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*Research Article*

# Efficient Computation of the Generalized Parabolic Cylinder Function Distribution

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This paper discusses the generalized parabolic cylinder function (or GPCF) distribution[i] (or GPCFD) probability density function (pdf) in a manner that is original and never presented before in the literature. For the GPCFD the closed form expression of the cumulative distribution function (cdf) is given by means of eight Kampé de Fériet functions (or hypergeometric series of two variables) and eight Srivastava functions (or hypergeometric series of three variables). Numerical results are derived for each case to validate the theoretical models presented in the paper.

*Index Terms*—Parabolic cylinder function, cumulative distribution function, hypergeometric series, Tricomi confluent hypergeometric series, gamma functions, incomplete gamma functions, pdf cdf analysis, Kampé de Fériet function, Srivastava functions.

## 1 Introduction

The parabolic cylinder function (PCF) [1] distributions (PCFDs) are distributions that involve the computation of PCF in their pdf or cdf.

The application of PCFDs is limited to gamma like properties as described in (Mathai and Saxena, 1968) [3]. In (Skjong and Madsen 1987) [4] a PCFD is known as the Pierson and Holmes distribution.

Even in Electrical and Computer Engineering the discussion of PCF applications is limited to a limited number of publications [5]-[8].

Moreover, the numerical evaluation of the PCF is also limited to special software such as (Wolfram 1999-2016) [9] or a special MATLAB toolbox (Cojocaru, 2009) [10].

Although the pdf of GPCFD was known, I found no references that discuss the cdf of GPCFD. These facts alone demonstrate that the cdf of GPCFD or its computation in closed form expression perhaps did not exist prior to this publication with the exception of Progri (2016, [11]).

Given the body of work (i.e., analysis, MATLAB functions, etc.) that I developed in the computation of the generalized Bessel function distribution Progri (2016, [12])-Progri (2019, [14]), the time was right to significantly enhance the work that I initially investigated in Progri (2016, [11]).

The real application that generated a very serious interest researching GPCFDs is because of adaptive GPS signal detection based on differential coherent accumulation (DCAC) and more specifically under the assumption with GPS signal present, the statistics are GPCFDs (Progri et al. 2016, [15], [16]).

This paper presents the first attempt to produce a closed form expression of the cdf of GPCFD that leads to an efficient MATLAB computation. Although I have produced a closed form expression of the cdf of GPCFD, which contains four Kampé de Fériet functions (Progri 2016, [12]) and six confluent hypergeometric series of two variables [17], I do not believe that this closed form expression is simplified enough to enable a fast numerical computation of the cdf of GPCFD in MATLAB.

After many attempts I was able to finally produce a closed form expression of the cdf of the GPCFD that requires the computation of eight Kampé de Fériet function (or double hypergeometric series) [12]-[14] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x,y,z]$ (see Appendix E of Progri (2019, [14]); i.e., it is the most computationally laborious function that I have ever come across with.

I would like to mention something very personal with regards to the methodology of solving such a difficult problem. There is a particular way of eliminating the singularity and performing the computation in the integration process that is laborious and it is the smart way; perhaps the only way that I was successful. Hence, I was thinking of the Golden Bear, Jack Nicklaus, the greatest golfer of all time and his masterpiece book, *Golf My Way*. Jack visualized and produced an efficient way of playing competitive golf at the highest level. Likewise, I have visualized an efficient way of solving difficult problems and it is Progri's way of solving it. I really hope the readers will enjoy because this is perhaps the most difficult computation I have come across in my entire life.

In its current form the efficient computation of the GPCFD cdf is not optimized for speed. The efficient computation of the closed needs three orders of magnitude improvement to be competitive with other forms of computation. I hope soon to create, originate, and develop a follow up journal paper that will discuss ways of optimizing this closed form expression for speed and memory.

This paper is organized as follows: in Sect. 2 the efficient computation of the GPCFD pdf is presented. The efficient computation of the GPCFD cdf is discussed in Sect. 3. Section IV contains numerical results; Conclusion is provided in Section V along with a list of references. This journal paper contains five appendices: Appendix A through Appendix E.

## 2 Efficient Computation of the GPCFD PDF

In this section the efficient computation of the GPCFD pdf is depicted. From 2016 until 2021 I developed a lot of efficient computational algorithms; therefore, from the accumulated body of knowledge I have produced the efficient computation of the GPCFD pdf that is presented here into two subsections: GPCFD pdf first main components that leads to the efficient computation of the GPCFD pdf.

### *2.1 GPCFD PDF First Main Components*

Before providing the efficient computation of the GPCFD pdf (see Progri (2016, [11])) we need to consider certain GPCFD main components or identities that lead to the efficient computation of the GPCFD first main component.

The PCD is defined in the integral form (see Gradshteyn, Ryzhik 2007, [17] pg. 1028 9.241 2.) as

$$D_p(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(-p)} \int_0^\infty e^{-yz-\frac{y^2}{2}} y^{-p-1} dy \ (p < 0) \tag{1}$$

and in the series expansion form as

$$D_p(z) = 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \sqrt{\pi} \left| \frac{{}_1F_1\left[\begin{matrix}-\frac{p}{2}\\ \frac{1}{2}\end{matrix}\middle| \frac{z^2}{2}\right]}{\Gamma\left(\frac{1-p}{2}\right)} - \frac{\sqrt{2} z\, {}_1F_1\left[\begin{matrix}\frac{1-p}{2}\\ \frac{3}{2}\end{matrix}\middle| \frac{z^2}{2}\right]}{\Gamma\left(-\frac{p}{2}\right)} \right. \tag{2}$$

From the definition of the Tricomi confluent hypergeometric function, $U(a,b,z)$, we have [19]

$$U(a,b,z) = \frac{\Gamma(1-b)\, {}_1F_1\left[\begin{matrix}a\\ b\end{matrix}\middle| z\right]}{\Gamma(a+1-b)} + \frac{\Gamma(b-1) z^{1-b}\, {}_1F_1\left[\begin{matrix}a+1-b\\ 2-b\end{matrix}\middle| z\right]}{\Gamma(a)} \tag{3}$$

If make the following substitutions:

$$a = -\frac{p}{2},\ b = \frac{1}{2}, \text{ and } z = \frac{z^2}{2} \tag{4}$$

From where we have

$$1 - b = \frac{1}{2},\ 2 - b = 2 - \frac{1}{2} = \frac{3}{2};\ b - 1 = -\frac{1}{2} \tag{5}$$

$$a + 1 - b = \frac{1}{2} - \frac{p}{2} = \frac{1-p}{2},\ z^{1-b} = \sqrt{\frac{z^2}{2}} = \frac{z}{\sqrt{2}} \tag{6}$$

Hence, substituting (4)-(6) into (3) yields

$$U\left(-\frac{p}{2},\frac{1}{2},\frac{z^2}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right){}_1F_1\left[\begin{array}{c|c}-\frac{p}{2} & \frac{z^2}{2}\\ \frac{1}{2} & \end{array}\right]}{\Gamma\left(\frac{1-p}{2}\right)}+\frac{\Gamma\left(-\frac{1}{2}\right)\frac{z}{\sqrt{2}}{}_1F_1\left[\begin{array}{c|c}\frac{1-p}{2} & \frac{z^2}{2}\\ \frac{3}{2} & \end{array}\right]}{\Gamma\left(-\frac{p}{2}\right)} \tag{7}$$

Next, from the particular-values of gamma function [20] we have:

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi};\ \Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi} \tag{8}$$

Substituting (8) into (7) produces,

$$U\left(-\frac{p}{2},\frac{1}{2},\frac{z^2}{2}\right)=\sqrt{\pi}\left|\frac{{}_1F_1\left[\begin{array}{c|c}-\frac{p}{2} & \frac{z^2}{2}\\ \frac{1}{2} & \end{array}\right]}{\Gamma\left(\frac{1-p}{2}\right)}-\frac{\sqrt{2}z\,{}_1F_1\left[\begin{array}{c|c}\frac{1-p}{2} & \frac{z^2}{2}\\ \frac{3}{2} & \end{array}\right]}{\Gamma\left(-\frac{p}{2}\right)}\right. \tag{9}$$

Comparing and contrasting (2) and (9) we have

$$D_p(z)=2^{\frac{p}{2}}e^{-\frac{z^2}{4}}U\left(-\frac{p}{2},\frac{1}{2},\frac{z^2}{2}\right) \tag{10}$$

From the integral definition of the Tricomi confluent hypergeometric function, $U(a,b,z)$, we have [19]

$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}dt,\ (a>0) \tag{11}$$

Next, substituting (4) into (11) produces

$$U\left(-\frac{p}{2},\frac{1}{2},z\right)=\frac{1}{\Gamma\left(-\frac{p}{2}\right)}\int_0^\infty e^{-zt}t^{-\frac{p}{2}-1}(1+t)^{\frac{p}{2}-\frac{1}{2}}dt \tag{12}$$

From Temme [21] we find another integral definition of PCB function, $U(a,z)$, as follows

$$U(a,z)=\frac{e^{-\frac{z^2}{4}}}{\Gamma\left(\frac{1}{2}+a\right)}\int_0^\infty t^{a-\frac{1}{2}}e^{-zt-\frac{t^2}{2}}dt \tag{13}$$

If we set

$$a=-\frac{1}{2}-p \tag{14}$$

then (13) becomes

$$U\left(-\frac{1}{2}-p,z\right)=\frac{e^{-\frac{z^2}{4}}}{\Gamma(-p)}\int_0^\infty t^{-1-p}e^{-zt-\frac{t^2}{2}}dt \tag{15}$$

Since (14) is identical to (1) then we have

$$D_p(z)=U\left(-\frac{1}{2}-p,z\right),\ p<0 \tag{16}$$

From Temme [21] we find another relation of PCB function, $U(a,z)$, and the Tricomi confluent hypergeometric function [19], $U(a,b,z)$, as follows

$$U(a,z)\equiv 2^{-\frac{1}{4}-\frac{1}{2}a}e^{-\frac{1}{4}z^2}U\left(\frac{1}{2}a+\frac{1}{4},\frac{1}{2};\frac{1}{2}z^2\right),\ p<0 \tag{17}$$

Hence, if we substitute (13) into (17) we obtain (10)

$$U\left(-\frac{1}{2}-p,z\right)\equiv 2^{\frac{p}{2}}e^{-\frac{1}{4}z^2}U\left(-\frac{p}{2},\frac{1}{2};\frac{1}{2}z^2\right),\ p<0 \tag{18}$$

Also, if we compare and contrast (1) with (10) we obtain another definition of the Tricomi confluent hypergeometric function [19], $U(a,b,z)$,

$$U\left(-\frac{p}{2},\frac{1}{2};\frac{z^2}{2}\right)=\frac{2^{-\frac{p}{2}}}{\Gamma(-p)}\int_0^\infty e^{-tz-\frac{t^2}{2}}t^{-p-1}dt\ \ p<0 \tag{19}$$

$$U\left(-\frac{p}{2},\frac{1}{2};\frac{z^2}{2}\right)=\frac{1}{\Gamma\left(-\frac{p}{2}\right)}\int_0^\infty e^{-\frac{z^2}{2}t}t^{-\frac{p}{2}-1}(1+t)^{\frac{p}{2}-\frac{1}{2}}dt \tag{20}$$

From (Gradshteyn, Ryzhik 2007, [17] pg. 899 8.351 4.) we have another definition of the upper incomplete gamma function by means of the Tricomi confluent hypergeometric function [19], $U(a,b,z)$,

$$\Gamma(\alpha,z)=z^\alpha e^{-z}U(1,1+\alpha;z)=e^{-z}U(1-\alpha,1-\alpha;z) \tag{21}$$

If we compare and contrast (21) with (10) we notice that the PCF, $D_p(z)$, is a generalization of the upper incomplete gamma function, $\Gamma(\alpha,z)$. Only for $p=-1$, we have

$$D_{-1}(z)=2^{-\frac{1}{2}}e^{-\frac{z^2}{4}}U\left(\frac{1}{2},\frac{1}{2},\frac{z^2}{2}\right)=2^{-\frac{1}{2}}e^{\frac{z^2}{4}}\Gamma\left(\frac{1}{2},\frac{z^2}{2}\right) \tag{22}$$

From (Gradshteyn, Ryzhik 2007, [17] pg. 902 8.359 3.) we have,

$$\Gamma\left(\frac{1}{2},\frac{z^2}{2}\right)=\sqrt{\pi}\left\{1-\left[\Phi\left(\frac{z}{\sqrt{2}}\right)\equiv \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right]\right\} \tag{23}$$

Next, substituting (23) into (22) produces,

$$D_{-1}(z)=2^{-\frac{1}{2}}e^{-\frac{z^2}{4}}U\left(\frac{1}{2},\frac{1}{2},\frac{z^2}{2}\right)=\sqrt{\frac{\pi}{2}}e^{\frac{z^2}{4}}\left[1-\Phi\left(\frac{z}{\sqrt{2}}\right)\right] \tag{24}$$

Equation (24) is identical to the result given by (Gradshteyn, Ryzhik 2007, [17] pg. 1030 9.254 1.).

