

Research Article

Confluent Hypergeometric Function Irregular Singularities

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The computation of the confluent hypergeometric function irregular singularities is performed in this paper. It is determined that the irregular singularities of the confluent hypergeometric function are exactly the same as those of the exponential function. The computation of the confluent hypergeometric function is performed by the Gauss power series expansion and by the incomplete Gamma function expansion. In both cases the computation of the confluent hypergeometric function irregular singularities results in identical analytical and numerical results. I strongly recommend that MATLAB modify the *hypergeom* and enable the correct computation of the irregular singularities of the confluent hypergeometric function.

Index Terms—Confluent hypergeometric functions, irregular singularities, incomplete Gamma function, Gauss power series, binomial expansion, L'Hôpital's rule, special cases, derivative, Bessel function.

1 Introduction

In this paper, the computation of irregular singularities of the confluent hypergeometric function [1] is performed. The confluent hypergeometric function [1] is a special case of the hypergeometric function [2] which is a special case of the generalized hypergeometric function [3].

Since, the incomplete gamma function [4] is a special case of the confluent hypergeometric function [1], and since the irregular singularities of the incomplete gamma function are already determined, one thinks that the irregular singularities of the confluent hypergeometric function should be determined also.

The question is how did I come across this need to compute

the irregular singularities of the confluent hypergeometric function?

I needed to compute the statistics of a new distribution model called the efficient computation of the generalized parabolic cylinder function distribution [5]. As I was working on the computation of the special cases of the generalized parabolic cylinder function distribution, I came across a very peculiar problem such that MATLAB [6] would give a not a number (NaN) on the computation of the irregular singularities of the confluent hypergeometric function [1].

It became clear to me that I needed to discuss this as a separate journal paper rather than as an appendix to an already laborious

journal paper.

The work discussed in this paper is entirely original, novel, and innovative. It is not based on any similar work presented in the literature. As such this work falls into the masterpiece marvels in analytical derivations and numerical computation series.

The main theme in this paper was also motivated by the ingenuity in discussing several methods such as the computation of the derivative and of the maneuvering of a previous journal paper on *hypergeometric function partial derivatives* [7].

The methods of the derivative and of the maneuvering discussed in this paper appear to be particularly useful during the computation of probability density functions (pdfs), or cumulative distribution functions (cdfs), or their statistics based on methods discussed in Progre 2021 [5].

This paper also makes use of a unique expansion of the confluent hypergeometric functions, M , via the incomplete gamma function by Muller, (2001, [8]).

After completing this investigation, I strongly recommend that MATLAB modify the *hypergeom* and enable the correct computation of the irregular singularities of the confluent hypergeometric function for the values of the parameters discussed in Sect. 2.

This paper is organized as follows: in Sect. 2, the definition of the irregular singularities of the confluent hypergeometric function is presented. In Sect. 3, the computation via the Gauss Power Series is discussed. Section 4 contains the computation of irregular singularities via the incomplete Gamma function expansion. In Sect. 5, the discussion of the computation of the special cases is treated. In Sect. 6 numerical MATLAB examples are considered. Conclusion is provided in Sect. 7 along acknowledgment and with a list of references.

2 Definition of Irregular Singularities

The computation of the confluent hypergeometric function irregular singularities is especially important because it enables to fill in a hole in the computation of the confluent hypergeometric function [1]. As of 2021, MATLAB [6] has almost no confidence in the computation of the irregular singularities of the confluent hypergeometric function [1].

Before, I discuss the computation of the irregular singularities of the confluent hypergeometric function [1] let us consider the formulation of the problem, which is the notation that is being used and the condition of the parameters.

Letting M denote a confluent hypergeometric function [1]

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] &\equiv \Phi \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] \\ &\equiv {}_1F_1 \left[\begin{matrix} a(t) \\ b(t) \end{matrix}; z \right] \\ &\equiv {}_1F_1 \left[\begin{matrix} a(t) \\ b(t) \end{matrix} \middle| z \right] \\ &\equiv {}_1F_1[a(t); b(t); z] \end{aligned} \quad (1)$$

where $a(t)$ and $b(t)$ are simple functions of t and $-\infty < z < \infty$.

There are applications that require the computation of the irregular singularities with respect to z and then their evaluation at $z \rightarrow \pm\infty$; i.e.,

$$\lim_{z \rightarrow \pm\infty} M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] \equiv M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \rightarrow \pm\infty \right] \quad (2)$$

Or

$$\lim_{z \rightarrow \pm\infty} \left. \frac{\partial \log \left\{ M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] \right\}}{\partial t} \right|_{t=0} \equiv \left. \frac{\partial \log \left\{ M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \rightarrow \pm\infty \right] \right\}}{\partial t} \right|_0 \quad (3)$$

Initially we assume that $a(t=0) \neq 0$, $b(t=0) \neq 0$, $\left. \frac{da(t)}{dt} \right|_{t=0} \neq 0$, and $\left. \frac{db(t)}{dt} \right|_{t=0} \neq 0$.

The immediate questions are: What will these singularities look like? How do we compute something like these?

Although there exists a short list of identities for particular values of $a(t)$, $b(t)$, z [1] in general there is no expression that shows how the computation of the irregular singularities of the M should be computationally and numerically determined with the help of other functions, leaving a “hole” in the computation of (2) in MATLAB.

Initially, let us compute the special cases solution for $a(t) = b(t) \neq 0$ $z \rightarrow \pm\infty$ and then approach the most general case.

First, for $z \rightarrow -\infty$

$$\lim_{z \rightarrow -\infty} M \left[\begin{matrix} a(t); \\ a(t); \end{matrix} z \right] \equiv M \left[\begin{matrix} a(t); \\ a(t); \end{matrix} z \rightarrow -\infty \right] \equiv e^{(z \rightarrow -\infty)} = 0 \quad (4)$$

Second, for $z \rightarrow \infty$

$$\lim_{z \rightarrow +\infty} M \left[\begin{matrix} a(t); \\ a(t); \end{matrix} z \right] \equiv M \left[\begin{matrix} a(t); \\ a(t); \end{matrix} z \rightarrow \infty \right] \equiv e^{(z \rightarrow \infty)} = \infty \quad (5)$$

There are two general cases we should consider for the computation of the irregular singularities of the confluent hypergeometric function M : (1) when $a(t) > b(t) > 0$ and (2) when $b(t) > a(t) > 0$. Of particular interests are the cases when $a(t)$, or $b(t)$, or $b(t) - a(t)$ are integers.

The cases when (3) $a(t) < b(t) < 0$ or (4) $b(t) < a(t) < 0$ or (5) $a(t) < 0 < b(t)$ or (6) $b(t) < 0 < a(t)$ are not

considered in this paper but we may consider them in a future publication.

This concludes the definition of the confluent hypergeometric function M irregular singularities at $z \rightarrow \pm\infty$. Next, we discuss the computation of these singularities via the Gauss power series.

3 Computation via Gauss Power Series

The confluent hypergeometric function M [1] was first introduced by Kummer (1837, [1], [9]), also known as the confluent hypergeometric function of the first kind, is a special case of the generalized hypergeometric function [2], [3]. In Progni (2016, pg. 65-67, [7]), we have already shown that hypergeometric function is also represented via Gauss power series, since, Gauss was the first to have systematically studied its series representation in 1813 [2] and we have shown that in fact Gauss power series is nothing more than the Binomial expansion (Arfken and Weber, 1995, pg. 317, [10]).

