m+1}(t) y_i^{2r}(t) dt \right]}{\left(\frac{3}{2}\right)_r}}{e^{a_2} \Gamma\left(-\frac{p}{2}\right)}$$

$$= \frac{2a\sqrt{2} \sum_{m,r=0}^{\infty} \frac{H_m(a_1) \left(\frac{p+1}{2}\right)_r {}_3F_2\left[\begin{matrix} 0.5m, y_i^{(x)}(b) \\ (-1)-r, 2.05m, 2rm!r! \end{matrix}; y_i^{m+2r+1}(t) dy_i(t) \right]}{\left(\frac{3}{2}\right)_r 2^r}}{(-1)^i e^{a_2} \Gamma\left(-\frac{p}{2}\right)}$$

$$= \frac{2a\sqrt{2} \sum_{m,r=0}^{\infty} \frac{H_m(a_1) \left(\frac{p+1}{2}\right)_r {}_3F_2\left[\begin{matrix} 0.5m, (-1)r \\ 2.05m, 2rm!r! \end{matrix}; y_i^{m+2r+2}(t) y_i^{(b)} \right]}{\left(\frac{3}{2}\right)_r 2^{0.5m} 2^r m!r!}}{(-1)^i e^{a_2} \Gamma\left(-\frac{p}{2}\right)}$$

$$= \frac{\frac{y_i^2(x)K_{1ie} - y_i^2(b)K_{1ieb} + y_i^3(x)K_{1io} - y_i^3(b)K_{1iob}}{2}}{2^{-1}a^{-1}2^{-0.5}e^{a_2}(-1)^i \Gamma\left(-\frac{p}{2}\right)}$$

$$= \frac{2a\sqrt{2} \left[\frac{y_i^2(x)K_{1ie} - y_i^2(b)K_{1ieb} + y_i^3(x)K_{1io} - y_i^3(b)K_{1iob}}{2} \right]}{e^{a_2}(-1)^i \Gamma\left(-\frac{p}{2}\right)} \quad (39)$$

$$K_{1ie} = \sum_{k,r=0}^{\infty} \frac{(1)_{k+r} H_{2k}(a_1) \left(\frac{p+1}{2}\right)_r z_{1i}^k z_{2i}^r}{(2)_{k+r} \left(\frac{1}{2}\right)_k \left(\frac{3}{2}\right)_r k! r!}$$

$$= F_{1:1;1}^{1:H_e;1} \left[\begin{matrix} 1: (a_1), \frac{p}{2} + 1; \\ 2: \frac{1}{2}, \frac{3}{2}; \end{matrix} ; z_{1i}, z_{2i} \right] \quad (40)$$

$$K_{1io} = \sum_{k,r=0}^{\infty} \frac{\left(\frac{3}{2}\right)_{k+r} H_{2k+1}(a_1) \left(\frac{p+1}{2}\right)_r z_{1i}^k z_{2i}^r}{\left(\frac{5}{2}\right)_{k+r} \left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_r k! r!}$$

$$= F_{1:1;1}^{1:H_o;1} \left[\begin{matrix} \frac{3}{2}: (a_1), \frac{p}{2} + 1; \\ 5: \frac{1}{2}, \frac{3}{2}, \frac{3}{2}; \end{matrix} ; z_{1i}, z_{2i} \right] \quad (41)$$

$$K_{1ieb} = \sum_{k,r=0}^{\infty} \frac{(1)_{k+r} H_{2k}(a_1) \left(\frac{p+1}{2}\right)_r z_{1b}^k z_{2b}^r}{(2)_{k+r} \left(\frac{1}{2}\right)_k \left(\frac{3}{2}\right)_r k! r!}$$

$$= F_{1:1;1}^{1:H_e;1} \left[\begin{matrix} 1: (a_1), \frac{p}{2} + 1; \\ 2: \frac{1}{2}, \frac{3}{2}; \end{matrix} ; z_{1b}, z_{2b} \right] \quad (42)$$

$$K_{1iob} = \sum_{k,r=0}^{\infty} \frac{\left(\frac{3}{2}\right)_{k+r} H_{2k+1}(a_1) \left(\frac{p+1}{2}\right)_r z_{1b}^k z_{2b}^r}{\left(\frac{5}{2}\right)_{k+r} \left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_r k! r!}$$

$$= F_{1:1;1}^{1:H_o;1} \left[\begin{matrix} \frac{3}{2}: (a_1), \frac{p}{2} + 1; \\ 5: \frac{1}{2}, \frac{3}{2}, \frac{3}{2}; \end{matrix} ; z_{1b}, z_{2b} \right] \quad (43)$$

For $i = 0$ we obtain $F_{p10}'(x)$ from (39) as follows

$$F_{p10}'(x) = \frac{\frac{y_0^2(x)K_{10e}-y_i^2(b)K_{10eb} + y_0^3(x)K_{10o}-y_i^3(b)K_{10ob}}{2} \sqrt[3]{6}}{2^{-1}a^{-1}2^{-0.5}e^{a_2}\Gamma(-\frac{p}{2})} \quad \text{vii} \quad (44)$$

The landmark computation of (44) can be employed in exactly (or identically) the same manner as in (29).

Likewise, for $i = 1$ we obtain $F_{p11}'(x)$ from (39) as follows

$$F_{p11}'(x) = -\frac{\frac{y_1^2(x)K_{11e}-y_i^2(b)K_{11eb} + y_1^3(x)K_{11o}-y_i^3(b)K_{11ob}}{2} \sqrt[3]{6}}{2^{-1}a^{-1}2^{-0.5}e^{a_2}\Gamma(-\frac{p}{2})} \quad \text{viii} \quad (45)$$

The optimized computation of (45) can be employed in exactly (or identically) the same manner as in (37).