Next, we exploit the connection between the Hermite polynomial, $H_k(N)$, PCF $D_p(z)$, and Tricomi confluent hypergeometric function, $U(a,b,z)$, (Gradshteyn, Ryzhik 2007, [17] pg. 1030 9.253) as follows:

$$D_n(z)=2^{-\frac{n}{2}}e^{-\frac{z^2}{4}}H_n\left(\frac{z}{\sqrt{2}}\right),\ n\ \text{natural number} \tag{25}$$

From where we obtain

$$H_n(z)=2^{\frac{n}{2}}e^{\frac{z^2}{2}}D_n\left(\sqrt{2}z\right),\ n\ \text{natural number} \tag{26}$$

Next, we substitute (10) into (26) yields

$$H_n(z)=2^{\frac{n}{2}}e^{\frac{z^2}{2}}D_n\left(\sqrt{2}z\right)=2^nU\left(-\frac{n}{2},\frac{1}{2},z^2\right) \tag{27}$$

Equation (27) is very important as it does not exist in explicit form in the Table of Integrals only in implicit form in (Gradshteyn, Ryzhik 2007, [17] pg. 997 8.953 1. & 2.).

## 2.2 *Efficient Computation of the GPCFD PDF*

Now that we have discussed the GPCFD main components we can discuss the efficient computation of the GPCFD pdf (see Progri (2016, [11] (1))

$$f_{PCD}(x) = \frac{d_0(x)\sum_{k=0}^{N-1}\frac{(-p-1)!\left[b_0(x)D_p[c_0(x)]+b_1(x)D_p[c_1(x)]\right]}{H_k(N)^{-1}a^p}}{C} \tag{28}$$

where

$$p = k - N;\ p < 0, \text{ negative integer} \tag{29}$$

First, we simplify the products by substituting (10) and Progri (2016, [11] (2)-(4)) we obtain:

$$b_i(x)D_p[c_i(x)] = 2^{\frac{p}{2}}U\left(-\frac{p}{2},\frac{1}{2},\frac{c_i^2(x)}{2}\right),\ i = \{0,1\} \tag{30}$$

Hence, substituting (30) into (28) yields,

$$f_{PCD}(x) = \frac{d_0(x)\sum_{k=0}^{N-1}\frac{(-p-1)!2^{\frac{p}{2}}\left[U\left(-\frac{p}{2},\frac{1}{2},\frac{c_0^2(x)}{2}\right)+U\left(-\frac{p}{2},\frac{1}{2},\frac{c_1^2(x)}{2}\right)\right]}{H_k(N)^{-1}a^p}}{C} \tag{31}$$

Next, substituting (27) into (31) produces,

$$f_{PCD}(x) = \frac{\sum_{k=0}^{N-1}\frac{(-p-1)!U\left(-\frac{k}{2},\frac{1}{2},N^2\right)\left[U\left(-\frac{p}{2},\frac{1}{2},\frac{c_0^2(x)}{2}\right)+U\left(-\frac{p}{2},\frac{1}{2},\frac{c_1^2(x)}{2}\right)\right]}{2^{-\frac{3k-N}{2}}a^p}}{d_0^{-1}(x)C} \tag{32}$$

Next, we simplify the computation of the components,

$$c_i(x) = \frac{a}{c^2} - (-1)^i\frac{x-b}{2a};\ i = \{0,1\} \tag{33}$$

The term $c_i^2(x)$ can be computed as follows:

$$c_i^2(x) = \frac{a^2}{c^4} + \frac{(x-b)^2}{4a^2} - (-1)^i\frac{x-b}{c^2};\ i = \{0,1\} \tag{34}$$

And then, $c_i^2(x)/2$ is simply

$$\frac{c_i^2(x)}{2} = \frac{a^2}{2c^4} + \frac{(x-b)^2}{8a^2} - (-1)^i\frac{x-b}{2c^2};\ i = \{0,1\} \tag{35}$$

Next, from (9), we compute the following sum

$$\begin{matrix} U\left[-\frac{p}{2},\frac{1}{2},\frac{c_0^2(x)}{2}\right] \\ U\left[-\frac{p}{2},\frac{1}{2},\frac{c_1^2(x)}{2}\right] \end{matrix} += \sqrt{\pi}\left|\begin{matrix} \frac{{}_1F_1\left[\begin{matrix}-\frac{p}{2}\\ \frac{1}{2}\end{matrix}\middle|\frac{c_0^2(x)}{2}\right]+{}_1F_1\left[\begin{matrix}-\frac{p}{2}\\ \frac{1}{2}\end{matrix}\middle|\frac{c_1^2(x)}{2}\right]}{\Gamma\left(\frac{1-p}{2}\right)} - \\ \frac{\frac{c_0(x)}{\sqrt{2}}{}_1F_1\left[\begin{matrix}\frac{1-p}{2}\\ \frac{3}{2}\end{matrix}\middle|\frac{c_0^2(x)}{2}\right]+\frac{c_1(x)}{\sqrt{2}}{}_1F_1\left[\begin{matrix}\frac{1-p}{2}\\ \frac{3}{2}\end{matrix}\middle|\frac{c_1^2(x)}{2}\right]}{2^{-1}\Gamma\left(-\frac{p}{2}\right)} \end{matrix}\right. \tag{36}$$

We cannot substitute (36) into (31) to produce the desired formula for the computation of the GPCFD pdf. The GPCFD pdf (31) contains a singularity as $x \to \pm\infty$ that needs to be eliminated.

$$\lim_{x\to\pm\infty} U\left[-\frac{p}{2},\frac{1}{2},\frac{c_0^2(x)}{2}\right] + U\left[-\frac{p}{2},\frac{1}{2},\frac{c_1^2(x)}{2}\right] = \infty \tag{37}$$

$$\lim_{x\to\pm\infty} d_0(x) = 0 \tag{38}$$

Hence, if we were to take the limit of the product of (37) with (38) the following yields

$$\lim_{x\to\pm\infty} d_0(x)\left\{U\left[-\frac{p}{2},\frac{1}{2},\frac{c_0^2(x)}{2}\right] + U\left[-\frac{p}{2},\frac{1}{2},\frac{c_1^2(x)}{2}\right]\right\} =$$

$$= 0 \times \infty = NaN \tag{39}$$

Hence, we cannot integrate directly (31) by directly substituting (36). We need to eliminate this singularity first. In order to eliminate this singularity we will use Kummer's first transformation.

$${}_1F_1\left[\begin{matrix}a\\ b\end{matrix}\middle|z\right] = e^z\,{}_1F_1\left[\begin{matrix}b-a\\ b\end{matrix}\middle|-z\right] \tag{40}$$

We already know what the right answer should be as $x \to \pm\infty$. The answer should be 0. Equation (40) explicitly shows that the singularity is in the computation of the exponential function. Why? In Progri (2021, [22]) the computation of the irregular singularities of the confluent hypergeometric function is discussed in great detail. As $x \to \pm\infty$ then $-z \to -\infty$ hence we have in Progri (2021, [22])

$$\lim_{z\to\infty} {}_1F_1\left[\begin{matrix}b-a\\ b\end{matrix}\middle|-z\right] = 0 \tag{41}$$

$$\lim_{z\to\infty} {}_1F_1\left[\begin{matrix}a\\ b\end{matrix}\middle|z\right] = \lim_{z\to\infty} e^z \lim_{z\to\infty} {}_1F_1\left[\begin{matrix}b-a\\ b\end{matrix}\middle|-z\right]$$

$$= \infty \times 0 = NaN \tag{42}$$

Since, $d_0(x)$ is an exponential function and $e^z$ is also an exponential function , their multiplication should result in the sum of exponents and hence should eliminate the singularity. It remains now just to perform the algebra.

First, let us make the substitution

$$v^2 = \frac{c_0^2(x)}{2} \tag{43}$$

$$w^2 = \frac{c_1^2(x)}{2} \tag{44}$$

Substituting (43) and (44) into (36) produces

$$U\left[-\frac{p}{2},\frac{1}{2},v^2\right]+U\left[-\frac{p}{2},\frac{1}{2},w^2\right]=\sqrt{\pi}\left[\frac{{}_1F_1\left[\begin{matrix}-\frac{p}{2}\\ \frac{1}{2}\end{matrix}\middle|v^2\right]+{}_1F_1\left[\begin{matrix}-\frac{p}{2}\\ \frac{1}{2}\end{matrix}\middle|w^2\right]}{\Gamma\left(\frac{1-p}{2}\right)}-\frac{v\,{}_1F_1\left[\begin{matrix}\frac{1-p}{2}\\ \frac{3}{2}\end{matrix}\middle|v^2\right]+w\,{}_1F_1\left[\begin{matrix}\frac{1-p}{2}\\ \frac{3}{2}\end{matrix}\middle|w^2\right]}{2^{-1}\Gamma\left(-\frac{p}{2}\right)}\right] \tag{45}$$

Substituting (40) into (45) yields

$$\begin{aligned}U\left[-\frac{p}{2},\frac{1}{2},v^2\right]+U\left[-\frac{p}{2},\frac{1}{2},w^2\right]&=\sqrt{\pi}\left[\frac{e^{v^2}{}_1F_1\left[\begin{matrix}\frac{p+1}{2}\\ \frac{1}{2}\end{matrix}\middle|-v^2\right]+e^{w^2}{}_1F_1\left[\begin{matrix}\frac{p+1}{2}\\ \frac{1}{2}\end{matrix}\middle|-w^2\right]}{\Gamma\left(\frac{1-p}{2}\right)}-\frac{ve^{v^2}{}_1F_1\left[\begin{matrix}\frac{p}{2}+1\\ \frac{3}{2}\end{matrix}\middle|-v^2\right]+we^{w^2}{}_1F_1\left[\begin{matrix}\frac{p}{2}+1\\ \frac{3}{2}\end{matrix}\middle|-w^2\right]}{2^{-1}\Gamma\left(-\frac{p}{2}\right)}\right]\\
&=\sqrt{\pi}\left[\begin{matrix}e^{v^2}\left[\frac{{}_1F_1\left[\begin{matrix}\frac{p+1}{2}\\ \frac{1}{2}\end{matrix}\middle|-v^2\right]}{\Gamma\left(\frac{1-p}{2}\right)}-\frac{v\,{}_1F_1\left[\begin{matrix}\frac{p}{2}+1\\ \frac{3}{2}\end{matrix}\middle|-v^2\right]}{2^{-1}\Gamma\left(-\frac{p}{2}\right)}\right]\\ +e^{w^2}\left[\frac{{}_1F_1\left[\begin{matrix}\frac{p+1}{2}\\ \frac{1}{2}\end{matrix}\middle|-w^2\right]}{\Gamma\left(\frac{1-p}{2}\right)}-\frac{w\,{}_1F_1\left[\begin{matrix}\frac{p}{2}+1\\ \frac{3}{2}\end{matrix}\middle|-w^2\right]}{2^{-1}\Gamma\left(-\frac{p}{2}\right)}\right]\end{matrix}\right]\end{aligned} \tag{46}$$

All it remains now is substituting (46) into (31) and eliminating the singularity.