For this reason, let us employ Gauss definition of the confluent hypergeometric function M for $|z| < \infty$ as follows [7]

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n] \Gamma[b(t)] z^n}{\Gamma[a(t)] \Gamma[b(t)+n] n!} \quad (6)$$

Equation (6) can be written as

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] &= \frac{\Gamma[b(t)]}{\Gamma[a(t)]} \sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n] z^n}{\Gamma[b(t)+n] n!} \\ &= \sum_{n=0}^{\infty} \frac{[a(t)]_n z^n}{[b(t)]_n n!} \\ &= 1 + \frac{a}{b} \frac{z}{1!} + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{z^3}{3!} + \dots \quad (7) \end{aligned}$$

where $[a(t)]_n$ and $[b(t)]_n$ are defined as the rising factorial or Pochhammer symbol [11].

When $a(t) > b(t) > 0$ and $-\infty < z < 0$ corresponds to the case of overdamped response in control system theory or linear circuit analysis or electromagnetic wave theory or quantum mechanics etc. from (7) we have

$$\frac{a}{b} \frac{z}{1!} < \frac{z}{1!} \quad (8)$$

$$\frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} > \frac{z^2}{2!} \quad (9)$$

$$\frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{z^3}{3!} < \frac{z^3}{3!} \quad (10)$$

If we were to add (8) through (10) side by side and then apply the total induction theorem, we would get:

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] \leq 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$= e^z, \quad -\infty < z < 0 \quad (11)$$

Where the symbol $\leq$ implies that for same values of z the confluent hypergeometric function, M , is greater than the exponential function, $\exp(z)$, and for some other the former it is smaller than the latter; i.e., it is implied that the confluent hypergeometric function, M , oscillates or waves or swings around the exponential function, $\exp(z)$.

For this case, the proof that the confluent hypergeometric function, M , converges to zero when $z \rightarrow -\infty$ is rather unique.

For this case we use Kummer's first transformation as follows:

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] = e^z M \left[\begin{matrix} b(t) - a(t); \\ b(t); \end{matrix} -z \right] \quad (12)$$

If $b(t) - a(t) = -m$, where m is a positive integer then (12) is reduced to the computation of the product of e^z with a polynomial as follows:

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] = e^z \sum_{n=0}^m \frac{(-1)^{m-n} \binom{m}{n} z^n}{(b(t))_n} \quad (13)$$

If we take the limit as $z \rightarrow -\infty$ and apply L'Hôpital's rule [12] of (13) we obtain

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] &= \lim_{z \rightarrow -\infty} e^z \sum_{n=0}^m \frac{(-1)^{m-n} \binom{m}{n} z^n}{(b(t))_n} \\ &= \sum_{n=0}^m \frac{(-1)^{m-n} \binom{m}{n} \lim_{z \rightarrow -\infty} e^z z^n}{(b(t))_n} = 0 \quad (14) \end{aligned}$$

If $b(t) - a(t) = c(t) < 0 \neq -m$ where m is a positive integer, then the expansion of the M via the incomplete gamma function should be used instead as given in Sect. 4.

When $a(t) > b(t) \gg 1 > 0$ and $-\infty < z < 0$ corresponds to the case of overdamped response in control system theory or linear circuit analysis or electromagnetic wave theory or quantum mechanics etc. from (12) we have

$$\frac{b-a}{b} \frac{(-z)}{1!} < \frac{1}{b} \frac{(-z)}{1!} < \frac{(-z)}{1!} \quad (15)$$

$$\frac{(b-a)(b-a+1)}{b(b+1)} \frac{(-z)^2}{2!} < \frac{1 \times 1}{b(b+1)} \frac{(-z)^2}{2!} < \frac{(-z)^2}{2!} \quad (16)$$

$$\frac{(b-a)(b-a+1)(b-a+2)}{b(b+1)(b+2)} \frac{(-z)^3}{3!} < \frac{1 \times 1 \times 2}{b(b+1)(b+2)} \frac{(-z)^3}{3!} < \frac{(-z)^3}{3!} \quad (17)$$

If we were to add (15) through (17) side by side and then apply the total induction theorem, we would get:

$$\begin{aligned} M \left[\begin{matrix} b(t) - a(t); \\ b(t); \end{matrix} -z \right] &\ll 1 + \frac{1}{b} \frac{(-z)}{1!} + \frac{1 \times 1}{b(b+1)} \frac{(-z)^2}{2!} + \dots \\ &= M \left[\begin{matrix} 1; \\ b(t); \end{matrix} -z \right] \\ &= \frac{(b-1)\gamma(b-1, -z)e^{-z}}{(-z)^{(b-1)}} \quad (18) \end{aligned}$$

Hence, if we take the limit as $z \rightarrow -\infty$ we obtain,

$$\lim_{z \rightarrow -\infty} M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] \ll e^z \frac{(b-1)\gamma(b-1, -z)e^{-z}}{(-z)^{(b-1)}}$$

$$\begin{aligned}
&= \lim_{z \rightarrow -\infty} \frac{(b-1)\gamma(b-1, -z)}{(-z)^{(b-1)}} = 0 \\
&= \lim_{z \rightarrow -\infty} \frac{(b-1)\Gamma(b-1)}{(-z)^{(b-1)}} = 0 \\
&= \Gamma(b) \lim_{z \rightarrow -\infty} \frac{1}{(-z)^{(b-1)}} = 0 \quad (19)
\end{aligned}$$

In fact this limit goes to zero as $z \rightarrow -\infty$.

If we take the derivative of M with respect to z we obtain the following:

$$\begin{aligned}
\frac{dM\left[\begin{smallmatrix} a(t); \\ b(t); \end{smallmatrix} z\right]}{dz} &= \frac{a}{b} + \frac{a(a+1)}{b(b+1)} \frac{z}{1!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{z^2}{2!} + \dots \\
&= \frac{a}{b} \left[1 + \frac{(a+1)}{(b+1)} \frac{z}{1!} + \frac{(a+1)(a+2)}{(b+1)(b+2)} \frac{z^2}{2!} + \dots \right] \\
&= \frac{a}{b} M\left[\begin{smallmatrix} a(t) + 1; \\ b(t) + 1; \end{smallmatrix} z\right] \quad (20)
\end{aligned}$$

If we were to compare and contrast (20) with the derivative of the exponential function which is also an exponential function, we get more insight about the nature of the exponential function and of the confluent hypergeometric function, M .

For the exponential function and for its derivative we get that as $z \rightarrow -\infty$

$$\lim_{z \rightarrow -\infty} e^z = \lim_{z \rightarrow -\infty} \frac{de^z}{dz} = 0 \quad (21)$$

both the exponential function and its derivative decrease by the same amount and stop decreasing as $z \rightarrow -\infty$.

In the case of the confluent hypergeometric function, M , although its derivative, M' , is not exactly the same as the function itself but it is of the same nature as the function, M , meaning that as the function decreases as $z \rightarrow -\infty$ so does its derivative.