Combining (36) with (44) and (37) with (45) into (19) and (20) we obtain

$$F_{p0}'(x) = F_{p00}''(x) + F_{p10}''(x) \quad x \geq b \quad (46)$$

$$F_{p1}'(x) = F_{p01}''(x) + F_{p11}''(x) \quad x \geq b \quad (47)$$

Substituting (46) and (47) into (18) produces,

$$F_p'(x) = F_{p0}'(x) + F_{p1}'(x) \\ = \sum_{i=0}^1 F_{p0i}''(x) + F_{p1i}''(x); \quad x \geq b \quad (48)$$

Next, substituting (48) into (17) we obtain the desired function

$$F'(x) = \frac{\sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2} \frac{k}{2}} F_p'(x)}{2^{\frac{3(N+1)}{2}} 3^{-\frac{N}{2}} a e^{\frac{-a^2}{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^{k_2 k_3} \frac{k}{2}} D_p\left(\frac{2a}{\sqrt{3c^2}}\right)} \quad (49)$$

It is particularly important to mention here that in (38) the function $F_p'(x)$ requires the computation of six pairs of the modified generalized Kampé de Fériet function (or double hypergeometric series) [7]-[10]; i.e., it is computationally optimized in contrast to (or significantly and computationally more efficient than) (151) in Progridi (2021, [6]) because the computational complexity of $F'(x)$ is equal to $\mathcal{O}[F'(x)] = N \times \mathcal{O}[F_p'(x)]$ and because we have reduced the computational complexity of $\mathcal{O}[F_p'(x)] = \mathcal{O}(8M^3 + 8M^2)$ for (151) in Progridi (2021, [6]) to $\mathcal{O}[F_p'(x)] = \mathcal{O}[6M^2]$.

Hence, Dr. Progridi has successfully produced an outstanding landmark solution that is considered optimized for speed and memory locations because he was able to reduce the computational complexity from $\mathcal{O}(8M^3 + 8M^2)$ to $\mathcal{O}[6M^2]$.

3.2 Special Cases of the Landmark GPCFD CDF

The direct integration a few special cases of the landmark GPCFD cdf is performed here.

The first special case corresponds to (7). For this special case

we have

$$F_{p0i}''(x) = \frac{y_i(x)K_{0ie}-y_i(b)K_{0ieb} + \frac{1}{2}\sqrt[3]{\frac{y_i^2(x)K_{0io}-y_i^2(b)K_{0iob}}{2}}}{2^{-1}a^{-1}e^{a_2}(-1)^{i+1}} \quad (50)$$

where

$$K_{0ie} = \sum_{k=0}^{\infty} \frac{H_{2k}(a_1) \frac{z_{1i}^k}{(\frac{3}{2})_k}}{k!} \\ = M\left[H_e(a_1); \frac{3}{2}; z_{1i}\right] \quad (51)$$

$$K_{0io} = \sum_{k=0}^{\infty} \frac{(1)_k H_{2k+1}(a_1) \frac{z_{1i}^k}{(\frac{3}{2})_k}}{(2)_k (\frac{3}{2})_k k!} \\ = M\left[1, H_o(a_1); 2, \frac{3}{2}; z_{1i}\right] \quad (52)$$

$$K_{0ieb} = \sum_{k=0}^{\infty} \frac{H_{2k}(a_1) \frac{z_{1b}^k}{(\frac{3}{2})_k}}{k!} \\ = M\left[H_e(a_1); \frac{3}{2}; z_{1b}\right] \quad (53)$$

$$K_{0iob} = \sum_{k=0}^{\infty} \frac{(1)_k H_{2k+1}(a_1) \frac{z_{1b}^k}{(\frac{3}{2})_k}}{(2)_k (\frac{3}{2})_k k!} \\ = M\left[1, H_o(a_1); 2, \frac{3}{2}; z_{1b}\right] \quad (54)$$

From (50) we observe that the optimized computation of $F_{p0i}''(x)$ requires the optimized computation of two pairs: one pair (51) with (52) and the other pair (53) with (54) for $F_{p00}''(x)$ and one pair (51) with (52) for $F_{p01}''(x)$ of the modified generalized confluent hypergeometric functions. This produces tremendous savings in computation.

On the other hand, $F_{p1i}''(x)$, can be computed from

$$F_{p1i}''(x) = \frac{2a\sqrt{2} \left| \frac{y_i^2(x)K_{1ie}-y_i^2(b)K_{1ieb} + y_i^3(x)K_{1io}-y_i^3(b)K_{1iob}}{2} \sqrt[3]{6} \right|}{e^{a_2}(-1)^i \Gamma(\frac{1}{2})} \quad (55)$$

where

$$K_{1ie} = F_{1:1;1}^{1:H_e;1} \left[\begin{matrix} 1: (a_1), \frac{1}{2}; \\ 2: \frac{1}{2}, \frac{3}{2}; \end{matrix} ; z_{1i}, z_{2i} \right] \quad (56)$$

$$K_{1io} = F_{1:1;1}^{1:H_o;1} \left[\begin{matrix} \frac{3}{2}: (a_1), \frac{1}{2}; \\ \frac{5}{2}: \frac{3}{2}, \frac{3}{2}; \end{matrix} ; z_{1i}, z_{2i} \right] \quad (57)$$

$$K_{1ieb} = F_{1:1;1}^{1:H_e;1} \left[\begin{matrix} 1: (a_1), \frac{1}{2}; \\ 2: \frac{1}{2}, \frac{3}{2}; \end{matrix} ; z_{1b}, z_{2b} \right] \quad (58)$$

$$K_{1iob} = F_{1:1;1}^{1:H_o;1} \left[\begin{matrix} \frac{3}{2} : (a_1), \frac{1}{2}; \\ \frac{5}{2} : \frac{3}{2}, \frac{3}{2}; \end{matrix} z_{1b}, z_{2b} \right] \quad (59)$$

The optimized computation of (55) requires only three pairs of the modified generalized Kampé de Fériet functions (or double hypergeometric series): two pairs come from twice (56) with (57) and one pair comes from (58) with (59) [7]-[10]. Therefore, the computational complexity of $F_{-1}'(x)$ will be $\mathcal{O}(3M^2 + 3M)$; i.e., fifty percent faster than for $N > 2$.