From (2) and (5) in Progri (2016, [11]) we have

$$d_0(x)=e^{-a_0(x)(x-b)}=e^{-\frac{(x-b)^2}{2a^2}} \tag{47}$$

Hence, the product of two exponential functions is equal to:

$$\begin{aligned}d_0(x)e^{v^2}&=e^{-\frac{(x-b)^2}{2a^2}+\frac{c_0^2(x)}{2}}=e^{-\frac{3(x-b)^2}{8a^2}+\frac{a^2}{2c^4}-\frac{x-b}{2c^2}}\\
&=e^{-\frac{1}{2}\left[\frac{\sqrt{3}(x-b)}{2a}+\frac{a}{\sqrt{3}c^2}\right]^2}e^{\frac{2a^2}{3c^4}}\\
&=e^{-\frac{1}{2}g_0^2(x)}e^{\frac{2a^2}{3c^4}}\end{aligned} \tag{48}$$

$$\begin{aligned}d_0(x)e^{w^2}&=e^{-\frac{(x-b)^2}{2a^2}+\frac{c_1^2(x)}{2}}=e^{-\frac{3(x-b)^2}{8a^2}+\frac{a^2}{2c^4}+\frac{x-b}{2c^2}}\\
&=e^{-\frac{1}{2}\left[\frac{\sqrt{3}(x-b)}{2a}-\frac{a}{\sqrt{3}c^2}\right]^2}e^{\frac{2a^2}{3c^4}}\\
&=e^{-\frac{1}{2}g_1^2(x)}e^{\frac{2a^2}{3c^4}}\end{aligned} \tag{49}$$

Clearly, from (48) and (49) if we were to take the limit

$$\lim_{x\to\pm\infty}d_0(x)e^{y^2}=\lim_{x\to\pm\infty}d_0(x)e^{z^2}=0 \tag{50}$$

Since, we have eliminated the singularity then we can substitute (46) into (31) and then perform the simplifications (48) and (49).

$$f_{PCD}(x)=\frac{\sqrt{\pi}\sum_{k=0}^{N-1}\frac{(-p-1)!2^{\frac{p}{2}}\left[e^{-\frac{1}{2}g_0^2(x)}V(p,v)+e^{-\frac{1}{2}g_1^2(x)}V(p,w)\right]}{H_k(N)^{-1}a^p}}{e^{\frac{-2a^2}{3c^4}}C} \tag{51}$$

where

$$V(p,q)=\frac{{}_1F_1\left[\begin{matrix}\frac{p+1}{2}\\ \frac{1}{2}\end{matrix}\middle|-q^2\right]}{\Gamma\left(\frac{1-p}{2}\right)}-\frac{q\,{}_1F_1\left[\begin{matrix}\frac{p}{2}+1\\ \frac{3}{2}\end{matrix}\middle|-q^2\right]}{2^{-1}\Gamma\left(-\frac{p}{2}\right)};\ q=\{v,w\} \tag{52}$$

Finally, substituting (56) of Progri (2016, [11]) we have into (5) produces the GPCFD pdf with no singularity that we can integrate and obtain the GPCFD cdf.

$$f_{PCD}(x)=\frac{\sum_{k=0}^{N-1}\frac{H_k(N)n!}{a^k}\sqrt{2}^k\left[e^{-\frac{1}{2}g_0^2(x)}V(p,v)+e^{-\frac{1}{2}g_1^2(x)}V(p,w)\right]}{2^{\frac{3(N+1)}{2}}3^{-\frac{N}{2}}ae^{\frac{-a^2}{3c^4}}\sum_{k=0}^{N-1}\frac{H_k(N)n!}{a^k}\left(\frac{\sqrt{3}}{2}\right)^k D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)} \tag{53}$$

Equation (53) has significantly improved advantage over (1) in Progri (2016, [11]) for three main reasons. First, (53) does not have any singularities; second, it simplifies the integration of (53) tremendously; and third, it enables an efficient computation of both the pdf and the cdf.

Next, we consider a special case which provides great insight into the nature of the GPCFD pdf and then later of the GPCFD cdf.

### 2.3 *Efficient Computation of a Special Case of the GPCFD PDF*

The main purpose of the efficient computation of a special case of the GPCFD pdf is not only to validate the computational work that was performed in the previous section but also to provide great insight into the nature of the GPCFD pdf and then later of the GPCFD cdf.

Hence, let us consider a special case when:

$$a=1;\ N=1\ \Rightarrow\ p=-1;\ n=0;\ k=0 \tag{54}$$

Substituting (54) into (53) yields

$$f_{PCD}(x)=\frac{e^{-\frac{1}{2}g_0^2(x)}V(p,v)+e^{-\frac{1}{2}g_1^2(x)}V(p,w)}{2^3 3^{-\frac{1}{2}}e^{\frac{-b^2}{3}}D_{-1}\left(\frac{2b}{\sqrt{3}}\right)} \tag{55}$$

where from (Gradshteyn, Ryzhik 2007, [17] pg. 1030 9.254 1.) we have,

$$D_{-1}\left(\frac{2b}{\sqrt{3}}\right)=e^{\frac{b^2}{3}}\sqrt{\frac{\pi}{2}}\left[1-\Phi\left(\frac{2b}{\sqrt{6}}\right)\right] \tag{56}$$

$$e^{-\frac{1}{2}g_0^2(x)} = e^{-\frac{1}{2}\left[\frac{\sqrt{3}(x-b)}{2}+\frac{b}{\sqrt{3}}\right]^2} = e^{-\frac{\left(\sqrt{3}x-\frac{b}{\sqrt{3}}\right)^2}{8}} \tag{57}$$

$$e^{-\frac{1}{2}g_1^2(x)} = e^{-\frac{1}{8}\left[\frac{\sqrt{3}(x-b)}{2}-\frac{b}{\sqrt{3}}\right]^2} = e^{-\frac{\left(\sqrt{3}x-\frac{5b}{\sqrt{3}}\right)^2}{8}} \tag{58}$$

from (Gradshteyn, Ryzhik 2007, [17] pg. 1027 9.236 1.) we have,

$$V(-1,q) = 1 - \frac{2q\,{}_1F_1\left[\begin{matrix}\frac{1}{2}\\ \frac{3}{2}\end{matrix}\middle| -q^2\right]}{\sqrt{\pi}} \equiv 1-\Phi(q) \tag{59}$$

Substituting (56)-(59) into (55) yields,

$$f_{PCD}(x) = \frac{e^{-\frac{\left(\sqrt{3}x-\frac{b}{\sqrt{3}}\right)^2}{8}}\left[1-\Phi\left(\frac{c_0(x)}{\sqrt{2}}\right)\right]+e^{-\frac{\left(\sqrt{3}x-\frac{5b}{\sqrt{3}}\right)^2}{8}}\left[1-\Phi\left(\frac{c_1(x)}{\sqrt{2}}\right)\right]}{4\times 3^{-0.5}\times\sqrt{2\pi}\left[1-\Phi\left(\frac{2b}{\sqrt{6}}\right)\right]} \tag{60}$$

From, (60) we obtain all the insight into the nature of the special case of the GPCFD pdf. For this special case the GPCFD pdf is reduced to a difference of a sum of two normally distributed pdfs with a weighted sum of two normally distributed pdfs weighted by error functions.

Therefore, not only is (60) very significant as an original expression not published anywhere else, but also because it provides great insight into the nature and the efficient computation of the GPCFD cdf.

This concludes the discussion of the computation of the GPCFD pdf. Next, we investigate the discussion on the efficient computation of the GPCFD CDF.

# 3 Efficient Computation of the GPCFD CDF

In this section the efficient computation of the GPCFD cdf is performed. From 2016 until 2021 I developed a lot of efficient computational algorithms [11]-[14]; therefore, from the accumulated body of knowledge I have produced the efficient computation of the GPCFD cdf into five subsections.

## *3.1 GPCFD First Main Component*

Before providing the closed form expressions of GPCFD CDF other main components discussed in Progri (2016, [11]) we need to consider certain GPCFD main components or identities that lead to the derivation of the GPCFD first main component.

What are the GPCFD main components identities that are employed for solving the GPCFD CDF first main component?

In Appendix A, we derive the complete steps for computing the integral (43) in Progri (2016, [11], pg. 76)

$$g_{0k,N}(a,b,c) = 2a\sqrt{\frac{2\pi}{4}}\left(1-\frac{1}{4}\right)^{\frac{p}{2}} e^{\frac{\frac{a^2}{c^4}}{4\left(1-\frac{1}{4}\right)}} D_p\left[\frac{a}{c^2}\left(1-\frac{1}{4}\right)^{-\frac{1}{2}}\right] \tag{61}$$

which is identical to (Gradshteyn, Ryzhik 2007, [17] pg. 842 7.724).

This result is so significant because it is considered the first main component and it gives the pathway for solving the GPCFD CDF other main components which are discussed next.

Now that we proved all the main identities of the PCF and derived the first main component in Appendix A, the integral (43) in Progri (2016, [11], pg. 76); let us develop the GPCFD cdf other main components.

Moreover, in Appendix B we have performed the direct integration and in Appendix C the alternative integration. Since, there are more one way for computing the closed form expression of the GPCFD cdf, it was necessary to fully investigate each and every path of integration and fully document their analytical results.

## *3.2 GPCFD CDF other Main Components*

In Appendix B we have performed the computation of the closed form expression of the GPCFD cdf as follows

$$F_{PCD}(x) = \int_{-\infty}^{x} f_{PCD}(t)dt \tag{62}$$ [ii]

where $f_{PCD}(x)$ is given by (53).

Since the function is symmetric around $x=b$ then we have to take advantage of the symmetry and not compute the entire integral.

$$F_{PCD}(x) = \left|\begin{matrix} \int_{-\infty}^{x} f_{PCD}(t)dt \equiv \int_{2b-x}^{\infty} f_{PCD}(t)dt & x \le b \\ 0.5 & x = b \\ 1-\int_{x}^{\infty} f_{PCD}(t)dt & x \ge b \end{matrix}\right. \tag{63}$$

Hence, it only makes sense to compute the following integral

$$F'(x) = \int_{x}^{\infty} f_{PCD}(t)dt;\ x \ge b > 0 \tag{64}$$

From Appendix B the function $F'(x)$

$$F'(x) = F_0{}'(x) + F_1{}'(x)$$

$$= \left|\begin{matrix} F_{00}{}'' + F_{10}{}'' \\ F_{01}{}'' + F_{11}{}'' \end{matrix}\right. + \left|\begin{matrix} \mathcal{F}_{00}(x) + \mathcal{F}_{10}(x) \\ -\mathcal{F}_{01}(x) - \mathcal{F}_{11}(x) \end{matrix}\right. \quad x \ge b \tag{65}$$

where the GPCFD other main components are the functions $F_{ii}{}''$ and $\mathcal{F}_{ii}(x)$ for $i=\{0,1\}$.

Hence, the efficient computation of two main components of the GPCFD cdf given by (90) and (158) of Progri (2016, [11]) is obtained here by means of the efficient computation of (65).

TABLE I: THE QUANTITATIVE COMPUTATIONAL PERFORMANCE

| GPCDF cdf | Integral (sec) | L. Ap. (ms) | CFE 1 (sec) | CFE 2 (sec) | Total (sec) |
|---|---|---|---|---|---|
| $a = 1$ | 2.83 | 0.2 | 102.25 | 9.88 | 114.96 |
| $a = 2$ | 2.88 | 0.2 | 96.81 | 9.32 | 109.03 |
| $b = 0.5$ | 2.73 | 0.5 | 100.49 | 9.66 | 112.88 |
| $N = 2$ | 6.75 | 0.8 | 213.95 | 20.85(*) | 241.54 |

Each function $F_{ii}''$ consists of two Kampé de Fériet function (or double hypergeometric series) [12]-[14] and each function $\mathcal{F}_{ii}(x)$ consists of two Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [14].

It is particularly important to mention here that in (65) the function $F'(x)$ requires the computation of eight Kampé de Fériet function (or double hypergeometric series) [12]-[14] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [14]); i.e., it is computationally the most laborious function that I have ever come across with.

For a special case when $a = 1;\ b = 1;\ N = 1\ \Rightarrow\ c = 1$ the computation of the closed form expression of the GPCFD cdf is reduced because four Kampé de Fériet function (or double hypergeometric series) [12]-[14] are reduced into two error functions and four Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [14]) are reduced into four Kampé de Fériet function (or double hypergeometric series) [12]-[14] as discussed in Appendix C.

Verification of the singularity points is performed in Appendix D. Integrals of the product of the error function with an exponential function is discussed in Appendix E.

Because this is such as important topic further investigation of the efficient computation of (65) will be produced in a separate publication that will make use of Appendix E.

## 4 Numerical, Theoretical Results

The numerical, theoretical results section contains a few simplified computational or quantitative examples.

### 4.1 *Examples*

Figure 1 presents the GPCFD pdf and four GPCFD cdfs: cdf 1 corresponds to the integral, cdf 2 to linear approximation (top), cdf 3 to a closed form expression via (151) using 100 terms and cdf 4 the same as cdf 3 using 30 terms (bottom). In Fig. 2 the absolute error of the cdf 1 with cdf 2 is shown (top); and cdf 1 with cdf 3 and cdf 3 with cdf 4 (bottom). The results of Fig. 3 are exactly the same as those of Fig. 2 (top); however, the absolute error of cdf 1 with cdf 3 is only shown (bottom).

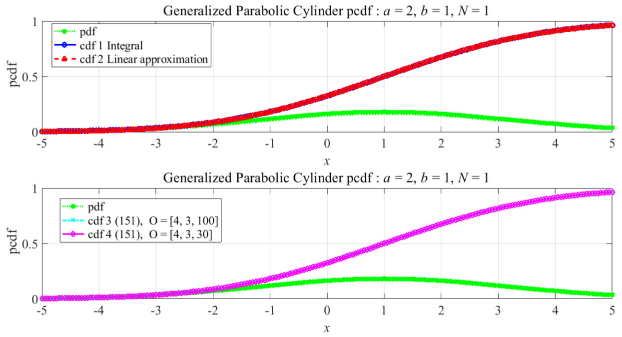


Fig. 1. The computation of the GPCFD pdf and cdf for $-5 \leq x \leq 5,\ a = 2,\ b = 1,\ N = 1$.