Although a more formal proof may be necessary to convince the sceptics, for this case, if we were to take the limit (4) we would get:

$$\lim_{z \rightarrow -\infty} M\left[\begin{smallmatrix} a(t); \\ b(t); \end{smallmatrix} z\right] = e^{(z \rightarrow -\infty)} = 0 \quad (22)$$

When $a(t) > b(t) > 0$ and then $z > 0$ then from (7) we have

$$\frac{a}{b} \frac{z}{1!} > \frac{z}{1!} \quad (23)$$

$$\frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} > \frac{z^2}{2!} \quad (24)$$

$$\frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{z^3}{3!} > \frac{z^3}{3!} \quad (25)$$

If we were to add (23) through (25) side by side and then apply the total induction theorem, we would get:

$$M\left[\begin{smallmatrix} a(t); \\ b(t); \end{smallmatrix} z\right] > 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$= e^z, \quad 0 < z < \infty \quad (26)$$

For this case if we were to take the limit (5) we would get

$$\lim_{z \rightarrow +\infty} M\left[\begin{smallmatrix} a(t); \\ b(t); \end{smallmatrix} z\right] > e^{(z \rightarrow +\infty)} = \infty \quad (27)$$

The derivative argument could have been used and we would have arrived at exactly the same result as in (27).

Since, we have determined that the irregular singularities the confluent hypergeometric function, M , for $a(t) > b(t) > 0$; it remains to do the same when $b(t) > a(t) > 0$.

When $b(t) > a(t) > 0$ and $-\infty < z < 0$ corresponds to the case of underdamped response; hence,

$$M\left[\begin{smallmatrix} a(t); \\ b(t); \end{smallmatrix} z\right] > e^z, \quad -\infty < z < 0 \quad (28)$$

However, if we take the limit when $z \rightarrow -\infty$ then we have the same result as in (16).

What happens when $b(t) > a(t) > 0$ and $0 < z < \infty$ corresponds to the case of a late response, i.e., the response grows behind the natural response. However, since we are interested in the value of the confluent hypergeometric function, M , as $z \rightarrow +\infty$, we will use the method of the derivative to arrive at the desired result.

If we compare and contrast (20) with the derivative of the natural logarithm function, $d \ln z / dz = 1/z$; it is clear that as $z \rightarrow +\infty$ the following is true:

$$\frac{dM\left[\begin{smallmatrix} a(t); \\ b(t); \end{smallmatrix} z\right]}{dz} = \frac{a}{b} M\left[\begin{smallmatrix} a(t) + 1; \\ b(t) + 1; \end{smallmatrix} z\right] \rightarrow \infty \gg 1/z \rightarrow 0 \quad (29)$$

Since the derivative of the confluent hypergeometric function, M' , grows much faster than the derivative of the natural logarithm function, $1/z$; hence, at some point in the z domain M will surpass and leave the $\ln z$ behind as illustrated in the numerical example section; therefore, the following will hold:

$$\lim_{z \rightarrow +\infty} M\left[\begin{smallmatrix} a(t); \\ b(t); \end{smallmatrix} z\right] \gg \ln z = \infty \quad (30)$$

We have analytically demonstrated that the irregular singularity points of the confluent hypergeometric function, M , are exactly the same as those of the exponential function for the values of the parameters discussed in Sect. 2. In the numerical examples section, we illustrate the physical interpretation of the confluent hypergeometric function, M .

4 Computation via the Incomplete Gamma function

The expansion of the confluent hypergeometric functions, M , via the incomplete gamma function [4] is not listed in the Tables of Integrals, Products, and Series [13], [14]. I have found,

however, a unique expansion of the confluent hypergeometric functions, M , via the incomplete gamma function by Muller, (2001, [8]).

According to Muller, (2001, [8]) when $b(t) > a(t) > 0$, $[a(t) \neq \text{Integer}^i]$ and $z > 0$ the following holds,

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] = e^z \frac{\Gamma(b)z^{-(b-a)}}{\Gamma(a)\Gamma(b-a)} \sum_{k=0}^{\infty} \frac{(1-a)_k \gamma(k+b-a, z)}{z^k k!} \quad (31)$$

Not only is the Muller expansion, (2001, [8]), important but there are several special cases that can be derived from the Muller expansion, (2001, [8]).

Before we can discuss the special cases, we exploit the Pochhammer (or the rising factorial) term [11] in (31) as follows:

$$\begin{aligned} (1-a)_k &= \begin{cases} 1, & k=0 \\ (1-a)(2-a) \cdots (k-a), & k \neq 0 \end{cases} \\ &= \prod_{i=1}^k (i-a), \quad k \neq 0 \end{aligned} \quad (32)$$

Clearly, if we set the left-hand side of (32) equal to zero we get

$$(1-a)_k = 0 \Leftrightarrow a = i, \forall i \in \{1, 2, \dots, k\} \quad (33)$$

Therefore, when $a(t)$ is a positive integer corresponds to a zero in the Pochhammer (or the rising factorial) term [11] in (31) that needs to be eliminated before we can perform the computation of the confluent hypergeometric function M via in (31) using maneuvering as beautifully illustrated in Progi 2016 [7], to eliminate the singularities for $a(t) = i, \forall i \in \{1, 2, \dots, k\}$. It turns out that this is a recursive process; i.e., the solution obtained for $a(t) = 1$ will be applied to the solution obtained for $a(t) = 2$, and so on and so forth. The detailed algorithm for this especially important computation is given below:

1. Let us start to obtain the solution when $a(t) = 1$. Equation (31) cannot be used directly for $a(t) = 1$ because it will lead to a singularity. We can rewrite (31) using maneuvering, as beautifully illustrated in Progi 2016 [7], to eliminate the singularity for $a(t) = 1$ as follows:

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] = \frac{\Gamma(b) \left[\gamma(b-a, z) + \frac{(1-a) \sum_{k=1}^{\infty} \frac{\Gamma(1-a+k) \gamma(k+b-a, z)}{z^k k!}}{\Gamma(2-a)} \right]}{z^{(b-a)} e^{-z} \Gamma(a) \Gamma(b-a)} \quad (34)$$

Next, we substitute $a(t) = 1$ in (34) then the following yields,

$$\begin{aligned} M \left[\begin{matrix} 1; \\ b(t); \end{matrix} z \right] &= e^z \Gamma(b) z^{1-b} \frac{\gamma(b-1, z)}{\Gamma(b-1)}; \\ &= e^z \Gamma(b) z^{1-b} P(b-1, z); \\ &= M_1[b(t), z]^{\text{ii}}; \quad b(t) > 1 \end{aligned} \quad (35)$$

where:

$$M_i[b(t), z] = e^z \Gamma(b) z^{(i-b)} P(b-i, z) \quad (36)$$

If we were to employ $b = s + 1$ in (36) then we get an identical expression in [4] (see Connection with Kummer's confluent hypergeometric function).