Similarly, the second special case corresponds to (10). For this special case we have

$$F_{poi}''(x) = \frac{y_i(x)K_{0ie}-y_i(b)K_{0ieb}+\frac{1}{2}\sqrt{\frac{3}{2}}[y_i^2(x)K_{0io}-y_i^2(b)K_{0iob}]}{2^{-1}a^{-1}(-1)^{i+1}e^{a_2}\Gamma(\frac{3}{2})} \quad (60)$$

where

$$K_{0ie} = F_{1:1;1}^{1:H_e;1} \left[\begin{matrix} \frac{1}{2} : (a_1), \frac{-1}{2}; \\ \frac{3}{2} : \frac{1}{2}, \frac{1}{2}; \end{matrix} z_{1i}, z_{2i} \right] \quad (61)$$

$$K_{0io} = F_{1:1;1}^{1:H_o;1} \left[\begin{matrix} 1 : (a_1), \frac{-1}{2}; \\ 2 : \frac{3}{2}, \frac{1}{2}; \end{matrix} z_{1i}, z_{2i} \right] \quad (62)$$

$$K_{0ieb} = F_{1:1;1}^{1:H_e;1} \left[\begin{matrix} \frac{1}{2} : (a_1), \frac{-1}{2}; \\ \frac{3}{2} : \frac{1}{2}, \frac{1}{2}; \end{matrix} z_{1b}, z_{2b} \right] \quad (63)$$

$$K_{0iob} = F_{1:1;1}^{1:H_o;1} \left[\begin{matrix} 1 : (a_1), \frac{-1}{2}; \\ 2 : \frac{3}{2}, \frac{1}{2}; \end{matrix} z_{1b}, z_{2b} \right] \quad (64)$$

The landmark computation of (60) requires only three pairs of the modified generalized Kampé de Fériet functions (or double hypergeometric series) [7]-[10]: two pairs come from twice (61) with (62) and the other pair comes from (63) and (64).

On the other hand, $F_{p1i}''(x)$, can be computed from

$$F_{p1i}''(x) = \frac{\frac{y_i^2(x)K_{0ie}-y_i^2(b)K_{0ieb}+y_i^3(x)K_{0io}-y_i^3(b)K_{0iob}}{2}}{2^{-1}a^{-1}2^{-0.5}e^{a_2}(-1)^i} \quad (65)$$

where

$$\begin{aligned} K_{0ie} &= \sum_{k=0}^{\infty} \frac{(1)_k H_{2k}(a_1) z_{1i}^k}{(2)_k (\frac{1}{2})_k k!} \\ &= M \left[1, H_e(a_1); 2, \frac{1}{2}; z_{1i} \right] \end{aligned} \quad (66)$$

$$K_{0io} = \sum_{k=0}^{\infty} \frac{H_{2k+1}(a_1) z_{1i}^k}{(\frac{5}{2})_k k!}$$

$$= M \left[H_o(a_1); \frac{5}{2}; z_{1i} \right] \quad (67)$$

$$\begin{aligned} K_{0ieb} &= \sum_{k=0}^{\infty} \frac{(1)_k H_{2k}(a_1) z_{1b}^k}{(2)_k (\frac{1}{2})_k k!} \\ &= M \left[1, H_e(a_1); 2, \frac{1}{2}; z_{1b} \right] \end{aligned} \quad (68)$$

$$\begin{aligned} K_{0iob} &= \sum_{k=0}^{\infty} \frac{H_{2k+1}(a_1) z_{1b}^k}{(\frac{5}{2})_k k!} \\ &= M \left[H_o(a_1); \frac{5}{2}; z_{1b} \right] \end{aligned} \quad (69)$$

From (65) we observe that the landmark computation of $F_{p1i}''(x)$ requires the optimized computation of two pairs for $F_{p10}''(x)$ (66) with (67) and one pair for $F_{p11}''(x)$ (68) and (69) of the modified generalized confluent hypergeometric functions. This simplification delivers tremendous savings in computation. Therefore, the computational complexity of $F_{-2}'(x)$ will be $\mathcal{O}(3M^2 + 3M)$; i.e., fifty percent faster than for $N > 2$.

For these two special cases we can achieve more than fifty percent savings in computational complexity.

This concludes the discussion on the special cases of the optimized GPCDF cdf. Next, we discuss in great detail a few numerical examples.

4 Numerical, Theoretical Results

In this section we compare and contrast the performance of the landmark GPCDF cdf with the efficient GPCDF cdf discussed in Progrid (2021, [6]). Since, we have made significant improvements to both algorithms this section validates the claims already discussed in Sects. 2, 3 and Appendix B.

4.1 Examples

For the numerical examples we employ exactly the same setup as the one in Numerical Examples Sect. Progrid (2021, [6]).

We ran the simulations on a Dell computer Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz using MATLAB 2020b [17] using a vector of $-5 \leq x \leq 5$ with a sample increment of $\Delta x = 0.05$ that leads to 200 points. Notice, that in this paper we increased the sample increment and have reduced the number of points so as to reduce the computation time but not the accuracy. Table 1 presents the computational duration times in either seconds or

TABLE I: THE QUANTITATIVE COMPUTATIONAL PERFORMANCE

GPCDF cdf:					
CFE 1 Efficient (151) vs. CFE 2 Landmark (49)					
$a = 1, b = 1, N = 1, \sum(\cdot) = 70$ terms					
Integral	L. Ap.	CFE 1	CFE 2	Total	Op.
(ms)	(ms)	(sec)	(sec)	(sec)	
58.5	0.6	385	23.7	408.8	1
53.4	0.6	355	22.7	378	2
56.6	0.4	166	21.8	1.88	3
54.0	0.4	6.7	0.1	6.8	4
$a = 2, b = 1, N = 1, \sum(\cdot) = 50$ terms					
Integral	L. Ap.	CFE 1	CFE 2	Total	Op.
(ms)	(ms)	(sec)	(sec)	(sec)	
53.6	0.5	144.25	16.53	160.85	1
54.3	0.4	132.82	15.84	148.72	2
56.8	0.4	63.34	15.45	71.86	3
63.2	0.4	3.63	0.074	3.77	4
$a = 2, b = 1, N = 1, \sum(\cdot) = 35$ terms					
Integral	L. Ap.	CFE 1	CFE 2	Total	Op.
(ms)	(ms)	(sec)	(ms)	(sec)	
53.8	0.4	1.83	41	1.92	4
$a = 1, b = 0.5, N = 1, \sum(\cdot) = 70$ terms					
Integral	L. Ap.	CFE 1	CFE 2	Total	Op.
(ms)	(ms)	(sec)	(ms)	(sec)	
48.5	0.5	6.15	104.6	6.29	4
$a = 2, b = 1, N = 2, \sum(\cdot) = 40$ terms					
Integral	L. Ap.	CFE 1	CFE 2	Total	Op.
(ms)	(ms)	(sec)	(ms)	(sec)	
91.3	0.7	4.37	138.4	4.6	4

milliseconds of the cdf 1 or the integral, cdf 2 or linear approximation, cdf 3 via Efficient (151) of Progrid (2021, [6]), and cdf 4 of the Landmark (49).