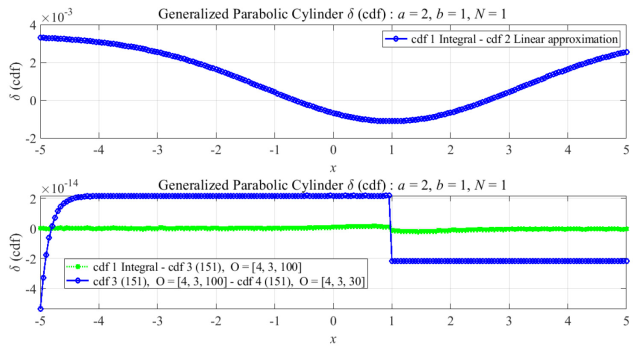


Fig. 2. The absolute error the GPCFD cdf for $-5 \leq x \leq 5$, $a = 2,\ b = 1,\ N = 1$.

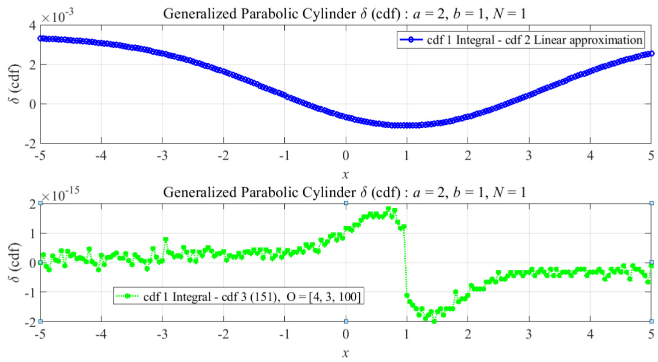


Fig. 3. The absolute error the GPCFD cdf for $-5 \leq x \leq 5$, $a = 2,\ b = 1,\ N = 1$.

Table 1 presents the computational duration times in either seconds or milliseconds on a Dell computer Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz using MATLAB 2020b [26] using a vector of $-5 \leq x \leq 5$ with a sample increment of 0.02 that leads to 500 points. A shown in Tab. 1 the cdf 1 or the integral takes 2.62 seconds, cdf 2 or linear approximation takes 0.2 milliseconds, cdf 3 via (151) using one hundred terms in the

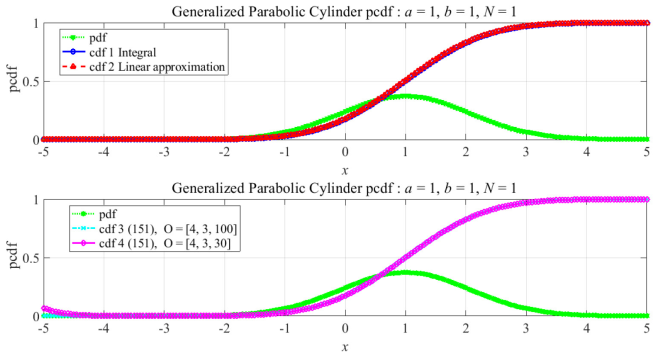


Fig. 4. The computation of the GPCFD pdf and cdf for $-5 \leq x \leq 5,\ a = 1,\ b = 1,\ N = 1$.

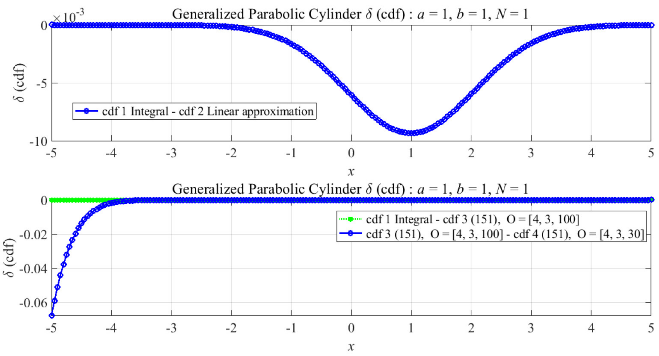


Fig. 5. The absolute error the GPCFD cdf for $-5 \leq x \leq 5$, $a = 1,\ b = 1,\ N = 1$.

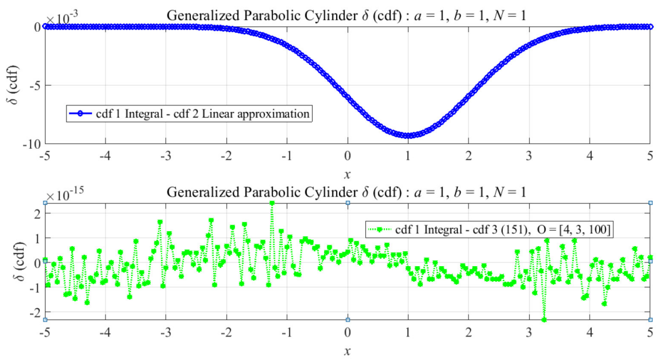


Fig. 6. Same as Fig. 4; however, the absolute error of cdf 1 integral – cdf 3 (151) for 100 terms is shown.

computation of eight Kampé de Fériet function (or double hypergeometric series) [12]-[14] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [14]) takes 101.8 seconds and cdf 4 via (151) using thirty terms of the same takes 9.67 seconds.

The second row of Tab. 1 corresponds of the quantitative results for $a = 2;\ b = 1;\ N = 1$. The third row corresponds to $a = 1;\ b = 0.5;\ N = 1$; and the final row corresponds to $a = 1;\ b = 1;\ N = 2$. With the exception of the linear approximations all the other duration times are doubled in the

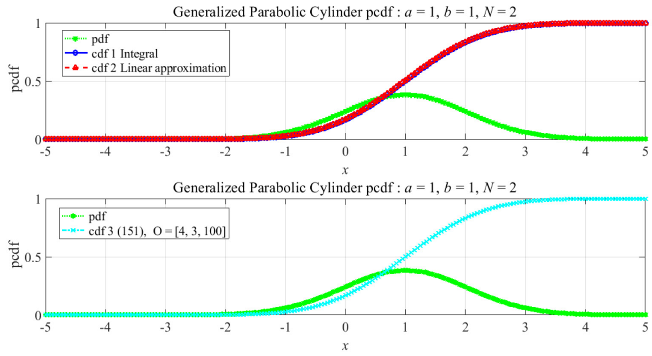


Fig. 7. The computation of the GPCFD pdf and cdf for $-5 \leq x \leq 5,\ a = 1,\ b = 1,\ N = 2$.

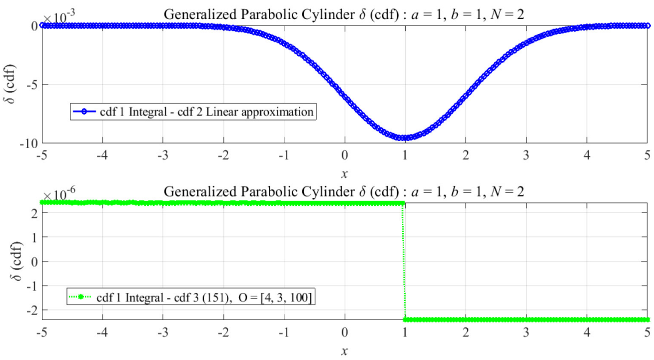


Fig. 8. The absolute error the GPCFD cdf for $-5 \leq x \leq 5$, $a = 1,\ b = 1,\ N = 2$.

last scenario as it is expected.

In the current non optimized form the numerical computation of the closed form expression of the GPCFD cdf requires two orders of magnitude more resources than direct integral. Its accuracy of the computation is sensitive to the number of terms in the computation of the eight Kampé de Fériet functions (or double hypergeometric series) [12]-[14] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [14]). When the number of terms is reduced by a third the computational duration time is reduced by an order of magnitude as indicated in Tab. 1 and Figs. 2. 3, 5, 6, and 8.

For $N = 1$ one hundred terms will produce an accuracy of fifteen orders of magnitude as opposed for $N = 2$ the same number of terms will produce an accuracy of six orders of magnitude.

The results of Figs. 4 through 6 and Fig. 7 and 8 are self-explanatory.

This concluded the discussion on examples and numerical, theoretical results.

## 5 Conclusions

We have been able to produce for the first time the complete discussion of the GPCFD pdf and cdf. We have showed that this is a valid pdf function and we have produced the valid normalized coefficient. This work is original and never published before in a journal paper.

We have been able to produce for the first time the closed form expression of the cdf of the GPCFD which contains eight Kampé de Fériet function (or double hypergeometric series) [12]-[14] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x,y,z]$ (see Appendix E of Progri (2019, [14]).

In the current non optimized form the numerical computation of the closed form expression of the GPCFD cdf requires two orders of magnitude more resources than direct integral. Its accuracy of the computation is sensitive to the number of terms in the computation of the eight Kampé de Fériet functions (or double hypergeometric series) [12]-[14] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x,y,z]$ (see Appendix E of Progri (2019, [14]). When the number of terms is reduced by a third the computational duration time is reduced by an order of magnitude.

Further work is needed to optimize the computation of (151) to make it competitive based on similar recursive algorithms discussed in Progri (2011, [24]) and further improvements of Appendix E of Progri (2019, [14]).

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This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless, to say I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus reminds me of my grandfather who I loved him very much.

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## 8 Appendix A: The Derivation of the Integral (43) in Progri 2016

In this section we derive the complete steps for the direct integration of (43) in Progri (2016, [11], pg. 76).

Next, let us try to directly solve the integral (43) in Progri (2016, [11], pg. 76).

$$g_{0k,N}(a,b,c) = 2a\int_{-\infty}^{\infty} e^{-\frac{\left(y-\frac{a}{c^2}\right)^2}{2^{\frac{1}{4}}}} e^{\frac{y^2}{4}} D_p(y)dy \tag{66}$$

Substituting (10) into (66) yields

$$g_{0k,N}(a,b,c) = 2^{\frac{p}{2}+1} a\int_{-\infty}^{\infty} e^{-\frac{\left(y-\frac{a}{c^2}\right)^2}{2^{\frac{1}{4}}}} U\left(-\frac{p}{2},\frac{1}{2},\frac{y^2}{2}\right)dy \tag{67}$$

Next, employing (19) into (67) yields,

$$g_{0k,N}(a,b,c) = \frac{2a\int_{-\infty}^{\infty} e^{-\frac{\left(y-\frac{a}{c^2}\right)^2}{2^{\frac{1}{4}}}} \int_0^{\infty} e^{-ty-\frac{t^2}{2}} t^{-p-1} dtdy}{\Gamma(-p)} \tag{68}$$

Next, changing the order of integration yields,

$$g_{0k,N}(a,b,c) = \frac{2a\int_0^{\infty}\int_{-\infty}^{\infty} e^{-\frac{\left(y-\frac{a}{c^2}\right)^2}{2^{\frac{1}{4}}}} e^{-ty}dye^{-\frac{t^2}{2}} t^{-p-1}dt}{\Gamma(-p)} \tag{69}$$

Next, let us consider the inside integral for a moment

$$e^{-2\left(y-\frac{a}{c^2}\right)^2} e^{-ty} = e^{-\left[2y^2-\left(4\frac{a}{c^2}-t\right)y+2\frac{a^2}{c^4}\right]} \tag{70}$$

Which is equivalent with

$$e^{-2\left(y-\frac{a}{c^2}\right)^2} e^{-ty} = e^{-\left[\sqrt{2}y-\frac{\left(4\frac{a}{c^2}-t\right)}{2\sqrt{2}}\right]^2} e^{-\left(\frac{at}{c^2}-\frac{t^2}{8}\right)} \tag{71}$$

Next, substituting (71) into (69) yields.

$$g_{0k,N}(a,b,c)=\frac{2a\int_0^{\infty}\int_{-\infty}^{\infty}e^{-\left[\sqrt{2}y-\frac{\left(4\frac{a}{c^2}-t\right)}{2\sqrt{2}}\right]^2}dy\frac{t^{-p-1}}{e^{\left(\frac{at}{c^2}-\frac{t^2}{8}\right)}e^{\frac{t^2}{2}}}dt}{\Gamma(-p)} \tag{72}$$

Now, we can solve the inside integral by making the substitution,

$$z=\sqrt{2}y-\frac{\left(4\frac{a}{c^2}-t\right)}{2\sqrt{2}};\ dz=\sqrt{2}dy;\ y\to|_{-\infty}^{\infty}\Rightarrow z\to|_{-\infty}^{\infty} \tag{73}$$

Hence, we have (Gradshteyn, Ryzhik 2007, [17] pg. 336 3.321 3.)

$$\frac{\int_{-\infty}^{\infty}e^{-z^2}dz}{\sqrt{2}}=\frac{\sqrt{\pi}}{\sqrt{2}} \tag{74}$$

It remains now to solve the outside integral,

$$g_{0k,N}(a,b,c)=\frac{2a\frac{\sqrt{\pi}}{\sqrt{2}}\int_0^{\infty}e^{-\left(\frac{at}{c^2}-\frac{t^2}{8}\right)}e^{-\frac{t^2}{2}}t^{-p-1}dt}{\Gamma(-p)} \tag{75}$$

Which can be written as

$$g_{0k,N}(a,b,c)=\frac{2a\frac{\sqrt{\pi}}{\sqrt{2}}\int_0^{\infty}e^{-\frac{t^2}{2}\left(1-\frac{1}{4}\right)-\frac{at}{c^2}}t^{-p-1}dt}{\Gamma(-p)} \tag{76}$$

Next, we make the substitution,

$$t'=t\sqrt{1-\tfrac{1}{4}};\ dt'=dt\sqrt{1-\tfrac{1}{4}};\ t\to|_0^{\infty}\Rightarrow t'\to|_0^{\infty} \tag{77}$$

Substituting (77) into (76) yields,

$$g_{0k,N}(a,b,c)=\frac{2a\frac{\sqrt{\pi}}{\sqrt{2}}\left(1-\frac{1}{4}\right)^{\frac{p}{2}}\int_0^{\infty}e^{-\frac{t'^2}{2}-\frac{at'}{c^2\sqrt{1-\frac{1}{4}}}}t'^{-p-1}dt'}{\Gamma(-p)} \tag{78}$$

Finally, from the definition of the PCF we recognize that (78) can be written as

$$g_{0k,N}(a,b,c)=2a\sqrt{\frac{2\pi}{4}}\left(1-\tfrac{1}{4}\right)^{\frac{p}{2}}e^{\frac{\frac{a^2}{c^4}}{4\left(1-\frac{1}{4}\right)}}D_p\left[\frac{a}{c^2}\left(1-\tfrac{1}{4}\right)^{-\frac{1}{2}}\right] \tag{79}$$

which is identical to (Gradshteyn, Ryzhik 2007, [17] pg. 842 7.724).