Similarly, to obtain the expression when $a(t) = 2$ we have to manipulate (34) to enable maneuvering as follows:

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] = \frac{\Gamma(b) \left[\gamma(b-a, z) + \frac{(1-a) \gamma(1+b-a, z)}{z} + \frac{(1-a)(2-a) \sum_{k=2}^{\infty} \frac{\Gamma(1-a+k) \gamma(k+b-a, z)}{z^k k!}}{\Gamma(3-a)} \right]}{z^{(b-a)} e^{-z} \Gamma(a) \Gamma(b-a)} \quad (37)$$

Next, we substitute $a(t) = 2$ in (37) then the following yields,

$$\begin{aligned} M \left[\begin{matrix} 2; \\ b(t); \end{matrix} z \right] &= \frac{\Gamma(b) \left[\gamma(b-2, z) - \frac{\gamma(b-1, z)}{z} \right]}{z^{(b-2)} e^{-z} \Gamma(2) \Gamma(b-2)} \\ &= e^z \Gamma(b) z^{(2-b)} \left[\frac{\gamma(b-2, z)}{\Gamma(b-2)} - z^{-1} \frac{(b-2) \gamma(b-1, z)}{\Gamma(b-1)} \right] \\ &= e^z \Gamma(b) \left[\frac{P(b-2, z)}{z^{(b-2)}} - \frac{(b-2) P(b-1, z)}{z^{(b-1)}} \right] \\ &= \frac{e^z \Gamma(b) P(b-2, z)}{z^{(b-2)}} - \frac{(b-2) e^z \Gamma(b) P(b-1, z)}{z^{(b-1)}} \\ &= \frac{e^z \Gamma(b) P(b-2, z)}{z^{(b-2)}} - (b-2) M \left[\begin{matrix} 1; \\ b(t); \end{matrix} z \right] \\ &= M_2[b(t), z] - (b-2) M_1[b(t), z]^{\text{iii}} \end{aligned} \quad (38)$$

If (36) was already published in [4], I have not seen (38) published anywhere else. Moreover, (38) is computed recursively as stated earlier.

Similarly, to obtain the expression when $a(t) = 3$ we have to manipulate (34) to enable maneuvering as follows:

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] = \frac{\Gamma(b) \left[\gamma(b-a, z) + \frac{(1-a) \gamma(1+b-a, z)}{z} + \frac{(1-a)(2-a) \gamma(2+b-a, z)}{z^2} + \frac{(1-a)(2-a)(3-a) \sum_{k=3}^{\infty} \frac{\Gamma(1-a+k) \gamma(k+b-a, z)}{z^k k!}}{\Gamma(4-a)} \right]}{z^{(b-a)} e^{-z} \Gamma(a) \Gamma(b-a)} \quad (39)$$

Next, we substitute $a(t) = 3$ in (39) then the following yields,

$$\begin{aligned} M \left[\begin{matrix} 3; \\ b(t); \end{matrix} z \right] &= \frac{\Gamma(b) \left[\gamma(b-3, z) - \frac{2\gamma(b-2, z)}{z} + \frac{\gamma(b-1, z)}{z^2} \right]}{z^{(b-3)} e^{-z} \Gamma(3) \Gamma(b-3)} \\ &= \frac{\Gamma(b)}{z^{(b-3)} e^{-z}} \left[\frac{\gamma(b-3, z)}{\Gamma(b-3)} - \frac{\gamma(b-2, z)}{z \Gamma(b-3)} + \frac{\gamma(b-1, z)}{z^2 \Gamma(b-3)} \right] \\ &= \frac{\Gamma(b)}{z^{(b-3)} e^{-z}} \left[\frac{P(b-3, z)}{2} - \frac{(b-3) P(b-2, z)}{z} + \frac{(b-2)(b-3) P(b-1, z)}{z^2} \right] \\ &= \left[\frac{1}{2} \frac{\Gamma(b) P(b-3, z)}{z^{(b-3)} e^{-z}} - \frac{(b-3) \Gamma(b) P(b-2, z)}{z^{(b-2)} e^{-z}} + \frac{(b-2)(b-3) \Gamma(b) P(b-1, z)}{2 z^{(b-1)} e^{-z}} \right] \end{aligned}$$

$$= \left[\frac{M_3[b(t), z]}{2} - (b-3)M_2[b(t), z] + \frac{(b-2)(b-3)}{2}M_1[b(t), z] \right]_{iv} \quad (40)$$

Equation (40) is computed recursively. Moreover, I have not seen it published anywhere else.

Similarly, to obtain the expression when $a(t) = n = \mathbb{N}$, and $b(t) > a(t)$ we have to exploit maneuvering to manipulate (40) as follows:

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] = \frac{\Gamma(b) \left[\frac{\gamma(b-a, z) + \frac{(1-a)\gamma(1+b-a, z)}{z} + \frac{(1-a)(2-a)\gamma(2+b-a, z)}{z^2} + \dots + \frac{(1-a)\dots(n-a)\sum_{k=n}^{\infty} \frac{\Gamma(1-a+k)\gamma(k+b-a, z)}{z^k k!}}{\Gamma(n+1-a)} \right]}{z^{(b-a)}e^{-z}\Gamma(a)\Gamma(b-a)} \quad (41)$$

Next, we substitute $a(t) = n$ in (41) then the following yields,

$$\begin{aligned} M \left[\begin{matrix} n; \\ b(t); \end{matrix} z \right] &= \frac{\Gamma(b) \left[\frac{\gamma(b-n, z) + \frac{(1-n)\gamma(1+b-n, z)}{z} + \frac{(1-n)\dots(-1)\gamma(b-1, z)}{z^{n-1}(n-1)!} \right]}{z^{(b-n)}e^{-z}\Gamma(n)\Gamma(b-n)} \\ &= \frac{\Gamma(b)}{z^{(b-n)}e^{-z}} \left[\frac{\gamma(b-n, z)}{\Gamma(n)\Gamma(b-n)} + \frac{(1-n)\gamma(b-n+1, z)}{z\Gamma(n)\Gamma(b-n)} + \dots + \frac{(-1)^{n-1}\gamma(b-1, z)}{z^{n-1}\Gamma(n)\Gamma(b-n)} \right] \\ &= \frac{\Gamma(b)}{z^{(b-n)}e^{-z}} \left[\frac{P(b-n, z)}{\Gamma(n)} + \frac{(1-n)(b-n)P(b-n+1, z)}{z\Gamma(n)} + \dots + \frac{(-1)^{n-1}(b-n)\dots(b-2)P(b-1, z)}{\Gamma(n)z^{n-1}} \right] \\ &= \left[\frac{1}{\Gamma(n)} \frac{\gamma(b-n, z)}{z^{(b-n)}e^{-z}} + \frac{(1-n)(b-n)\gamma(b-n+1, z)}{\Gamma(n)z^{(b-n+1)}e^{-z}} + \dots + \frac{(b-n)\dots(b-2)P(b-1, z)\Gamma(b)}{(-1)^{1-n}\Gamma(n)z^{(b-1)}e^{-z}} \right] \\ &= \left[\frac{M_n[b(t), z]}{\Gamma(n)} + \frac{(1-n)(b-n)M_{n-1}[b(t), z]}{\Gamma(n)} + \dots + \frac{(b-n)\dots(b-2)M_1[b(t), z]}{(-1)^{1-n}\Gamma(n)} \right]_{iv} \quad (42) \end{aligned}$$

Equation (42) is computed recursively, and I have not seen it published anywhere else.

If we substitute $n = \{1, 2, 3\}$ in (42) we obtain (36), (38), and (40) respectively.