The first four rows in Tab. 1 correspond to the computation times for values of $a = 1$; $b = 1$; $N = 1$ for seventy terms for each option that is used for the computation of the Kampé de Fériet functions (or double hypergeometric series) [7]-[10] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progrid (2019, [10])). Based on the data shown in these four rows we clearly notice a significant reduction of the computation time as the option order increases from one to four with one being the worst option and four being the best option. Detailed discussion on the nature of these options and of the improvements of the algorithms for each option will be presented in a separate journal paper. We also

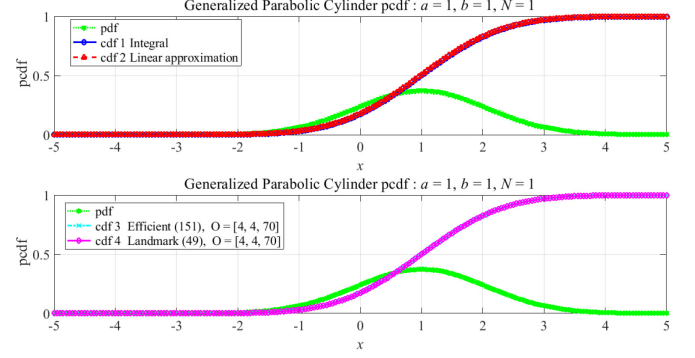


Fig. 1. The computation of the GPCDF pdf and cdf for $-5 \leq x \leq 5$, $a = 1, b = 1, N = 1$.

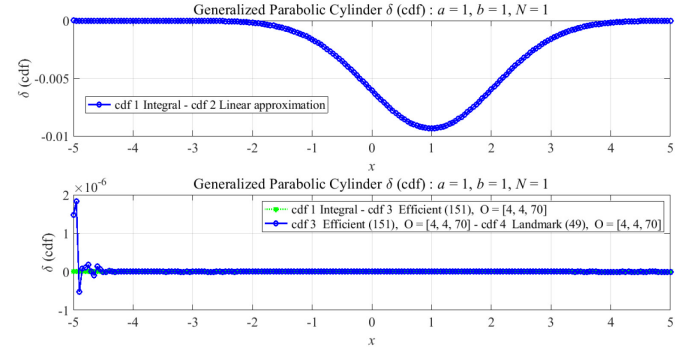


Fig. 2. The absolute error the GPCDF cdf for $-5 \leq x \leq 5$, $a = 1, b = 1, N = 1$.

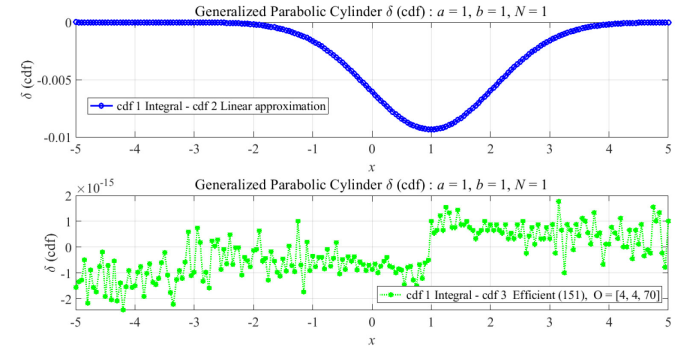


Fig. 3. The absolute error the GPCDF cdf for $-5 \leq x \leq 5$, $a = 1, b = 1, N = 1$.

observe a significant reduction of the computation times of the Landmark (49) vs. the Efficient (151) of Progrid (2021, [6]), which is because the Landmark (49) employs a significantly simpler expansion and computation than the efficient (151) of Progrid (2021, [6]).

The rest of Tab. 1 is really self-explanatory as we ran and recorded the computation times for various values of $a = \{2, 1, 1, 2\}$; $b = \{1, 0.5, 1, 1\}$; $N = \{1, 1, 2, 2\}$.

The reader might ask the following question Why are the integral computations reduced so much in this paper vs. the

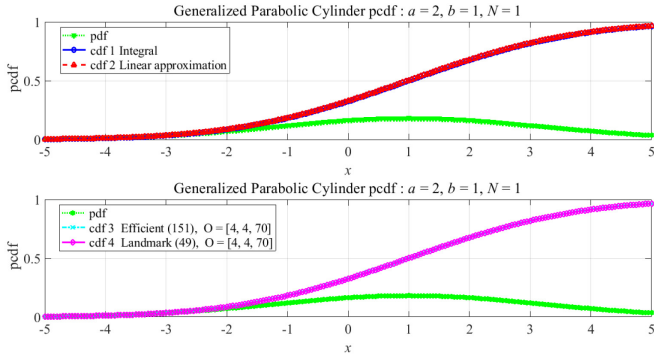


Fig. 4. The computation of the GPCFD pdf and cdf for $-5 \leq x \leq 5$, $a = 2$, $b = 1$, $N = 1$.

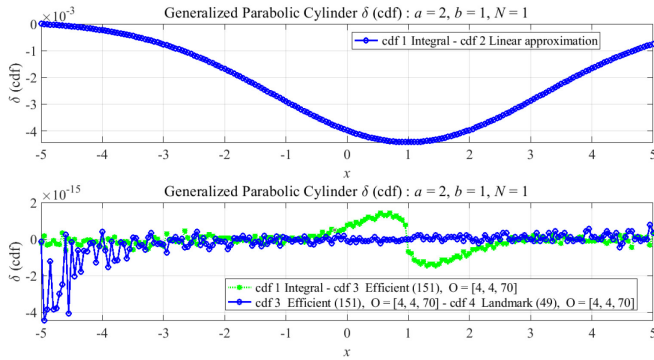


Fig. 5. The absolute error the GPCFD cdf for $-5 \leq x \leq 5$, $a = 2$, $b = 1$, $N = 1$.

same shown in Tab. 1 of Progrid (2021, [6]). This is because here we have employed the Landmark GPCFD pdf (8) and (11) that have reduced the computation times of the integral by fifty times.

The computation of the Landmark GPCFD cdf via (49) is just as efficient the computation of the Landmark integral using the Landmark GPCFD pdf (8) and (11) which is the fastest because it is the integration of elementary functions in the order of sub seconds.

The Landmark (optimized algorithm which is op. 4) has superior performance than any other algorithm. The Efficient algorithm (151) (op. 4) has superior performance than the Landmark (49) (op. 1-3) whose details will be presented in a separate journal paper.

The Efficient (151) Progrid (2021, [6]) op. 4 and the Landmark (49) op. 4 are the two fastest and the most efficient algorithms for the computation of the GPCFD cdf in closed form expression.