This concludes the derivations of Appendix A.

# 9 Appendix B: The Direct Integration

The direct integration is given by

$$F_{PCD}(x)=\int_{-\infty}^{x}f_{PCD}(t)dt \tag{80}$$

where $f_{PCD}(x)$ is given by (53).

Since the function is symmetric around $x=b$ then we have to take advantage of the symmetry and not compute the entire integral.

$$F_{PCD}(x)=\left|\begin{array}{cc}\int_{-\infty}^{x}f_{PCD}(t)dt\equiv\int_{2b-x}^{\infty}f_{PCD}(t)dt & x\le b\\ 0.5 & x=b\\ 1-\int_{x}^{\infty}f_{PCD}(t)dt & x\ge b\end{array}\right. \tag{81}$$

Hence, it only makes sense to compute the following integral

$$F'(x)=\int_x^{\infty}f_{PCD}(t)dt;\ x\ge b>0 \tag{82}$$

Substituting (53) into (82) and after some elementary algebra like changing the order of summation and integration we have

$$F_{PCD}{}'(x)=\frac{\sum_{k=0}^{N-1}\frac{H_k(N)n!}{a^k2^{-\frac{k}{2}}}\int_x^{\infty}\left[e^{-\frac{g_0^2(t)}{2}}V(p,v)+e^{-\frac{g_1^2(t)}{2}}V(p,w)\right]dt}{2^{\frac{3(N+1)}{2}}3^{-\frac{N}{2}}ae^{\frac{-a^2}{3c^4}}\sum_{k=0}^{N-1}\frac{H_k(N)n!}{a^k2^k3^{-\frac{k}{2}}}D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)} \tag{83}$$

It remains to solve two integrals,

$$F_i{}'=\int_x^{\infty}e^{-\frac{g_i^2(t)}{2}}V\left(p,\frac{c_i(t)}{\sqrt{2}}\right)dt;\ i=\{0,1\}$$

$$=\begin{array}{c}\int_x^{\infty}e^{-\frac{g_i^2(t)}{2}}\frac{{}_1F_1\left[\begin{array}{c}\frac{p+1}{2}\\ \frac{1}{2}\end{array}\middle|-\frac{c_i^2(t)}{2}\right]}{\Gamma\left(\frac{1-p}{2}\right)}dt+\\ -\int_x^{\infty}e^{-\frac{g_i^2(t)}{2}}\frac{\frac{c_i(t)}{\sqrt{2}}{}_1F_1\left[\begin{array}{c}\frac{p}{2}+1\\ \frac{3}{2}\end{array}\middle|-\frac{c_i^2(t)}{2}\right]}{2^{-1}\Gamma\left(-\frac{p}{2}\right)}dt\end{array};\ i=\{0,1\} \tag{84}$$

which then produces four integrals,

$$F_{0i}{}'(x)=\frac{\int_x^{\infty}e^{-\frac{g_i^2(t)}{2}}{}_1F_1\left[\begin{array}{c}\frac{p+1}{2}\\ \frac{1}{2}\end{array}\middle|-\frac{c_i^2(t)}{2}\right]dt}{\Gamma\left(\frac{1-p}{2}\right)};\ i=\{0,1\} \tag{85}$$

$$F_{1i}{}'(x)=\frac{-\sqrt{2}\int_x^{\infty}c_i(t)e^{-\frac{g_i^2(t)}{2}}{}_1F_1\left[\begin{array}{c}\frac{p}{2}+1\\ \frac{3}{2}\end{array}\middle|-\frac{c_i^2(t)}{2}\right]dt}{\Gamma\left(-\frac{p}{2}\right)};\ i=\{0,1\} \tag{86}$$

If we make the substitution

$$c_i(t)=y=\frac{2a^2+(-1)^ibc^2}{2ac^2}-(-1)^i\frac{t}{2a};\ i=\{0,1\} \tag{87}$$

Then we have

$$dy=(-1)^{i+1}\frac{dt}{2a};\ t\to|_x^{\infty}\Rightarrow\ y\to|_{y_i}^{(-1)^{i+1}\infty} \tag{88}$$

where $y_i$ is the value of $y$ corresponding to the subscript $i$. If we make the substitution, $y$, then the $g_i(t)$ becomes

$$-\frac{1}{2}\left[\frac{\sqrt{3}(x-b)}{2a}+\frac{a}{\sqrt{3}c^2}\right]^2=-\frac{3}{2}\left(-y+\frac{4a}{3c^2}\right)^2 \tag{89}$$

We substitute (87)-(89) into (85) we have

$$F_{0i}{}'(x) = \frac{2a\int_{y_i}^{(-1)^{i+1}\infty} e^{-\frac{3\left(y-\frac{4a}{3c^2}\right)^2}{2}} {}_1F_1\left[\begin{array}{c|c} \frac{p+1}{2} & -\frac{y^2}{2} \\ \frac{1}{2} & \end{array}\right] dy}{(-1)^{i+1}\Gamma\left(\frac{1-p}{2}\right)} \tag{90}$$

Next, we expand the exponent of the exponential function as follows:

$$\frac{3\left(y-\frac{4a}{3c^2}\right)^2}{2} = \frac{3}{2}y^2 - \frac{4a}{c^2}y + a_2 \tag{91}$$

where

$$a_2 = 8 \times 3^{-1}a^2c^{-4} \tag{92}$$

We substitute (91) into (90) we have

$$F_{0i}{}'(x) = \frac{2ae^{-a_2}\int_{y_i}^{(-1)^{i+1}\infty} e^{-\frac{3}{2}y^2} e^{\frac{4a}{c^2}y} {}_1F_1\left[\begin{array}{c|c} \frac{p+1}{2} & -\frac{y^2}{2} \\ \frac{1}{2} & \end{array}\right] dy}{(-1)^{i+1}\Gamma\left(\frac{1-p}{2}\right)} \tag{93}$$

Next, we perform the series expansion of the confluent hypergeometric function and of the exponential function with the $y$ exponent we obtain

$$F_{0i}{}'(x) = \frac{2a\int_{y_i}^{(-1)^{i+1}\infty} e^{-\frac{3}{2}y^2} \sum_{m,r=0}^{\infty} \frac{\left(\frac{p+1}{2}\right)_m (-1)^m \frac{2^{2r}a^r y^{2m+r}}{2^m c^{2r}}}{\left(\frac{1}{2}\right)_m m!r!} dy}{(-1)^{i+1}e^{a_2}\Gamma\left(\frac{1-p}{2}\right)}$$

$$= \frac{2a\sum_{m,r=0}^{\infty} \frac{\left(\frac{p+1}{2}\right)_m (-1)^m \frac{2^{2r}a^r}{2^m c^{2r}}}{\left(\frac{1}{2}\right)_m m!r!} \int_{y_i}^{(-1)^{i+1}\infty} y^{2m+r} e^{-\frac{3}{2}y^2} dy}{(-1)^{i+1}e^{a_2}\Gamma\left(\frac{1-p}{2}\right)}$$

$$= \frac{2a\sum_{r=0}^{\infty} \frac{\frac{2^{2r}a^r}{c^{2r}}}{r!} \int_{y_i}^{(-1)^{i+1}\infty} y^r e^{-\frac{3}{2}y^2} dy}{(-1)^{i+1}e^{a_2}\Gamma\left(\frac{1-p}{2}\right)} \tag{94}$$

Finally, it remains to solve the integral

$$G_{0i}{}'(x) = \int_{y_i}^{(-1)^{i+1}\infty} y^{2m+r} e^{-\frac{3}{2}y^2} dy \tag{95}$$

Next, we make the substitution

$$m' = 2m + r, \text{ a positive integer; } \mu = \frac{3}{2} > 0 \tag{96}$$

Then we have

$$G_{0i}{}'(x) = \int_{y_i}^{(-1)^{i+1}\infty} y^{m'} e^{-\mu y^2} dy \tag{97}$$

Next, we make the substitution for the integral in (97) we have

$$z = \mu y^2;\ \frac{dz}{2\sqrt{\mu}\sqrt{z}} = dy;\ y \to |_{y_i}^{(-1)^{i+1}\infty};\ \Rightarrow z \to |_{\mu y_i^2 > 0}^{\infty} \tag{98}$$

$$G_{0i}{}'(x) = \mu^{-\frac{m'}{2}-1} \frac{(-1)^r \int_{\mu y_i^2}^{\infty} z^{\frac{m'}{2}-\frac{1}{2}} e^{-z} dz}{2} \tag{99}$$

where

$$y_0 = \left(\frac{2a^2+bc^2}{2ac^2} \equiv \frac{b'}{2a}\right) - \frac{x}{2a}; \tag{100}$$

$$y_1 = \frac{2a^2-bc^2}{2ac^2} + \frac{x}{2a}; \tag{101}$$

$$q = \frac{m'}{2} + \frac{1}{2} \tag{102}$$

Finally, we obtain

$$G_{00}{}'(x) = -\frac{\mu^{-\frac{m'}{2}-1} |(-1)^r\Gamma(q) + \gamma(q,x_3) \quad x \geq b}{2} \tag{103}$$

$$G_{01}{}'(x) = \frac{\mu^{-\frac{m'}{2}-1} |\Gamma(q) - \gamma(q,x_6) \quad x \geq b}{2} \tag{104}$$

where

$$x_3 = \frac{3y_0^2}{2} \tag{105}$$

$$x_6 = \frac{3y_1^2}{2} \tag{106}$$

Substituting (103) and (104) into (94) we have

$$F_{00}{}' = \frac{\sum_{m,r=0}^{\infty} \frac{\left(\frac{p+1}{2}\right)_m (-1)^m \frac{2^{2.5r}a^r}{c^{2r}}}{\left(\frac{1}{2}\right)_m 3^{\frac{m'}{2}} m!r!} |(-1)^r\Gamma(q) + \gamma(q,x_3) \quad x \geq b}{2^{-1}a^{-1}\sqrt{6}e^{a_2}\Gamma\left(\frac{1-p}{2}\right)} \tag{107}$$

$$F_{01}{}' = \frac{2a\sum_{m,r=0}^{\infty} \frac{\left(\frac{p+1}{2}\right)_m (-1)^m \frac{2^{2.5r}a^r}{c^{2r}}}{\left(\frac{1}{2}\right)_m 3^{\frac{m'}{2}} m!r!} |\Gamma(q) - \gamma(q,x_6) \quad x \geq b}{\sqrt{6}e^{a_2}\Gamma\left(\frac{1-p}{2}\right)} \tag{108}$$

Next, we can write the gamma function as follows:

$$\Gamma(q) = \Gamma\left(m + \frac{r}{2} + \frac{1}{2}\right) = \left|\begin{array}{ll} \Gamma\left(m + j + \frac{1}{2}\right) & r = 2j \\ \Gamma(m + j + 1) & r = 2j + 1 \end{array}\right.$$

$$= \left|\begin{array}{ll} \left(\frac{1}{2}\right)_{m+j}\Gamma\left(\frac{1}{2}\right) & r = 2j \\ (1)_{m+j}\Gamma(1) & r = 2j + 1 \end{array}\right. \tag{109}$$