Similarly, we can employ a similar transformation from Muller expansion, (2001, [8]) to obtain the expansion of the M for negative values of z as follows

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} -z \right] = \frac{\Gamma(b) \left[\gamma(a, z) + \frac{(a-b+1)\sum_{k=1}^{\infty} \frac{\Gamma(a-b+1+k)\gamma(k+a, z)}{z^k k!}}{\Gamma(a-b+2)} \right]}{z^a\Gamma(b-a)\Gamma(a)} \quad (43)$$

Next, we substitute $a(t) = 1$ in (43) then the following yields,

$$M \left[\begin{matrix} 1; \\ b(t); \end{matrix} -z \right] = \frac{(b-1) \left[\gamma(1, z) + \frac{(2-b)\sum_{k=1}^{\infty} \frac{\Gamma(2-b+k)\gamma(k+1, z)}{z^k k!}}{\Gamma(3-b)} \right]}{z} \quad (44)$$

Equation (44) is valid only when $b(t) \neq \mathbb{N}$

If $b(t) = 2$ then (44) is simply:

$$M \left[\begin{matrix} 1; \\ 2; \end{matrix} -z \right] = \frac{\gamma(1, z)}{z}; \quad z > 0 \quad (45)$$

If $b(t) = 3$ then (44) is simply:

$$M \left[\begin{matrix} 1; \\ 3; \end{matrix} -z \right] = \frac{2[\gamma(1, z) - \frac{\gamma(2, z)}{z}]}{z}; \quad z > 0 \quad (46)$$

One can obtain more identities using the same techniques as discussed earlier.

2. When $b(t) = a(t) + 1$ the following holds

$$\Gamma[b(t) - a(t)] = \Gamma(1) = 1 \quad (47)$$

$$x^{-[b(t)-a(t)]} = x^{-1} \quad (48)$$

$$\Gamma[b(t)] = a(t)\Gamma[a(t)] = 1 \quad (49)$$

$$\gamma(k + b - a, z) = \gamma(k + 1, z) \quad (50)$$

Substituting (47) through (50) into (31) the following holds

$$M \left[\begin{matrix} a(t); \\ a(t) + 1; \end{matrix} z \right] = e^z a z^{-1} \sum_{k=0}^{\infty} \frac{(1-a)_k \gamma(k+1, z)}{z^k k!} \quad (51)$$

Note that (51) is different from Olver et al. 2010, [13] pg. 328 ex. 13.6.5)

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ a(t) + 1; \end{matrix} -z \right] &= e^{-z} M \left[\begin{matrix} 1; \\ a(t) + 1; \end{matrix} z \right] \\ &= a(t) z^{-a(t)} \gamma[a(t), z] \quad (52) \end{aligned}$$

If we compare and contrast (51) with (52) a new identity is obtained for $a(t) \neq 1 > 0$:

$$M \left[\begin{matrix} 1; \\ a(t) + 1; \end{matrix} -z \right] = a z^{-1} \sum_{k=0}^{\infty} \frac{(1-a)_k \gamma(k+1, z)}{z^k k!} \quad (53)$$

Equation (51) cannot be used directly for values of $a(t) = 1$. For that we need to apply maneuvering technique as beautifully illustrated in Progi 2016 [7]; hence, (51) can be written as

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ a(t) + 1; \end{matrix} z \right] &= \frac{e^z a z^{-1} \sum_{k=0}^{\infty} \frac{\Gamma(1-a+k)\gamma(k+1, z)}{z^k k!}}{\Gamma(1-a)} \\ &= \frac{e^z a z^{-1} \sum_{k=0}^{\infty} \frac{\Gamma(1-a+k)\gamma(k+1, z)}{z^k k!}}{\Gamma(1-a)} \\ &= e^z a \frac{\gamma(1, z) + \frac{(1-a)\sum_{k=1}^{\infty} \frac{\Gamma(1-a+k)\gamma(k+1, z)}{z^k k!}}{\Gamma(2-a)}}{z} \quad (54) \end{aligned}$$

If we substitute $a(t) = 1$ in (54) we obtain:

$$\begin{aligned} M \left[\begin{matrix} 1; \\ 2; \end{matrix} z \right] &= e^z z^{-1} \gamma(1, z) \\ &= e^z z^{-1} (1 - e^{-z}) \\ &= z^{-1} (e^z - 1); \quad z \neq 0 \quad (55) \end{aligned}$$

Equation (55) is identical to what is given in [1] which validates the method of maneuvering.

3. What is the correct expansion when $b(t) = 2 > a(t) \neq 1$ and $z > 0$? From the Euler integral representation, we have

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ 2; \end{matrix} z \right] &= \frac{\Gamma(2)}{\Gamma(a)\Gamma(2-a)} \int_0^1 e^{-zt} t^{a-1} (1-t)^{1-a} dt \\ &= \frac{\Gamma(2)}{\Gamma(a)\Gamma(2-a)} \int_0^1 e^{-zt} t^{a-1} (1-t)^{1-a} dt \end{aligned} \quad (56)$$

Next, we expand the function $(1-t)^{1-a}$ using the generalized binomial expansion as follows:

$$\begin{aligned} (1-t)^{1-a} &= \sum_{k=0}^{\infty} \binom{1-a}{k} (-t)^k \\ &= \sum_{k=0}^{\infty} (a-1)_k (-1)^k \frac{(-1)^k t^k}{k!} \\ &= \sum_{k=0}^{\infty} (a-1)_k \frac{t^k}{k!} \end{aligned} \quad (57)$$

Next, substituting (57) into (56) the following is obtained,

$$M \left[\begin{matrix} a(t); \\ 2; \end{matrix} z \right] = \frac{\Gamma(2)}{\Gamma(a)\Gamma(2-a)} \int_0^1 e^{-zt} t^{a-1} \sum_{k=0}^{\infty} (a-1)_k \frac{t^k}{k!} dt \quad (58)$$

Next, changing the order of summation and integration yields,

$$M \left[\begin{matrix} a(t); \\ 2; \end{matrix} z \right] = \frac{\Gamma(2)}{\Gamma(a)\Gamma(2-a)} \sum_{k=0}^{\infty} (a-1)_k \frac{\int_0^1 e^{-zt} t^{a+k-1} dt}{k!} \quad (59)$$

Making the substitution $zt = y$, $zdt = dy$, $t \rightarrow \frac{y}{z}$, $\Rightarrow y \rightarrow \frac{z}{z}$ produces,

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ 2; \end{matrix} z \right] &= \frac{\Gamma(2)}{\Gamma(a)\Gamma(2-a)} \sum_{k=0}^{\infty} \frac{(a-1)_k}{z^{a+k-2}} \frac{\int_0^z y^{a+k-1} e^{-y} dy}{k!} \\ &= \frac{\Gamma(2)}{\Gamma(a)\Gamma(2-a)} \sum_{k=0}^{\infty} \frac{(a-1)_k}{z^{a+k-2}} \frac{\gamma(a+k, z)}{k!} \end{aligned} \quad (60)$$

Equation (60) is a new original expression not published anywhere else. It is only true when $b(t) = 2 > a(t) \neq 1$. Equation (60) cannot be derived by plugging into the Muller's (2001, [8]) (4.6) $b(t) = 2$ because then we would have

$$M \left[\begin{matrix} a(t); \\ 2; \end{matrix} z \right] = e^z \frac{\Gamma(2)z^{-(2-a)}}{\Gamma(a)\Gamma(2-a)} \sum_{k=0}^{\infty} \frac{(1-a)_k}{z^k} \frac{\gamma(k+2-a, z)}{k!} \quad (61)$$

Clearly, (61) is different from (60).