Figure 1 presents the GPCFD pdf and four GPCFD cdfs cdf 1 corresponds to the integral, cdf 2 to linear approximation (top), cdf 3 to a closed form expression via (151) Progrid (2021, [6]) op. 4 using seventy terms and cdf 4 Landmark (49) using

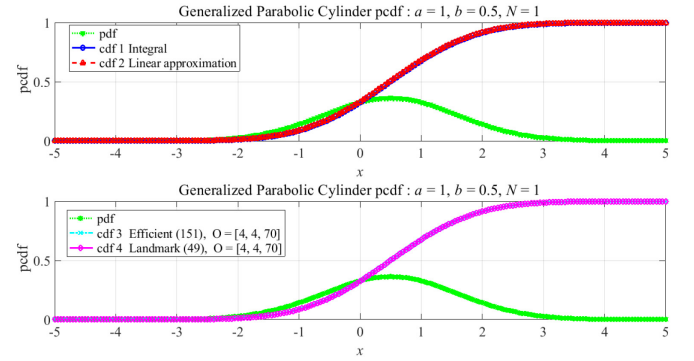


Fig. 6. The computation of the GPCFD pdf and cdf for $-5 \leq x \leq 5$, $a = 1$, $b = 0.5$, $N = 1$.

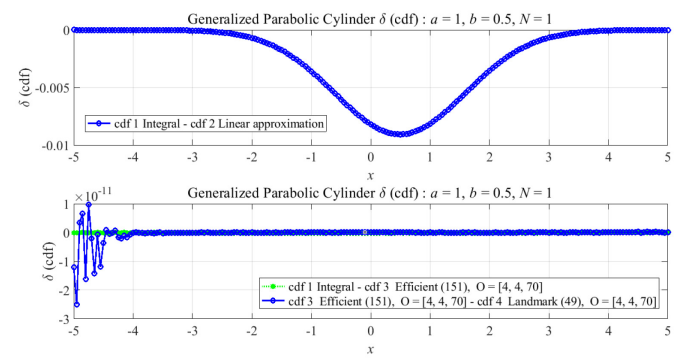


Fig. 7. The absolute error the GPCFD cdf for $-5 \leq x \leq 5$, $a = 1$, $b = 0.5$, $N = 1$.

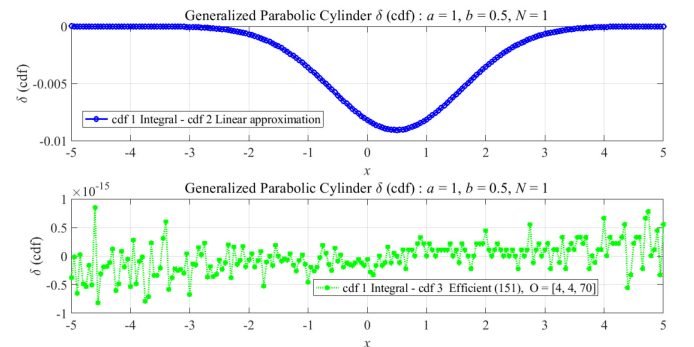


Fig. 8. Same as Fig. 4; however, the absolute error of cdf 1 integral – cdf 3 (151) for seventy terms is shown.

the same number of terms (bottom). In Fig. 2 the absolute error of the cdf 1 with cdf 2 is shown (top); and cdf 1 with cdf 3 and cdf 3 with cdf 4 (bottom). The results of Fig. 3 are exactly the same as those of Fig. 2 (top); however, the absolute error of cdf 1 with cdf 3 is only shown (bottom).

Figures 4 and 5 present the same result as Figs. 1 and 2 but for values of $a = 2$; $b = 1$; $N = 1$.

The results of Figs. 6 through 10 are self-explanatory. We have also removed a bias in the computation of the Efficient GPCFD cdf for values of $N \geq 2$.

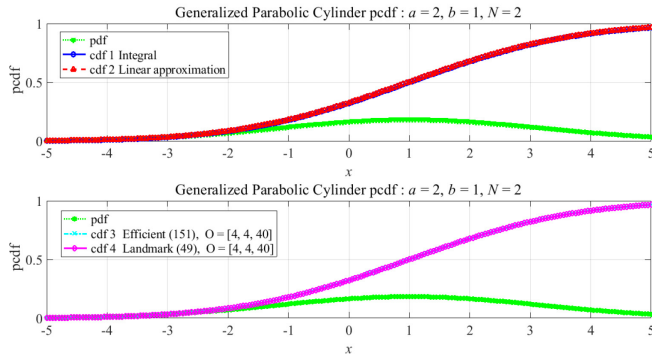


Fig. 9. The computation of the GPCFD pdf and cdf for $-5 \leq x \leq 5$, $a = 2$, $b = 1$, $N = 2$.

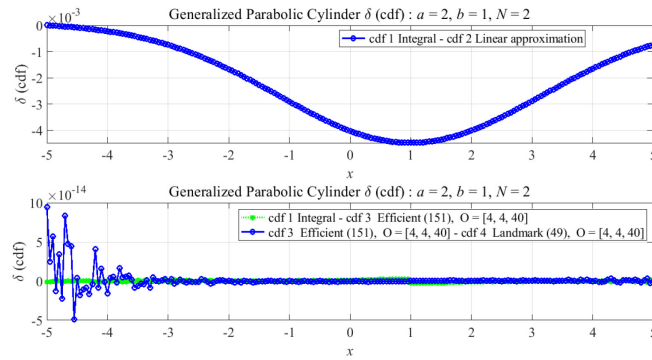


Fig. 10. The absolute error the GPCFD cdf for $-5 \leq x \leq 5$, $a = 2$, $b = 1$, $N = 2$.

This concluded the discussion on examples and numerical, theoretical results.

5 Conclusions

From the title and the content of the paper, Dr. Progri has produced more than a *masterpiece* he has produced an *iconic* derivation of the closed form expression of the GPCFD by means of the orthogonal polynomials such as Hermite polynomials and has made use of all the tools in the toolbox to improve the computational times by two to three and a half orders of magnitude.

The Landmark GPCFD cdf in op. 4 is the fastest implementation of the integration of the landmark GPCFD pdf.

Simultaneously, Dr. Progri has also made extraordinary improvements of the optimization of the efficient algorithm presented in Progri (2021, [6]) whose details will be published in a separate journal paper.