Next, we write the function $F_{0i}{}''$ as follows

$$F_{0i}{}'' = \frac{\dfrac{\sum_{m,j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{m+j}\left(\frac{p+1}{2}\right)_m a_1^m a_2^j}{\left(\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_j m!\ j!}}{2^{-1}a^{-1}\sqrt{\pi^{-1}}\sqrt{6}} - \dfrac{\sum_{m,j=0}^{\infty} \frac{(1)_{m+j}\left(\frac{p+1}{2}\right)_m a_1^m a_2^j}{\left(\frac{1}{2}\right)_m\left(\frac{3}{2}\right)_j m!\ j!}}{(-1)^{-i}a_2^{-1}c^{-2}}}{e^{a_2}\Gamma\left(\frac{1-p}{2}\right)} \tag{110}$$

Where

$$a_1 = -3^{-1} \tag{111}$$

The above formula (110) can be written with the help of the Kampé de Fériet function (or double hypergeometric series) [12]-[14] as follows

$$F_{0i}{}'' = \frac{\frac{2a\sqrt{\pi}}{\sqrt{6}} F^{1:1;0}_{0:1;1}\left[\begin{matrix}\frac{1}{2}:\frac{p+1}{2};-; \\ -:\frac{1}{2};\frac{1}{2};\end{matrix}\, a_1, a_2\right] - (-1)^i a_2 c^2 F^{1:1;0}_{0:1;1}\left[\begin{matrix}1:\frac{p+1}{2};-; \\ -:\frac{1}{2};\frac{3}{2};\end{matrix}\, a_1, a_2\right]}{e^{a_2}\Gamma\left(\frac{1-p}{2}\right)} \quad (112)$$

Next, we can write the lower incomplete gamma function based on the expansion (Gradshteyn, Ryzhik 2007, [17] pg. 899 8.35 8.351 2.) as follows:

$$\gamma(q, x_3) = \frac{x_3^q e^{-x_3} \sum_{n=0}^{\infty} \frac{(1)_n}{(q+1)_n} \frac{x_3^n}{n!}}{q} \quad (113)$$

Next, the Pochhammer symbol [25], in the summation of the incomplete gamma function, can be written as:

$$(q+1)_n = \left| \begin{matrix} \frac{\left(\frac{3}{2}\right)_{m+j+n}}{\left(\frac{3}{2}\right)_{m+j}} & r = 2j \\ \frac{(2)_{m+j+n}}{(2)_{m+j}} & r = 2j+1 \end{matrix} \right. \quad (114)$$

Next, the parameter $q$ can be written with the help of the Pochhammer function [25] as follows:

$$q = \left| \begin{matrix} m + j + \frac{1}{2} = \frac{\left(m+\frac{3}{2}\right)_j \left(m+\frac{1}{2}\right)}{\left(m+\frac{1}{2}\right)_j} = \frac{\left(\frac{3}{2}\right)_{m+j}}{\left(\frac{1}{2}\right)_{m+j}} \frac{1}{2} & r = 2j \\ m + j + 1 = \frac{(m+2)_j (m+1)}{(m+1)_j} = \frac{(2)_{m+j}}{(1)_{m+j}} & r = 2j+1 \end{matrix} \right. \quad (115)$$

Next, substituting (114) and (115) into (113) we obtain for $r = 2j$

$$\gamma(q, x_3) = 2\frac{\left(\frac{1}{2}\right)_{m+j} x_3^q \sum_{n=0}^{\infty} \frac{(1)_n}{\left(\frac{3}{2}\right)_{m+j+n}} \frac{x_3^n}{n!}}{e^{x_3}} \quad (116)$$

and for $r = 2j+1$

$$\gamma(q, x_3) = \frac{(1)_{m+j} x_3^q \sum_{n=0}^{\infty} \frac{(1)_n}{(2)_{m+j+n}} \frac{x_3^n}{n!}}{e^{x_3}} \quad (117)$$

Next, we write the function $\mathcal{F}_{00}$ as follows

$$\mathcal{F}_{00} = \frac{\left| \begin{matrix} 2ay_0 \sum_{m,j,n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{m+j} \left(\frac{p+1}{2}\right)_m (1)_n \frac{x_1^m x_2^j x_3^n}{m!\ j!\ n!}}{\left(\frac{3}{2}\right)_{m+j+n} \left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_j} + \\ x_2 c^2 \sum_{m,j,n=0}^{\infty} \frac{(1)_{m+j} \left(\frac{p+1}{2}\right)_m (1)_n \frac{x_1^m x_2^j x_3^n}{m!\ j!\ n!}}{(2)_{m+j+n} \left(\frac{1}{2}\right)_m \left(\frac{3}{2}\right)_j} \end{matrix} \right.}{e^{a_2 + x_3}\Gamma\left(\frac{1-p}{2}\right)} \quad (118)$$

where

$$x_1 = \frac{-y_0^2}{2} \quad (119)$$

$$x_2 = \frac{4a^2 y_0^2}{c^4} \quad (120)$$

Equation (155) can be written efficiently by the help of the Srivastava’s Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [14]) as follows

$$\mathcal{F}_{00} = \frac{\frac{F^{(3)}\left[\begin{matrix}-::\frac{1}{2};-;-:\frac{p+1}{2};-;1; \\ \frac{3}{2}::-;-;-:-;\frac{1}{2};\frac{1}{2};-;\end{matrix}\, x_1, x_2, x_3\right]}{2^{-1}a^{-1}y_0^{-1}} + \frac{F^{(3)}\left[\begin{matrix}-::1;-;-:\frac{p+1}{2};-;1; \\ 2::-;-;-:\frac{1}{2};\frac{3}{2};-;\end{matrix}\, x_1, x_2, x_3\right]}{x_2^{-1}c^{-2}}}{e^{a_2 + x_3}\Gamma\left(\frac{1-p}{2}\right)} \quad (121)$$

Finally, the functions $F_{00}{}'(x)$ and $F_{01}{}'(x)$ can be written as follows

$$F_{00}{}'(x) = F_{00}{}'' + \mathcal{F}_{00}(x) \quad (122)$$

$$F_{01}{}'(x) = F_{01}{}'' - \mathcal{F}_{01}(x) \quad (123)$$

where

$$\mathcal{F}_{01} = \frac{\frac{F^{(3)}\left[\begin{matrix}-::\frac{1}{2};-;-:\frac{p+1}{2};-;1; \\ \frac{3}{2}::-;-;-:\frac{1}{2};\frac{1}{2};-;\end{matrix}\, x_4, x_5, x_6\right]}{2^{-1}a^{-1}y_1^{-1}} + \frac{F^{(3)}\left[\begin{matrix}-::1;-;-:\frac{p+1}{2};-;1; \\ 2::-;-;-:\frac{1}{2};\frac{3}{2};-;\end{matrix}\, x_4, x_5, x_6\right]}{x_5^{-1}c^{-2}}}{e^{a_2 + x_6}\Gamma\left(\frac{1-p}{2}\right)} \quad (124)$$

$$x_4 = \frac{-y_1^2}{2} \quad (125)$$

$$x_5 = \frac{4a^2 y_1^2}{c^4} \quad (126)$$

Similarly, if we substitute (87)-(89) into (86) we have

$$F_{1i}{}'(x) = \frac{2\sqrt{2}a \int_{y_i}^{(-1)^{i+1}\infty} y e^{-\frac{3\left(y - \frac{4a}{3c^2}\right)^2}{2}} {}_1F_1\left[\begin{matrix}\frac{p}{2}+1 \\ \frac{3}{2}\end{matrix}\middle| -\frac{y^2}{2}\right] dy}{(-1)^i \Gamma\left(-\frac{p}{2}\right)} \quad (127)$$

We substitute (91) into (127) we have

$$F_{1i}{}'(x) = \frac{2\sqrt{2}a e^{-a_2} \int_{y_i}^{(-1)^{i+1}\infty} y e^{-\frac{3}{2}y^2} e^{\frac{4a}{c^2}y} {}_1F_1\left[\begin{matrix}\frac{p}{2}+1 \\ \frac{3}{2}\end{matrix}\middle| -\frac{y^2}{2}\right] dy}{(-1)^i \Gamma\left(-\frac{p}{2}\right)} \quad (128)$$

Next, we perform the series expansion of the confluent hypergeometric function and of the exponential function with the $y$ exponent we obtain

$$F_{1i}{}'(x) = \frac{2\sqrt{2}a \int_{y_i}^{(-1)^{i+1}\infty} e^{-\frac{3}{2}y^2} \sum_{m,r=0}^{\infty} \frac{\left(\frac{p}{2}+1\right)_m (-1)^m \frac{2^{2r} a^r y^{n\prime}}{2^m c^{2r}}}{\left(\frac{3}{2}\right)_m m!r!} dy}{(-1)^i e^{a_2}\Gamma\left(-\frac{p}{2}\right)}$$

$$= \frac{2\sqrt{2}a \sum_{m,r=0}^{\infty} \frac{\left(\frac{p}{2}+1\right)_m (-1)^m \frac{2^{2r} a^r}{2^m c^{2r}}}{\left(\frac{3}{2}\right)_m m!r!} \int_{y_i}^{(-1)^{i+1}\infty} y^{n\prime} e^{-\frac{3}{2}y^2} dy}{(-1)^i e^{a_2}\Gamma\left(-\frac{p}{2}\right)} \quad (129)$$

where

$$m'' = 2m + r + 1, \text{ a positive integer} \quad (130)$$

Finally, it remains to solve the integral

$$G_{1i}{}'(x) = \int_{y_i}^{(-1)^{i+1}\infty} y^{m''} e^{-\frac{3}{2}y^2} dy \quad (131)$$

Since, the remaining integral is already solved then we have

$$G_{10}'(x) = -\frac{\left(\frac{2}{3}\right)^q \left|(-1)^r\Gamma(q')+\gamma(q',x_3) \quad x\geq b\right.}{\sqrt{6}} \tag{132}$$

$$G_{11}'(x) = \frac{\left(\frac{2}{3}\right)^q \left|\Gamma(q')-\gamma(q',x_6) \quad x\geq b\right.}{\sqrt{6}} \tag{133}$$

Where

$$q' = m + \frac{r}{2} + 1 = q + \frac{1}{2} \tag{134}$$

Finally, substituting in (132), (133), into (129) yields

$$F_{10}' = \frac{4a\sum_{m,r=0}^{\infty}\frac{\left(\frac{p}{2}+1\right)_m (-1)^m \frac{2^{2.5r}a^r}{3^q c^{2r}}\left|(-1)^r\Gamma(q')+\gamma(q'x_3) \quad x\geq b\right.}{\left(\frac{3}{2}\right)_m m!r!}}{\sqrt{6}e^{a_2}\Gamma\left(-\frac{p}{2}\right)} \tag{135}$$

$$F_{11}' = \frac{-4a\sum_{m,r=0}^{\infty}\frac{\left(\frac{p}{2}+1\right)_m (-1)^m \frac{2^{2.5r}a^r}{3^q c^{2r}}\left|\Gamma(q')-\gamma(q',x_6) \quad x\geq b\right.}{\left(\frac{3}{2}\right)_m m!r!}}{\sqrt{6}e^{a_2}\Gamma\left(-\frac{p}{2}\right)} \tag{136}$$

We can write the gamma function as follows:

$$\Gamma(q') = \Gamma\left(m + \frac{r}{2} + 1\right) = \begin{vmatrix}\Gamma(m+j+1) & r = 2j \\ \Gamma\left(m+j+\frac{3}{2}\right) & r = 2j+1\end{vmatrix}$$

$$= \begin{vmatrix}(1)_{m+j}\Gamma(1) & r = 2j \\ \left(\frac{3}{2}\right)_{m+j}\Gamma\left(\frac{3}{2}\right) & r = 2j+1\end{vmatrix} \tag{137}$$

The above formula can be written with the help of the Kampé de Fériet function (or double hypergeometric series) [12]-[14] as follows

$$F_{1i}'' = \frac{-\frac{4a}{3\sqrt{2}}F_{0:1;1}^{1:1;0}\left[\begin{matrix}1:\frac{p}{2}+1;-; \\ -:\frac{3}{2},\frac{1}{2};\end{matrix} a_1,a_2\right] + \frac{(-1)^i a_2\sqrt{\pi}}{c^{-2}\sqrt{3}}F_{0:1;1}^{1:1;0}\left[\begin{matrix}\frac{3}{2}:\frac{p}{2}+1;-; \\ -:\frac{3}{2},\frac{3}{2};\end{matrix} a_1,a_2\right]}{e^{a_2}\Gamma\left(-\frac{p}{2}\right)} \tag{138}$$

Next, we can write the lower incomplete gamma function based on the expansion (Gradshteyn, Ryzhik 2007, [17] pg. 899 8.35 8.351 2.) as follows:

$$\gamma(q',x_3) = \frac{x_3^{q'}e^{-x_3}\sum_{n=0}^{\infty}\frac{(1)_n}{(q'+1)_n}\frac{x_3^n}{n!}}{q'} \tag{139}$$