Finally, let us discuss the computation of the irregular singularities

First, $b(t) > a(t) > 0$, $[a(t) \neq \text{Integer}]$ and $z < 0$

$$\begin{aligned} \lim_{z \rightarrow -\infty} M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] &= \lim_{z \rightarrow +\infty} M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} -z \right] \\ &= \frac{\Gamma(b) \left[\lim_{z \rightarrow +\infty} \frac{\gamma(a, z)}{z^a} + \frac{(a-b+1)}{\Gamma(a-b+2)} \sum_{k=1}^{\infty} \frac{\Gamma(a-b+1+k)}{k!} \lim_{z \rightarrow +\infty} \frac{\gamma(k+a, z)}{z^{a+k}} \right]}{\Gamma(b-a)\Gamma(a)} \\ &= \frac{\Gamma(b) \left[\lim_{z \rightarrow +\infty} \frac{\Gamma(a)}{z^a} + \frac{(a-b+1)}{\Gamma(a-b+2)} \sum_{k=1}^{\infty} \frac{\Gamma(a-b+1+k)}{k!} \lim_{z \rightarrow +\infty} \frac{\Gamma(k+a)}{z^{a+k}} \right]}{\Gamma(b-a)\Gamma(a)} \end{aligned}$$

$$\begin{aligned} &= \frac{\Gamma(b) \left[\lim_{z \rightarrow +\infty} \frac{1}{z^a} + \sum_{k=1}^{\infty} \frac{(a-b+1)_k \Gamma(a)_k}{k!} \lim_{z \rightarrow +\infty} \frac{1}{z^{a+k}} \right]}{\Gamma(b-a)} \\ &= 0 + 0 = 0 \end{aligned} \quad (62)$$

Second, when $b(t) > a(t) > 0$, $[a(t) \neq \text{Integer}]$ and $z > 0$ and if we were to apply the L'Hôpital's rule [12]

$$\begin{aligned} \lim_{z \rightarrow +\infty} M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} z \right] &= \frac{\Gamma(b) \sum_{k=0}^{\infty} \frac{(1-a)_k}{k!} \lim_{z \rightarrow +\infty} \frac{e^z \gamma(k+b-a, z)}{z^{b-a+k}}}{\Gamma(a)\Gamma(b-a)} \\ &= \frac{\Gamma(b) \sum_{k=0}^{\infty} \frac{(1-a)_k \Gamma(k+b-a)}{k!} \lim_{z \rightarrow +\infty} \frac{e^z}{z^{b-a+k}}}{\Gamma(a)\Gamma(b-a)} \\ &= \frac{\Gamma(b) \sum_{k=0}^{\infty} \frac{(1-a)_k (b-a)_k \Gamma(b-a)}{k!} \lim_{z \rightarrow +\infty} \frac{e^z}{z^{b-a+k}}}{\Gamma(a)\Gamma(b-a)} \\ &= \frac{\Gamma(b) \sum_{k=0}^{\infty} \frac{(1-a)_k (b-a)_k}{k!} \lim_{z \rightarrow +\infty} \frac{e^z}{z^{b-a+k}}}{\Gamma(a)} = \infty \end{aligned} \quad (63)$$

We have shown that the irregular singularities of the confluent hypergeometric function are the same as when we compute them using the Gauss power series in Sect. 3.

This concludes the derivations of the computation of the confluent hypergeometric function via the incomplete gamma function. Next, we consider the computation of the irregular singularities of the confluent hypergeometric function for several special cases.

5 Special Cases

Now, that we have considered the general formulation and the solution of the problem let us illustrate them using several special cases. The special cases include the exponential function, the elementary function, the Bessel function, the incomplete gamma function, and the error function.

1. *Exponential Function:* Of particular importance is the special case when $0 < a(t) = b(t) < \infty$. For this special case the confluent hypergeometric function, M , reduces to an exponential function [1] as follows:

$$M \left[\begin{matrix} a(t); \\ b(t) \equiv a(t); \end{matrix} z \right] = e^z \quad (64)$$

As $z \rightarrow -\infty$ we get from the right-hand side and the left-hand side the following

$$\lim_{z \rightarrow -\infty} M \left[\begin{matrix} a(t); \\ a(t); \end{matrix} z \right] = \lim_{z \rightarrow -\infty} e^z = 0 \quad (65)$$

As $z \rightarrow 0$ we get from the left-hand side we obtain the following

$$\lim_{z \rightarrow 0} M \left[\begin{matrix} a(t); \\ a(t); \end{matrix} z \right] = \lim_{z \rightarrow 0} e^z = 1 \quad (66)$$

As $z \rightarrow +\infty$ we get from the right-hand side and the left-hand side the following

$$\lim_{z \rightarrow +\infty} M \left[\begin{matrix} a(t); \\ a(t); \end{matrix} z \right] = \lim_{z \rightarrow +\infty} e^z = +\infty \quad (67)$$

2. *Elementary Functions.* For example, for the special case when $a(t) = 1$ and $b(t) = 2$, the confluent hypergeometric function, M , reduces to an elementary function [1] as follows:

$$M \left[\begin{matrix} 1; \\ 2; \end{matrix} z \right] = \frac{e^z - 1}{z}; \quad z \neq 0 \quad (68)$$

As $z \rightarrow -\infty$ we get from the right-hand side and the left-hand side the following

$$\lim_{z \rightarrow -\infty} M \left[\begin{matrix} 1; \\ 2; \end{matrix} z \right] = \lim_{z \rightarrow -\infty} \frac{e^z - 1}{z} = 0 \quad (69)$$

As $z \rightarrow 0$ we get from the left-hand side we obtain the following

$$\lim_{z \rightarrow 0} M \left[\begin{matrix} 1; \\ 2; \end{matrix} z \right] = 1 \quad (70)$$

From the right-hand side we obtain the following irregular singularity

$$\lim_{z \rightarrow 0} \frac{e^z - 1}{z} = \frac{0}{0} = \text{NaN} \quad (71)$$

This singularity can be easily eliminated by means of the L'Hôpital's rule [12] as follows:

$$\lim_{z \rightarrow 0} \frac{e^z - 1}{z} = \lim_{z \rightarrow 0} \frac{\frac{d}{dz} e^z - 1}{\frac{d}{dz} z} = \lim_{z \rightarrow 0} e^z = 1 \quad (72)$$

As $z \rightarrow +\infty$ we get from the right-hand side we obtain the following

$$\lim_{z \rightarrow +\infty} M \left[\begin{matrix} 1; \\ 2; \end{matrix} z \right] = \frac{e^z - 1}{z} = \frac{\infty}{\infty} = \text{NaN} \quad (73)$$

Similarly to (71), this singularity can be eliminated by means of the L'Hôpital's rule [12] as follows:

$$\lim_{z \rightarrow +\infty} \frac{e^z - 1}{z} = \lim_{z \rightarrow +\infty} \frac{\frac{d}{dz} e^z - 1}{\frac{d}{dz} z} = \lim_{z \rightarrow +\infty} e^z = +\infty \quad (74)$$

Hence, for this elementary function we have clearly determined that the confluent hypergeometric function singularities follow the $M[1, 2; z]$ are identical to the exponential function singularities for $z = \{-\infty, 0, +\infty\}$.