The Landmark GPCFD cdf in op. 4 is fifty to sixty times faster than the Efficient GPCFD cdf in op. 4 which is fifty to sixty times faster than the Efficient GPCFD cdf in op. 1; i.e.,

the best option (Landmark GPCFD cdf in op. 4) is anywhere from twenty-five hundred to thirty-six hundred times faster than the worst option (Efficient GPCFD cdf in op. 1).

6 Acknowledgement

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I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [17] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications. Vol. I* (Not yet available in print) [15].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless, to say I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [24] reminds me of my grandfather who I loved him very much.

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8 Appendix A: The Derivation of (12)

In this section we derive the complete steps for (12).

From the definition of the confluent hypergeometric function we have

$${}_1F_1 \left[\begin{matrix} -\frac{1}{2} \\ \frac{1}{2} \end{matrix} \middle| -z^2 \right] = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (-z^2)^n}{\left(\frac{1}{2}\right)_n n!} \quad (70)$$

Next, the Pochhammer symbol, $(-1/2)_n$, or the rising factorial can be written as

$$\left(-\frac{1}{2}\right)_n = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(-\frac{1}{2})} \left[1 = \frac{\left(n-\frac{1}{2}\right)}{\left(n-\frac{1}{2}\right)}\right] = \left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} = \left(\frac{1}{2}\right)_n\right] \frac{\Gamma(\frac{1}{2})}{\left(n-\frac{1}{2}\right)\Gamma(-\frac{1}{2})} \quad (71)$$

Next, employing the identity of (71) into (70) yields,

$$\sum_{n=0}^{\infty} \frac{\left(n-\frac{1}{2}\right)\Gamma(-\frac{1}{2})\left(\frac{1}{2}\right)_n (-z^2)^n}{\Gamma(\frac{1}{2})\left(\frac{1}{2}\right)_n n!} = \sum_{n=0}^{\infty} \frac{(-z^2)^n}{n!} = e^{-z^2} \quad (72)$$

Next, the left-hand side of (72) can be written as,

$$\begin{aligned} -2 \sum_{n=0}^{\infty} \frac{\left(n-\frac{1}{2}\right)\left(\frac{1}{2}\right)_n (-z^2)^n}{\left(\frac{1}{2}\right)_n n!} &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (-z^2)^n}{\left(\frac{1}{2}\right)_n n!} \\ &- 2 \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (-z^2)^n}{\left(\frac{1}{2}\right)_n (n-1)!} \end{aligned} \quad (73)$$

Next, it remains to evaluate the series

$$\begin{aligned} -2 \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (-z^2)^n}{\left(\frac{1}{2}\right)_n (n-1)!} &= 2z^2 \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{n+1} (-z^2)^n}{\left(\frac{1}{2}\right)_{n+1} n!} \\ &= -2\sqrt{\pi}z \frac{z}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (-z^2)^n}{\left(\frac{z}{2}\right)_n n!} \\ &= -\sqrt{\pi}z\Phi(z) \end{aligned} \quad (74)$$

Substituting (74) into (73) and then into (72) yields

$${}_1F_1 \left[\begin{matrix} -\frac{1}{2} \\ \frac{1}{2} \end{matrix} \middle| -z^2 \right] - \sqrt{\pi}z\Phi(z) = e^{-z^2} \quad (75)$$

Or the other equivalent expression

$${}_1F_1 \left[\begin{matrix} -\frac{1}{2} \\ \frac{1}{2} \end{matrix} \middle| -z^2 \right] = e^{-z^2} + \sqrt{\pi}z\Phi(z) \quad (76)$$

This concludes the derivations of Appendix A.

9 Appendix B: Computation of Special Cases of the Efficient GPCFD CDF

In this appendix we revisit the computation of the special cases of the efficient GPCFD cdf. First let us review the efficient computation of the GPCFD cdf as given by (151) and (152) and then (81) in Proгри (2021, [6]).

We rewrite (152) of Proгри (2021, [6]) as follows:

$$F_{PCD}'(x) = \frac{\sum_{k=0}^{N-1} \frac{H_k(N)n!}{k} F_p'(x)}{a^{k_2} \frac{2}{3} \frac{3(N+1)}{2} \frac{N}{2} \frac{-a^2}{3c^4} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{k} D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)} \quad (77)$$

Where

$$F_p'(x) = \begin{cases} F_{p00}'' + F_{p10}'' + \mathcal{F}_{p00}(x) + \mathcal{F}_{p10}(x) \\ + F_{p01}'' + F_{p11}'' - \mathcal{F}_{p01}(x) - \mathcal{F}_{p11}(x); \end{cases} x \geq b \quad (78)$$

$$F_{p0i}'' = \frac{\frac{2a\sqrt{\pi}}{\sqrt{6}} K_{p0i1}'' - (-1)^i a_2 c^2 K_{p0i2}''}{e^{a_2 \Gamma\left(\frac{1-p}{2}\right)}} \quad (79)$$

$$K_{p0i1}'' = F_{0:1;1}^{1:1;0} \left[\begin{matrix} \frac{1}{2}; \frac{p+1}{2}; -; \\ -; \frac{1}{2}; \frac{1}{2}; \end{matrix} a_1, a_2 \right] \quad (80)$$

$$K_{p0i2}'' = F_{0:1;1}^{1:1;0} \left[\begin{matrix} 1; \frac{p+1}{2}; -; \\ -; \frac{1}{2}; \frac{3}{2}; \end{matrix} a_1, a_2 \right] \quad (81)$$

$$a_2 = 8 \times 3^{-1} a^2 c^{-4} = a_1^2 \quad (82)$$

$$a_1 = 2 \times 2^{0.5} \times 3^{-0.5} a c^{-2} \quad (83)$$

$$F_{p1i}'' = \frac{\frac{4a}{3\sqrt{2}} K_{p0i1}'' + \frac{(-1)^i a_2 c^2 \sqrt{\pi}}{\sqrt{3}} K_{p1i2}''}{e^{a_2 \Gamma\left(\frac{1-p}{2}\right)}} \quad (84)$$

$$K_{p1i1}'' = F_{0:1;1}^{1:1;0} \left[\begin{matrix} 1; \frac{p}{2} + 1; -; \\ -; \frac{3}{2}; \frac{1}{2}; \end{matrix} a_1, a_2 \right] \quad (85)$$

$$K_{p1i2}'' = F_{0:1;1}^{1:1;0} \left[\begin{matrix} \frac{3}{2}; \frac{p}{2} + 1; -; \\ -; \frac{3}{2}; \frac{3}{2}; \end{matrix} a_1, a_2 \right] \quad (86)$$