Next, the Pochhammer symbol [25] in the summation of the incomplete gamma function can be written as:

$$(q'+1)_n = \begin{vmatrix}\frac{(2)_{m+j+n}}{(2)_{m+j}} & r = 2j \\ \frac{\left(\frac{5}{2}\right)_{m+j+n}}{\left(\frac{5}{2}\right)_{m+j}} & r = 2j+1\end{vmatrix} \tag{140}$$

Next, the parameter $q'$ can be written with the help of the Pochhammer symbol [25] as follows:

$$q' = \begin{vmatrix}m+j+1 = \frac{(m+2)_j(m+1)}{(m+1)_j} = \frac{(2)_{m+j}}{(1)_{m+j}} & r = 2j \\ m+j+\frac{3}{2} = \frac{\left(m+\frac{5}{2}\right)_j\left(m+\frac{3}{2}\right)}{\left(m+\frac{3}{2}\right)_j} = \frac{\left(\frac{5}{2}\right)_{m+j}}{\left(\frac{3}{2}\right)_{m+j}}\frac{3}{2} & r = 2j+1\end{vmatrix} \tag{141}$$

Next, substituting (140) and (141) into (139) we obtain for $r = 2j$

$$\gamma(q',x_3) = \frac{(1)_{m+j}x_3^{q'}\sum_{n=0}^{\infty}\frac{(1)_n}{(2)_{m+j+n}}\frac{x_3^n}{n!}}{e^{x_3}} \tag{142}$$

and for $r = 2j+1$

$$\gamma(q',x_3) = \frac{2}{3}\frac{\left(\frac{3}{2}\right)_{m+j}x_3^{q'}\sum_{n=0}^{\infty}\frac{(1)_n}{\left(\frac{5}{2}\right)_{m+j+n}}\frac{x_3^n}{n!}}{e^{x_3}} \tag{143}$$

Next, we write the function $\mathcal{F}_{10}$ as follows

$$\mathcal{F}_{10} = \frac{\left|\begin{matrix}\sqrt{2}ay_0^2\sum_{m,j,n=0}^{\infty}\frac{(1)_{m+j}\left(\frac{p}{2}+1\right)_m(1)_n\frac{x_1^m x_2^j x_3^n}{m!\ j!\ n!}}{(2)_{m+j+n}\left(\frac{3}{2}\right)_m\left(\frac{1}{2}\right)_j}+ \\ \frac{2^{3.5}a^2}{3c^2}y_0^3\sum_{m,j,n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_{m+j}\left(\frac{p}{2}+1\right)_m(1)_n\frac{x_1^m x_2^j x_3^n}{m!\ j!\ n!}}{\left(\frac{5}{2}\right)_{m+j+n}\left(\frac{3}{2}\right)_m\left(\frac{3}{2}\right)_j}\end{matrix}\right.}{e^{a_2+x_3}\Gamma\left(-\frac{p}{2}\right)} \tag{144}$$

Equation (180) can be written efficiently by the help of the Srivastava's Triple Hypergeometric Series $F^{(3)}[x,y,z]$ (see Appendix E of Progri (2019, [14]) as follows

$$\mathcal{F}_{10} = \frac{\frac{F^{(3)}\left[\begin{matrix}-::1;-;-:\frac{p}{2}+1;-;1; \\ 2::-;-;-:\frac{3}{2},\frac{1}{2};-;\end{matrix} x_1,x_2,x_3\right]}{2^{-0.5}a^{-1}y_0^{-2}} + \frac{F^{(3)}\left[\begin{matrix}-::\frac{3}{2};-;-:\frac{p}{2}+1;-;1; \\ \frac{5}{2}::-;-;-:\frac{3}{2},\frac{3}{2};-;\end{matrix} x_1,x_2,x_3\right]}{3c^2 2^{-3.5}a^{-2}y_0^{-3}}}{e^{a_2+x_3}\Gamma\left(-\frac{p}{2}\right)} \tag{145}$$

Finally, the functions $F_{10}'(x)$ and $F_{11}'(x)$ can be written as follows

$$F_{10}'(x) = F_{10}'' + |\mathcal{F}_{10}(x) \quad x \geq b \tag{146}$$

$$F_{11}'(x) = F_{11}'' - |\mathcal{F}_{11}(x) \quad x \geq b \tag{147}$$

where

$$\mathcal{F}_{11} = \frac{\frac{F^{(3)}\left[\begin{matrix}-::1;-;-:\frac{p}{2}+1;-;1; \\ 2::-;-;-:\frac{3}{2},\frac{1}{2};-;\end{matrix} x_4,x_5,x_6\right]}{-2^{-0.5}a^{-1}y_1^{-2}} + \frac{F^{(3)}\left[\begin{matrix}-::\frac{3}{2};-;-:\frac{p}{2}+1;-;1; \\ \frac{5}{2}::-;-;-:\frac{3}{2},\frac{3}{2};-;\end{matrix} x_4,x_5,x_6\right]}{-3c^2 2^{-3.5}a^{-2}y_1^{-3}}}{e^{a_2+x_6}\Gamma\left(-\frac{p}{2}\right)} \tag{148}$$

Combining (122) with (146) and (123) with (147) into (84) we obtain

$$F_0' = F_{00}'' + F_{10}'' + |\mathcal{F}_{00}(x) + \mathcal{F}_{10}(x) \quad x \geq b \tag{149}$$

$$F_1' = F_{01}'' + F_{11}'' - |\mathcal{F}_{01}(x) + \mathcal{F}_{11}(x) \quad x \geq b \tag{150}$$

Substituting (149) and (150) into (82) yields,

$$F'(x) = F_0'(x) + F_1'(x)$$

$$= \begin{vmatrix} F_{00}{}'' + F_{10}{}'' \\ F_{01}{}'' + F_{11}{}'' \end{vmatrix} + \begin{vmatrix} \mathcal{F}_{00}(x) + \mathcal{F}_{10}(x) \\ -\mathcal{F}_{01}(x) - \mathcal{F}_{11}(x) \end{vmatrix} \quad x \geq b \tag{151}$$

It is particularly important to mention here that in (151) the function $F'(x)$ requires the computation of eight Kampé de Fériet function (or double hypergeometric series) [12]-[14] and eight Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [14]); i.e., it is the most computationally laborious function that I have ever come across with.

Next, substituting (151) into (81) we obtain

$$F_{PCD}{}'(x) = \frac{\sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^k 2^{-\frac{k}{2}}} F'(x)}{2^{\frac{3(N+1)}{2}} 3^{-\frac{N}{2}} a e^{\frac{-a^2}{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N)n!}{a^k 2^k 3^{-\frac{k}{2}}} D_{k-N}\left(\frac{2a}{\sqrt{3c^2}}\right)} \tag{152}$$

We have arrived at a rather unique expression of the direct integration. Since the direct integration is computationally particularly laborious; hence, we should a special case in the following appendix to see how the computation of (151) is reduced to less computationally laborious functions.

## 10 Appendix C: The Direct Integration of a Special Case

The direct integration of a special case is performed here.

For this special case we have

$$a = 1;\ b = 1;\ N = 1 \ \Rightarrow\ c = 1 \tag{153}$$

From (107) we have

$$F_{0i}{}'' = \frac{\frac{2\times 1\sqrt{\pi}}{\sqrt{6}} F^{1:1;0}_{0:1;1}\left[\begin{matrix} \frac{1}{2}:\frac{-1+1}{2};-; \\ -:\frac{1}{2},\frac{1}{2}; \end{matrix} a_1, a_2\right] - (-1)^i a_2 c^2 F^{1:1;0}_{0:1;1}\left[\begin{matrix} 1:\frac{-1+1}{2};-; \\ -:\frac{1}{2},\frac{3}{2}; \end{matrix} a_1, a_2\right]}{e^{a_2}\Gamma\left(\frac{1-(-1)}{2}\right)}$$

$$= \frac{\frac{2\sqrt{\pi}}{\sqrt{6}} e^{a_2} - (-1)^i a_2 F\left[\begin{matrix} 1; \\ \frac{3}{2}; \end{matrix} a_2\right]}{e^{a_2}} = 2\sqrt{\frac{\pi}{6}}\left[-(-1)^i \Phi\left(\frac{2}{\sqrt{3}}\right)\right] \tag{154}$$

From (121) and (124) we obtain the simplified functions $\mathcal{F}_{00}$ and $\mathcal{F}_{01}$ respectively as follows

$$\mathcal{F}_{00} = \frac{\frac{F^{(3)}\left[\begin{matrix} -::\frac{1}{2};-;-:\frac{-1+1}{2};-;1; \\ \frac{3}{2}::-;-;-:-;\frac{1}{2},\frac{1}{2};-; \end{matrix} x_1, x_2, x_3\right]}{2a^{-1}y_0^{-1}} + \frac{F^{(3)}\left[\begin{matrix} -::1;-;-:\frac{-1+1}{2};-;1; \\ 2::-;-;-:\frac{1}{2},\frac{3}{2};-; \end{matrix} x_1, x_2, x_3\right]}{x_2^{-1}c^{-2}}}{e^{a_2+x_3}\Gamma\left(\frac{1-(-1)}{2}\right)}$$

$$= \frac{2y_0 F^{0:0;1}_{1:0;0}\left[\begin{matrix} -:-;1; \\ \frac{3}{2}:-;-; \end{matrix} x_2, x_3\right] + 4y_0^2 F^{0:1;1}_{1:1;0}\left[\begin{matrix} -:1;1; \\ 2:\frac{3}{2};-; \end{matrix} x_2, x_3\right]}{e^{\frac{8}{3}+x_3}} \tag{155}$$

$$\mathcal{F}_{01} = \frac{2y_1 F^{0:0;1}_{1:0;0}\left[\begin{matrix} -:-;1; \\ \frac{3}{2}:-;-; \end{matrix} x_5, x_6\right] + 4y_1^2 F^{0:1;1}_{1:1;0}\left[\begin{matrix} -:1;1; \\ 2:\frac{3}{2};-; \end{matrix} x_5, x_6\right]}{e^{\frac{8}{3}+x_6}} \tag{156}$$

From (138) we obtain

$$F_{1i}{}'' = \frac{-\frac{4}{3\sqrt{2}} F^{1:1;0}_{0:1;1}\left[\begin{matrix} 1:\frac{1}{2};-; \\ -:\frac{3}{2},\frac{1}{2}; \end{matrix} a_1, a_2\right] + \frac{(-1)^i 8\sqrt{\pi}}{3\sqrt{3}} F^{1:1;0}_{0:1;1}\left[\begin{matrix} \frac{3}{2}:\frac{1}{2};-; \\ -:\frac{3}{2},\frac{3}{2}; \end{matrix} a_1, a_2\right]}{e^{a_2}\Gamma\left(\frac{1}{2}\right)} \tag{157}$$

From (145) and (148) we obtain we obtain simplified functions $\mathcal{F}_{10}$ and $\mathcal{F}_{11}$ respectively as follows

$$\mathcal{F}_{10}(x) = \frac{\frac{F^{(3)}\left[\begin{matrix} -::1;-;-:\frac{1}{2};-;1; \\ 2::-;-;-:\frac{3}{2},\frac{1}{2};-; \end{matrix} x_1, x_2, x_3\right]}{2^{-0.5}y_0^{-2}} + \frac{F^{(3)}\left[\begin{matrix} -::\frac{3}{2};-;-:\frac{1}{2};-;1; \\ \frac{5}{2}::-;-;-:\frac{3}{2},\frac{3}{2};-; \end{matrix} x_1, x_2, x_3\right]}{3\times 2^{-3.5}y_0^{-3}}}{e^{a_2+x_3}\Gamma\left(\frac{1}{2}\right)} \tag{158}$$

$$\mathcal{F}_{11}(x) = \frac{\frac{F^{(3)}\left[\begin{matrix} -::1;-;-:\frac{1}{2};-;1; \\ 2::-;-;-:\frac{3}{2},\frac{1}{2};-; \end{matrix} x_4, x_5, x_6\right]}{-2^{-0.5}y_1^{-2}} + \frac{F^{(3)}\left[\begin{matrix} -::\frac{3}{2};-;-:\frac{1}{2};-;1; \\ \frac{5}{2}::-;-;-:\frac{3}{2},\frac{3}{2};-; \end{matrix} x_4, x_5, x_6\right]}{-3\times 2^{-3.5}y_1^{-3}}}{e^{a_2+x_6}\Gamma\left(\frac{1}{2}\right)} \tag{159}$$

We obtain tremendous savings in this computation of the function, $F'(x)$, (151) because four Kampé de Fériet function (or double hypergeometric series) [12]-[14] are reduced in two error functions and four Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E of Progri (2019, [14]) are reduced into four Kampé de Fériet function (or double hypergeometric series) [12]-[14].

Because this is such as important topic further investigation of the special cases on the GPCF cdf will be produced in a separate publication.