3. *The Bessel Functions.* For example, for the special case when $b(t) = 2a(t)$, the confluent hypergeometric function reduces to a Bessel function of the first kind [1] as follows:

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ 2a(t); \end{matrix} z \right] &= e^{\frac{z}{2}} {}_1F_1 \left[\begin{matrix} -; \\ a(t) + \frac{1}{2}; \end{matrix} \frac{z^2}{16} \right] \\ &= e^{\frac{z}{2}} \left(\frac{z}{4} \right)^{\frac{1}{2} - a(t)} \Gamma \left[a(t) + \frac{1}{2} \right] I_{a(t) - \frac{1}{2}} \left(\frac{z}{2} \right) \end{aligned} \quad (75)$$

where I is the Bessel function of the first kind (see Gradshteyn,

Ryzhik, 2007 [14] pg. 919 ex. 8.445) given by:

$$I_{a(t) - \frac{1}{2}} \left(\frac{z}{2} \right) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{4} \right)^{a(t) - \frac{1}{2} + 2k}}{k! \Gamma \left[a(t) - \frac{1}{2} + k + 1 \right]} \quad (76)$$

Next, substituting (76) into (75) yields,

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ 2a(t); \end{matrix} z \right] &= \frac{e^{\frac{z}{2}} \Gamma \left[a(t) + \frac{1}{2} \right] \sum_{k=0}^{\infty} \frac{\left(\frac{z}{4} \right)^{a(t) - \frac{1}{2} + 2k}}{k! \Gamma \left[a(t) - \frac{1}{2} + k + 1 \right]}}{\left(\frac{z}{4} \right)^{a(t) - \frac{1}{2}}} \\ &= e^{\frac{z}{2}} \Gamma \left[a(t) + \frac{1}{2} \right] \sum_{k=0}^{\infty} \frac{\left(\frac{z}{4} \right)^{2k}}{k! \Gamma \left[a(t) + \frac{1}{2} + k \right]} \\ &= e^{\frac{z}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{4} \right)^{2k}}{\left[a(t) + \frac{1}{2} \right]_k k!} \\ &= e^{\frac{z}{2}} M \left[\begin{matrix} -; \\ a(t) + \frac{1}{2}; \end{matrix} \frac{z^2}{16} \right] \end{aligned} \quad (77)$$

As $z \rightarrow -\infty$ we get from the right-hand side and the left-hand side the following

$$\lim_{z \rightarrow -\infty} M \left[\begin{matrix} a(t); \\ 2a(t); \end{matrix} z \right] = \text{NaN} \quad (78)$$

$$\lim_{z \rightarrow -\infty} e^{\frac{z}{2}} M \left[\begin{matrix} -; \\ a(t) + \frac{1}{2}; \end{matrix} \frac{z^2}{16} \right] = 0 \times \text{NaN} = \text{NaN} \quad (79)$$

In order to evaluate this limit, we have to use the approximated formula from ([1] and Olver et al. 2010, [13] pg. 323 ex. 13.2.23) for $a(t) \neq 0, -1, -2, \dots$

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ 2a(t); \end{matrix} z \right] &\sim \Gamma[2a(t)] \left\{ \frac{e^{\frac{z}{2}} z^{a(t) - 2a(t)}}{\Gamma[a(t)]} + \frac{(-z)^{-a(t)}}{\Gamma[2a(t) - a(t)]} \right\} \\ &= \frac{\Gamma[2a(t)] [e^{\frac{z}{2}} z^{-a(t)} + (-z)^{-a(t)}]}{\Gamma[a(t)]} \end{aligned} \quad (80)$$

As $z \rightarrow -\infty$ from (80) we obtain the following:

$$\lim_{z \rightarrow -\infty} M \left[\begin{matrix} a(t); \\ 2a(t); \end{matrix} z \right] = \lim_{z \rightarrow -\infty} \frac{\Gamma[2a(t)] [e^{\frac{z}{2}} z^{-a(t)} + (-z)^{-a(t)}]}{\Gamma[a(t)]} = 0 \quad (81)$$

Equation (80) is exactly the answer we were supposed to get.

However, from MATLAB we get NaN from the left-hand side. Therefore, since we know what the answer is MATLAB here is wrong.

Next, we evaluate the case when $z \rightarrow 0$ we get from the left-hand side the following:

$$\lim_{z \rightarrow 0} M \left[\begin{matrix} a(t); \\ 2a(t); \end{matrix} z \right] = 1 \quad (82)$$

From the right-hand side we obtain the following irregular singularity

$$\lim_{z \rightarrow 0} e^{\frac{z}{2}} M \left[\begin{matrix} -; \\ a(t) + \frac{1}{2}; \end{matrix} \frac{z^2}{16} \right] = 1 \times 1 = 1 \quad (83)$$

For $0 \leq a(t) < \infty$ as $z \rightarrow +\infty$ then from the right side of

(75) we have

$$\lim_{z \rightarrow +\infty} e^{\frac{z}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{4}\right)^{2k}}{\left[a(t) + \frac{1}{2}\right]_k k!} = \infty \quad (84)$$

However, from MATLAB we get NaN from the left-hand side. Therefore, since we know what the answer is MATLAB here is wrong.

4. *The Incomplete Gamma Functions.* This case occurs when $b(t) - a(t)$ is a positive integer or $a(t)$ is a positive integer we have from Kummer's first transformation [1]:

$$M \left[\begin{matrix} a(t); \\ b(t); \end{matrix} -z \right] = e^{-z} M \left[\begin{matrix} b(t) - a(t); \\ b(t); \end{matrix} z \right] \quad (85)$$

As $-z \rightarrow -\infty$ we get from the right-hand side and the left-hand side the following

$$M \left[\begin{matrix} a(t); \\ a(t) + 1; \end{matrix} -z \right] = e^{-z} M \left[\begin{matrix} 1; \\ a(t) + 1; \end{matrix} z \right] = a(t) z^{-a(t)} \gamma[a(t), z] \quad (86)$$

As $-z \rightarrow -\infty$ we get from the right-hand side and the left-hand side the following

$$\lim_{z \rightarrow +\infty} M \left[\begin{matrix} a(t); \\ a(t) + 1; \end{matrix} -z \right] = \text{NaN} \quad (87)$$

$$\lim_{z \rightarrow +\infty} e^{-z} M \left[\begin{matrix} 1; \\ a(t) + 1; \end{matrix} z \right] = 0 \times \text{NaN} = \text{NaN} \quad (88)$$

However, from (86) we obtain,

$$\lim_{z \rightarrow +\infty} a(t) z^{-a(t)} \gamma[a(t), z] = a(t) \times 0 \times 1 = 0 \quad (89)$$

This means that the following limit should be equal to zero; as shown below:

$$\lim_{z \rightarrow -\infty} M \left[\begin{matrix} a(t); \\ a(t) + 1; \end{matrix} z \right] = 0, \quad \forall a(t) \in \mathbb{N} \quad (90)$$

Next, we evaluate the case when $z \rightarrow 0$ we get from the left-hand side the following:

$$\lim_{z \rightarrow 0} M \left[\begin{matrix} a(t); \\ a(t) + 1; \end{matrix} z \right] = 1 \quad (91)$$

$$\lim_{z \rightarrow 0} e^{-z} M \left[\begin{matrix} 1; \\ a(t) + 1; \end{matrix} z \right] = 1 \times 1 = 1 \quad (92)$$

From the right-hand side of (86) we obtain the following irregular singularity

$$\lim_{z \rightarrow 0} a(t) z^{-a(t)} \gamma[a(t), z] = \frac{0}{0} = \text{NaN} \quad (93)$$

Hence, the righthand side can only be employed for $z \neq 0$.