$$\mathcal{F}_{p00}(x) = \frac{2ay_0 S_{p001}(x) + x_2 c^2 S_{p002}(x)}{e^{a_2 + x_3 \Gamma\left(\frac{1-p}{2}\right)}} \quad (87)$$

$$S_{p001}(x) = F^{(3)} \left[\begin{matrix} -:: \frac{1}{2}; -; -; \frac{p+1}{2}; -; 1; \\ \frac{3}{2} :: -; -; -; \frac{1}{2}; \frac{1}{2}; -; \end{matrix} x_1, x_2, x_3 \right] \quad (88)$$

$$S_{p002}(x) = F^{(3)} \left[\begin{matrix} -:: 1; -; -; \frac{p+1}{2}; -; 1; \\ 2 :: -; -; -; \frac{1}{2}; \frac{3}{2}; -; \end{matrix} x_1, x_2, x_3 \right] \quad (89)$$

$$\mathcal{F}_{p01}(x) = \frac{2ay_1 S_{p011}(x) + x_5 c^2 S_{p012}(x)}{e^{a_2 + x_6 \Gamma\left(\frac{1-p}{2}\right)}} \quad (90)$$

$$S_{p011}(x) = F^{(3)} \left[\begin{matrix} -:: \frac{1}{2}; -; -; \frac{p+1}{2}; -; 1; \\ \frac{3}{2} :: -; -; -; \frac{1}{2}; \frac{1}{2}; -; \end{matrix} x_4, x_5, x_6 \right] \quad (91)$$

$$S_{p012}(x) = F^{(3)} \left[\begin{matrix} -:: 1; -; -; \frac{p+1}{2}; -; 1; \\ 2 :: -; -; -; \frac{1}{2}; \frac{3}{2}; -; \end{matrix} ; x_4, x_5, x_6 \right] \quad (92)$$

$$\mathcal{F}_{p10}(x) = \frac{\sqrt{2}ay_0^2 S_{p101}(x) + \frac{8\sqrt{2}a^2}{3c^2} y_0^3 S_{p102}(x)}{e^{a_2+x_3} \Gamma\left(-\frac{p}{2}\right)} \quad (93)$$

$$S_{p101}(x) = F^{(3)} \left[\begin{matrix} -:: 1; -; -; \frac{p}{2} + 1; -; 1; \\ 2 :: -; -; -; \frac{3}{2}; \frac{1}{2}; -; \end{matrix} ; x_1, x_2, x_3 \right] \quad (94)$$

$$S_{p102}(x) = F^{(3)} \left[\begin{matrix} -:: \frac{3}{2}; -; -; \frac{p}{2} + 1; -; 1; \\ \frac{5}{2} :: -; -; -; \frac{3}{2}; \frac{3}{2}; -; \end{matrix} ; x_1, x_2, x_3 \right] \quad (95)$$

$$\mathcal{F}_{p11} = \frac{-\sqrt{2}ay_1^2 S_{p111}(x) - \frac{8\sqrt{2}a^2}{3c^2} y_1^3 S_{p112}(x)}{e^{a_2+x_6} \Gamma\left(-\frac{p}{2}\right)} \quad (96)$$

$$S_{p111}(x) = F^{(3)} \left[\begin{matrix} -:: 1; -; -; \frac{p}{2} + 1; -; 1; \\ 2 :: -; -; -; \frac{3}{2}; \frac{1}{2}; -; \end{matrix} ; x_4, x_5, x_6 \right] \quad (97)$$

$$S_{p112}(x) = F^{(3)} \left[\begin{matrix} -:: \frac{3}{2}; -; -; \frac{p}{2} + 1; -; 1; \\ \frac{5}{2} :: -; -; -; \frac{3}{2}; \frac{3}{2}; -; \end{matrix} ; x_4, x_5, x_6 \right] \quad (98)$$

The efficient implementation of (81) in Proгри (2021, [6]) requires the computation of nearly twenty equations from which eight are Kampé de Fériet function (or double hypergeometric series) [7]-[10] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Proгри (2019, [10]).

For special cases of $N = \{1, 2\}$ there is a significant reduction in the computation of (77) through (96) that we discuss here.

First, for the special case when $N = 1$; i.e., $p = -1$ four Kampé de Fériet functions get reduced to elementary functions and four Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ get reduced to four Kampé de Fériet functions as follows:

$$K_{-10i1}'' = e^{a_2} \quad (99)$$

$$K_{-10i2}'' = M \left[1; \frac{3}{2}; a_2 \right] \quad (100)$$

$$F_{-10i}'' = \frac{\frac{2a\sqrt{\pi}}{\sqrt{6}} K_{-10i1}'' - (-1)^i a_2 c^2 K_{-10i2}''}{e^{a_2}}$$

$$= \frac{2a\sqrt{\pi}}{\sqrt{6}} - (-1)^i c^2 a_2 e^{-a_2} M \left[1; \frac{3}{2}; a_2 \right]$$

$$= \frac{2a\sqrt{\pi}}{\sqrt{6}} - (-1)^i \frac{\sqrt{\pi a_2}}{2} c^2 \Phi(\sqrt{a_2})$$

$$= \frac{2a\sqrt{\pi}}{\sqrt{6}} - (-1)^i \frac{2a\sqrt{\pi}}{\sqrt{6}} \Phi(a_1) \quad (101)$$

$$S_{-1001}(x) = \sum_{j,n=0}^{\infty} \frac{(1)_n}{\left(\frac{3}{2}\right)_{j+n}} \frac{x_2^j x_3^n}{j! n!} = F_{1:0;0}^{0:0;1} \left[\begin{matrix} -; -; 1; \\ \frac{3}{2}; -; -; \end{matrix} ; x_2, x_3 \right] \quad (102)$$

$$S_{-1002}(x) = \sum_{j,n=0}^{\infty} \frac{(1)_j (1)_n}{(2)_{j+n} \left(\frac{3}{2}\right)_j} \frac{x_2^j x_3^n}{j! n!} = F_{1:1;0}^{0:1;1} \left[\begin{matrix} -; 1; 1; \\ 2; \frac{3}{2}; -; \end{matrix} ; x_2, x_3 \right] \quad (103)$$

$$S_{-1011}(x) = F_{1:0;0}^{0:0;1} \left[\begin{matrix} -; -; 1; \\ \frac{3}{2}; -; -; \end{matrix} ; x_5, x_6 \right] \quad (104)$$

$$S_{-1012}(x) = F_{1:1;0}^{0:1;1} \left[\begin{matrix} -; -; 1; 1; \\ 2; \frac{3}{2}; -; \end{matrix} ; x_5, x_6 \right] \quad (105)$$

The rest of the equations are exactly the same.