This concludes the derivations of Appendix C.

## 11 Appendix D: Verification of the Singularity Points of the GPCF CDF

In this appendix we perform the verification of the singularity points of the GPCF cdf.

First, when $x \to +\infty$ we have from (87) we have

$$y_0^2 = \left(\frac{2a^2+bc^2}{2ac^2} - \frac{x}{2a}\right)^2 \to +\infty \tag{160}$$

$$y_1^2 = \left(\frac{2a^2-bc^2}{2ac^2} + \frac{x}{2a}\right)^2 \to +\infty \tag{161}$$

Substituting into (160) and (161) into (151) we obtain

$$F'(x \to \infty) = F_0{}'(x \to \infty) + F_1{}'(x \to \infty)$$

$$= \begin{vmatrix} F_{00}'' + F_{10}'' \\ F_{01}'' + F_{11}'' \end{vmatrix} + \begin{vmatrix} \mathcal{F}_{00}(x \to \infty) + \mathcal{F}_{10}(x \to \infty) \\ -\mathcal{F}_{01}(x \to \infty) - \mathcal{F}_{11}(x \to \infty) \end{vmatrix}$$

$$= \begin{vmatrix} F_{00}'' + F_{10}'' \\ F_{01}'' + F_{11}'' \end{vmatrix} - \begin{vmatrix} F_{00}'' + F_{10}'' \\ F_{01}'' + F_{11}'' \end{vmatrix} = 0 \qquad (162)$$

Second, when $x \to -\infty$ from the symmetry property (81) we have exactly the same answer.

It only remains to verify the value for $x = b$. From (160) and (161) we have

$$y_0^2 = \frac{a^2}{c^4} \qquad (163)$$

$$y_1^2 = \frac{a^2}{c^4} = y_0^2 \qquad (164)$$

$$F'(x=b) = F_0'(x=b) + F_1'(x=b)$$

$$= \begin{vmatrix} F_{00}'' + F_{10}'' \\ F_{01}'' + F_{11}'' \end{vmatrix} + \begin{vmatrix} \mathcal{F}_{00}(x=b) - \mathcal{F}_{01}(x=b) \\ +\mathcal{F}_{10}(x=b) - \mathcal{F}_{11}(x=b) \end{vmatrix}$$

$$= F_{00}'' + F_{10}'' + F_{01}'' + F_{11}'' \qquad (165)$$

It only remains to verify that if we substitute (165) into (83) we obtain

$$F_{PCD}^{\prime(b)} = \frac{\sum_{k=0}^{N-1} \frac{H_k(N) n!}{a^k 2^{-\frac{k}{2}}} F'(b)}{2^{\frac{3(N+1)}{2}} 3^{-\frac{N}{2}} a e^{\frac{-a^2}{3c^4}} \sum_{k=0}^{N-1} \frac{H_k(N) n!}{a^k 2^k 3^{-\frac{k}{2}}} D_{k-N}\left(\frac{2a}{\sqrt{3}c^2}\right)} = 0.5 \qquad (166)$$

Because this is such as important topic further investigation of the special cases and special points of the GPCF cdf will be produced in a separate publication.

This concludes the derivations of Appendix D.

## 12 Appendix E: Integrals of the Product of the Error Function with an Exponential Function

In this appendix we perform the analytical derivation of an important class of integrals of the product of the error function with an exponential function as follows:

$$\int_a^b \Phi(\alpha z + \beta) e^{-(\alpha' z + \beta')^2} dz \qquad (167)$$

If we assume that $\alpha' \neq \alpha$ and $\beta' \neq \beta$ then if we make the substitution

$$y = \alpha z + \beta;\ dy = \alpha dz;\ z \to |_a^b \Rightarrow y \to |_{\alpha a + \beta}^{\alpha b + \beta} \qquad (168)$$

Hence, the integral (167) becomes

$$\int_a^b \frac{\Phi(\alpha z + \beta)}{e^{(\alpha' z + \beta')^2}} dz = \frac{\int_{a'}^{b'} \Phi(y) e^{-\left[\frac{\alpha'}{\alpha}(y-\beta)+\beta'\right]^2} dy}{\alpha} \qquad (169)$$

where

$$a' = \alpha a + \beta \qquad (170)$$

$$b' = \alpha b + \beta \qquad (171)$$

Next, we observe the exponent

$$\frac{\alpha'}{\alpha}(y-\beta) + \beta' = \frac{\alpha'}{\alpha} y + \beta' - \frac{\alpha'}{\alpha}\beta \qquad (172)$$

$$\tilde{\alpha} = \frac{\alpha'}{\alpha} \qquad (173)$$

$$\tilde{\beta} = \beta' - \frac{\alpha'}{\alpha}\beta \qquad (174)$$

$$\int_a^b \frac{\Phi(\alpha z + \beta)}{e^{(\alpha' z + \beta')^2}} dz = \frac{\int_{a'}^{b'} \Phi(y) e^{-(\tilde{\alpha} y + \tilde{\beta})^2} dy}{\alpha} \qquad (175)$$

Next, we utilize the series expansion of the error function

$$\Phi(y) = \frac{2}{\sqrt{\pi}} e^{-y^2} \sum_{k=0}^{\infty} \frac{(1)_k}{\left(\frac{3}{2}\right)_k} \frac{y^{2k+1}}{k!} \qquad (176)$$

Next, we perform the expansion of the exponent

$$\left(\tilde{\alpha} y + \tilde{\beta}\right)^2 = \tilde{\alpha}^2 y^2 + 2\tilde{\alpha}\tilde{\beta} y + \tilde{\beta}^2 \qquad (177)$$

Hence, the product of two exponential functions will be

$$e^{-(\tilde{\alpha} y + \tilde{\beta})^2} e^{-y^2} = e^{-\gamma'' y^2} e^{-2\tilde{\alpha}\tilde{\beta} y} e^{-\tilde{\beta}^2} \qquad (178)$$

where

$$\alpha'' = \tilde{\alpha}^2 + 1 \qquad (179)$$

$$\beta'' = \tilde{\beta}^2 \qquad (180)$$

Next, employing the series expansion of the exponential function we expand the exponential function that has the $y^2$ as follows:

$$e^{-\alpha'' y^2} = \sum_{m=0}^{\infty} (-1)^m \frac{\alpha''^m y^{2m}}{m!} \qquad (181)$$

Next, substituting (176)-(181) into (175) we obtain:

$$\int_a^b \frac{\Phi(\alpha z + \beta)}{e^{(\alpha' z + \beta')^2}} dz = \frac{\sum_{k,m=0}^{\infty} \frac{(1)_k (-1)^m \alpha''^m \int_{a'}^{b'} y^{2k+2m+1} e^{-2\tilde{\alpha}\tilde{\beta} y} dy}{\left(\frac{3}{2}\right)_k k! m!}}{2^{-1}\sqrt{\pi} e^{\beta''^2} \alpha} \qquad (182)$$

If we assume that $a' > 0$ and $b' > 0$ and if we set

$$n = 2k + 2m + 1, \text{ a positive integer} \qquad (183)$$

$$\mu = 2\tilde{\alpha}\tilde{\beta};\ \mathrm{Re}\mu > 0 \qquad (184)$$

then we have

$$\int_{a'}^{b'} y^n e^{-\mu y} dy = \int_0^{b'} y^n e^{-\mu y} dy - \int_0^{a'} y^n e^{-\mu y} dy \quad (185)$$

Next, from (Gradshteyn, Ryzhik 2007, [17] pg. 340 3.35 3.351 1.) we obtain:

$$\int_{a'}^{b'} y^n e^{-\mu y} dy = \frac{\gamma(n+1,\mu b') - \gamma(n+1,\mu a')}{\mu^{n+1}} \quad (186)$$

Finally., substituting (186) into (182) yields,

$$\int_a^b \frac{\Phi(\alpha z+\beta)}{e^{(\alpha' z+\beta')^2}} dz = \frac{2\sum_{k,m=0}^{\infty} \frac{(1)_k}{\left(\frac{3}{2}\right)_k} \frac{\frac{(-1)^m {\alpha''}^m [\gamma(n+1,\mu b') - \gamma(n+1,\mu a')]}{\mu^{n+1}}}{k!m!}}{\sqrt{\pi} e^{{\beta''}^2} \alpha} \quad (187)$$

Let us consider a special case when

$$a = 0,\ b = \infty;\ \alpha = 1,\ \beta = 0 \quad (188)$$

Then we have

$$a' = \alpha a + \beta = 0 \quad (189)$$

$$b' = \alpha b + \beta = \infty \quad (190)$$

$$\int_0^{\infty} \Phi(z) e^{-(\alpha' z+\beta')^2} dz = \frac{2\sum_{k,m=0}^{\infty} \frac{(1)_k}{\left(\frac{3}{2}\right)_k} \frac{\frac{(-1)^m {\alpha''}^m (2k+2m+1)!}{\mu^{2k}\mu^{2m}}}{k!m!}}{\sqrt{\pi} e^{{\beta''}^2} \mu^2} \quad (191)$$

From the properties of the gamma function (Gradshteyn, Ryzhik 2007, [17] pg. 896 8.335 1. [doubling formula]) we have

$$(2k+2m+1)! = \Gamma(2w) = \frac{2^{2w-1}\Gamma(w)\Gamma\left(w+\frac{1}{2}\right)}{\sqrt{\pi}} \quad (192)$$

Substituting (229) into (228) we obtain

$$\int_0^{\infty} \frac{\Phi(z)}{e^{(\alpha' z+\beta')^2}} dz = \frac{4\sum_{k,m=0}^{\infty} \frac{(1)_k}{\left(\frac{3}{2}\right)_k} \frac{\frac{(-1)^m {\alpha''}^m 2^{2k} 2^{2m} \Gamma(w)\Gamma\left(w+\frac{1}{2}\right)}{\mu^{2k}\mu^{2m}}}{k!m!}}{\pi e^{{\beta''}^2} \mu^2} \quad (193)$$

It remains to write the following gamma functions

$$\Gamma(w) = (k+m)! = (1)_{k+m} \quad (194)$$

$$\Gamma\left(w+\frac{1}{2}\right) = \left(\frac{1}{2}\right)_{k+m} \Gamma\left(\frac{1}{2}\right) \quad (195)$$

Finally, substituting (231) and (232) into (230) we obtain

$$\int_0^{\infty} \frac{\Phi(z)}{e^{(\alpha' z+\beta')^2}} dz = \frac{\sum_{k,m=0}^{\infty} \frac{(1)_{k+m}\left(\frac{1}{2}\right)_{k+m}(1)_k}{\left(\frac{3}{2}\right)_k} \frac{\left(\frac{2^2}{\mu^2}\right)^k}{k!} \frac{\left(\frac{-2^2\alpha''}{\mu^2}\right)^m}{m!}}{4^{-1}\sqrt{\pi} e^{{\beta''}^2} \mu^2} \quad (196)$$

The above formula can be written with the help of the Kampé de Fériet function (or double hypergeometric series) [12]-[14] as follows

$$\int_0^{\infty} \frac{\Phi(z)}{e^{(\alpha' z+\beta')^2}} dz = \frac{4}{\sqrt{\pi}} \frac{e^{-{\beta''}^2} F_{0:1;0}^{2:1;0}\left[\begin{matrix}1,\frac{1}{2}:1;-; \\ -:\frac{3}{2};-;\end{matrix}\ \frac{2^2}{\mu^2}, \frac{-2^2\alpha''}{\mu^2}\right]}{\mu^2} \quad (197)$$

This concludes the derivations of Appendix E.

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[i] The parabolic cylinder functions are a class of functions sometimes called Weber functions. There are a number of slightly different definitions in use by various authors [9]. The credit for introducing this function in the "Indoor adaptive GNSS signal acquisition. Part 1: theory and simulations" paper (see Progri et al. 2016, [15], [16]) goes to my co-author P. Huang since 2010 or early 2011. The credit for actually naming the function and performing the computations and the research goes to Dr. Progri since January of 2016. The PCFs are used to compute the PCFDs which are a special class of a multivariate Gaussian distribution. In this paper we will only show that PCFDs produces valid pdfs. We will neither show how to compute the main statistics nor will we show how to compute the closed form expression of the PCFDs cdf and later its inverse. The computation of the PCFDs pdf was performed using the MATLAB toolbox developed by Cojocaru [10] in January 2009. As far as I can tell the PCFDs are introduced for the first time in the navigation community and as of January 2016 there were only four papers [5]-[8] that had discussion on parabolic cylinder functions in the IEEE Xplore; hence, Progri et al. 2016 [15], [16] presented a unique opportunity to introduce and discuss the PCFDs in a navigation related topic namely adaptive GPS signal detection based on DCAC.

[ii] Typically the material in Appendix B belongs to this section; however, since the material is so mathematically laborious; i.e., it is extremely hard to follow the material and since the level of computation is so difficult it is best to keep it in the appendix where it belongs.