Second, for the special case when $N = 2$; i.e., $p = -2$ four Kampé de Fériet functions get reduced to elementary functions and four Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ get reduced to four Kampé de Fériet functions as follows:

$$K_{-21i1}'' = M \left[1; \frac{1}{2}; a_2 \right]$$

$$= e^{a_2} M \left[-\frac{1}{2}; \frac{1}{2}; -a_2 \right] \text{ (Kummer Trans.)}$$

$$= 1 + \sqrt{\pi} \sqrt{a_2} e^{a_2} \Phi(\sqrt{a_2}) \text{ (See (76))}$$

$$= 1 + \sqrt{\pi} a_1 e^{a_2} \Phi(a_1) \quad (106)$$

$$K_{-21i2}'' = e^{a_2} \quad (107)$$

In (106) we have applied Kummer's first transformation and the result of (76) in Appendix A as follows:

$$M \left[1; \frac{1}{2}; z^2 \right] = 1 + \sqrt{\pi} z e^{z^2} \Phi(z) \quad (108)$$

Equation (108) is an original equation not readily found in Tables of Integrals, Series, and Products.

Substituting (106) and (107) into (82) produces:

$$\begin{aligned} F_{-21i}'' &= \frac{(-1)^i a_2 c^2 \sqrt{\pi}}{\sqrt{3}} - \frac{4a}{3\sqrt{2}} e^{-a_2} M \left[1; \frac{1}{2}; a_2 \right] \\ &= \frac{(-1)^i a_2 c^2 \sqrt{\pi}}{\sqrt{3}} - \frac{4a[e^{-a_2} + \sqrt{\pi} a_1 \Phi(a_1)]}{3\sqrt{2}} \end{aligned} \quad (109)$$

The simplifications of the four Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ are given by:

$$S_{-2101}(x) = \sum_{j,n=0}^{\infty} \frac{(1)_j (1)_n \frac{x_2^j x_3^n}{j! n!}}{(2)_{j+n} \left(\frac{1}{2}\right)_j} = F_{1:1;0}^{0:1;1} \left[\begin{matrix} -; 1; 1; \\ 2; \frac{1}{2}; -; \end{matrix} x_2, x_3 \right] \quad (110)$$

$$S_{-2102}(x) = \sum_{m,j,n=0}^{\infty} \frac{(1)_n \frac{x_2^j x_3^n}{j! n!}}{\left(\frac{5}{2}\right)_{j+n}} = F_{1:0;0}^{0:0;1} \left[\begin{matrix} -; -; 1; \\ 5; -; -; \end{matrix} x_2, x_3 \right] \quad (111)$$

$$S_{-2111}(x) = F_{1:1;0}^{0:1;1} \left[\begin{matrix} -; 1; 1; \\ 2; \frac{1}{2}; -; \end{matrix} x_5, x_6 \right] \quad (112)$$

$$S_{-2112}(x) = F_{1:0;0}^{0:0;1} \left[\begin{matrix} -; -; 1; \\ 5; -; -; \end{matrix} x_5, x_6 \right] \quad (113)$$

The rest of the equations are exactly the same.

This concludes the derivations for the computation of the special cases of the efficient GPCFD cdf given by (151) and (152) and then (81) in Proгри (2021, [6]).

ⁱ The parabolic cylinder functions (PCFs) are a class of functions sometimes called Weber functions. The computation of both the generalized PCF distribution (GPCFD) pdf and cdf has been an objective of mine since 2016 (see Proгри 2016, [3]). Having basically exhausted the search of every possible avenue for their efficient and optimized, quantitative computation in 2021 I finally came to realization that there must be a special path that reduces the triple hypergeometric series to a double hypergeometric series (see Proгри 201, [6]). In this publication I was very fortunate to have found this special path by means of the expansion of the Hermite polynomials. The collection of these three journal papers is a collection of masterpieces because they describe how important is the path of integration in order to achieve a fast efficient and optimization computation of the GPCFD pdf and cdf.

ⁱⁱ Hermite polynomials were defined by Pierre-Simon Laplace in 1810, though in scarcely recognizable form and studied in detail by Pafnuty Chebyshev in 1859. Chebyshev's work was overlooked, and they were named later after Charles Hermite, who wrote on the polynomials in 1864, describing them as new. They were consequently not new, although Hermite was the first to define the multidimensional polynomials in his later 1865 publications [2].

ⁱⁱⁱ In 2016 Dr. Proгри stated: "We can only compute Kampé de Fériet functions indirectly as part of the integration or integral. The direct computation of the Kampé de Fériet function will be considered in a future publication," (Proгри (2016, [8])).

^{iv} If this were to be a symphony it would sound like Beethoven third symphony (premiered in 1804 [20]) or Brahms fourth symphony (premiered in 1885 [21]) considered by Richard Strauss as a "*giant work, great in concept and invention*", if this were a competition in golf, tennis, basketball, or American football it would feel like Bob Jones [22] winning the Grand Slam in 1930 or Rod Laver [23] winning the Grand Slam in 1962 and 1969, Jack Nicklaus [24] 1980 season, Michael Jordan [25] 1990-1993 season, Tiger Wood's [26] 2000 slam, or Roger Federer 2006 season [27], Rafael Nadal [28] winning the Roland Garros, Novak Djokovic [29] 2020 season or Tom Brady winning all the Superbowl [30]. Dr. Proгри created the journal of geolocation, geoinformation, and geo-intelligence specifically to produce high quality publications at the level of a grand slam performance.

^v In order to perform the efficient, optimized, and recursive computation of the generalized Kampé de Fériet function (or double hypergeometric series) [7]-[10] Dr. Proгри created the MATLAB function called kamdeferh2(a,b,c,d,z,op) which computes simultaneously two generalized Kampé de Fériet function in four options using the Hermite polynomials in the numerator. For a detailed discussion on the super-efficient implementation of the Kampé de Fériet function or double hypergeometric series the reader should refer to Appendix A of Proгри (2018, [9]).

^{vi} Ditto v.

^{vii} Ditto v.

^{viii} Ditto v.