Next, before we evaluate the limit when $z \rightarrow +\infty$, we look at the following function:

$$\begin{aligned} M \left[\begin{matrix} a(t); \\ a(t) + 1; \end{matrix} z \right] &= 1 + \frac{a}{a+1} \frac{z}{1!} + \frac{a}{a+2} \frac{z^2}{2!} + \frac{a}{a+3} \frac{z^3}{3!} + \dots \\ &= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots - \frac{1}{a+1} \frac{z}{1!} - \frac{1}{a+2} \frac{z^2}{2!} - \frac{1}{a+3} \frac{z^3}{3!} - \dots \\ &= e^z - z \left[\frac{1}{a+1} \frac{1}{1!} + \frac{1}{a+2} \frac{z}{1!} + \frac{1}{a+3} \frac{z^2}{2!} + \dots \right] \end{aligned} \quad (94)$$

As seen in (94) the confluent hypergeometric function M is smaller than the exponential function, which corresponds to the underdamped response. The larger the $a(t)$ the closer M gets to the exponential function. Only if $a(t) \rightarrow \infty$ then $M \rightarrow 1$. For all other values of $a(t) < \infty$, we have the following:

$$\lim_{z \rightarrow +\infty} M \left[\begin{matrix} a(t); \\ a(t) + 1; \end{matrix} z \right] = +\infty \quad (95)$$

5. *The Error Functions.* This case occurs when $a(t) = 1/2$ is and $b(t) = 3/2$ we have:

$$\begin{aligned} M \left[\begin{matrix} \frac{1}{2}; \frac{3}{2}; \\ -z^2 \end{matrix} \right] &= \frac{\sqrt{\pi}}{2z} \text{erf}(z) = \frac{\sqrt{\pi}}{2z} e^{-z^2} \sum_{k=0}^{\infty} \frac{z^k z^{2k+1}}{(2k+1)!!} \\ &= e^{-z^2} M \left[\begin{matrix} 1; \frac{3}{2}; \\ z^2 \end{matrix} \right] \end{aligned} \quad (96)$$

Next, let us consider the case when $z \rightarrow -\infty$. If we were to apply the answer directly for this case, we would get the following:

$$\lim_{z \rightarrow -\infty} M \left[\begin{matrix} \frac{1}{2}; \frac{3}{2}; \\ -z^2 \end{matrix} \right] = \text{NaN} \quad (97)$$

Therefore, we need to apply the error function formula we get

$$\lim_{z \rightarrow -\infty} \frac{\sqrt{\pi}}{2z} \text{erf}(z) = 0 \times 1 = 0 \quad (98)$$

Equation (98) gives the desired answer.

Next, we evaluate the case when $z \rightarrow 0$ we get from the left-hand side the following:

$$\lim_{z \rightarrow 0} M \left[\begin{matrix} \frac{1}{2}; \frac{3}{2}; \\ -z^2 \end{matrix} \right] = 1 \quad (99)$$

$$\lim_{z \rightarrow 0} e^{-z^2} M \left[\begin{matrix} 1; \frac{3}{2}; \\ z^2 \end{matrix} \right] = 1 \times 1 = 1 \quad (100)$$

From the right-hand side of (96) we obtain the following irregular singularity

$$\lim_{z \rightarrow 0} \frac{\sqrt{\pi}}{2z} \text{erf}(z) = \frac{0}{0} = \text{NaN} \quad (101)$$

Hence, (96) can be utilized only for values of $z \neq 0$.

Here we conclude the discussion on the computation of the irregular singularities of the special cases of the confluent hypergeometric function. Next, we consider several numerical examples in MATLAB.

TABLE I: THE EXPONENTIAL FUNCTION

$M \begin{bmatrix} a=1; \\ b=a; z \end{bmatrix}$	$z \rightarrow -\infty$	$z = -10$	$z = 0$	$z = 10$	$z \rightarrow +\infty$
<i>hypergeom</i>	0	4.541e-5	1	2.202e4	∞
<i>phypergeom</i>	0	4.541e-5	1	2.202e4	∞
<i>exp (64)</i>	0	4.541e-5	1	2.202e4	∞

TABLE II: THE ELEMENTARY FUNCTIONS

$M \begin{bmatrix} a=1; \\ b=2; z \end{bmatrix}$	$z \rightarrow -\infty$	$z = -10$	$z = 0$	$z = 10$	$z \rightarrow +\infty$
<i>hypergeom</i>	0	0.09999	1	2.202e3	NaN
<i>phypergeom</i>	0	0.09999	1	2.202e3	∞
<i>ElmF (68)</i>	0	0.09999	NaN	2.202e3	NaN

TABLE III: THE MODIFIED BESSEL FUNCTION

$M \begin{bmatrix} a=1/2; \\ b=2a; z \end{bmatrix}$	$z \rightarrow -\infty$	$z = -10$	$z = 0$	$z = 10$	$z \rightarrow +\infty$
<i>hypergeom</i>	NaN	0.18354	1	4.043e3	NaN
<i>phypergeom</i>	0	0.18354	1	4.043e3	∞
<i>MBF</i>	NaN	0.18354	1	4.043e3	∞

TABLE IV: THE INCOMPLETE GAMMA FUNCTION

$M \begin{bmatrix} a=1; \\ b=2; z \end{bmatrix}$	$z \rightarrow -\infty$	$z = -10$	$z = 0$	$z = 10$	$z \rightarrow +\infty$
<i>hypergeom</i>	0	0.09999	1	2.202e3	NaN
<i>phypergeom</i>	0	0.09999	1	2.202e3	∞
<i>IGF</i>	0	0.09999	NaN	2.202e3	NaN

TABLE V: THE ERROR FUNCTION

$M \begin{bmatrix} a; \\ b; -z^2 \end{bmatrix}$	$z \rightarrow -\infty$	$z = -10$	$z = 0$	$z = 10$	$z \rightarrow +\infty$
<i>hypergeom</i>	NaN	0.28025	1	0.28025	NaN
<i>phypergeom</i>	0	0.28025	1	0.28025	0
<i>ErrF</i>	0	0.28025	NaN	0.28025	NaN

6 Numerical Examples

There are several MATLAB examples that we have considered in this journal paper.

Exponential Function: Of particular importance is the special case when $0 < a = b < \infty$ see (64). For this special case in Tab. I we perform the computation of the irregular singularities and other values of $z = \{-\infty, -10, 0, 10, +\infty\}$.

The computation was performed using, *hypergeom*, which is the MATLAB function for computing the generalized hypergeometric function, *phypergeom*, which is the Progi implementation of the generalized hypergeometric function, and *exp*, which the exponential function.

As seen from Tab. I, there is perfect agreement between the three ways of implementation or computation of the confluent hypergeometric functions and their corresponding singularities.

Elementary Functions. For example, for the special case when $a(t) = 1$ and $b(t) = 2$, the confluent hypergeometric function, M , reduces to an elementary function [1] as given by (68) see the MATLAB numerical computation in Tab. II.

It is clear as discussed in (68) through (74) that MATLAB fails to compute the correct answer for $z \rightarrow +\infty$ see Tab. II.

The Bessel Functions. For example, for the special case when $b(t) = 2a(t)$, the confluent hypergeometric function reduces to a Bessel function of the first kind [1] see (75) and Tab. III.

It is clear as discussed in (75) through (84) that MATLAB fails to compute the correct answer for $z \rightarrow \pm\infty$ see Tab. III.

The Incomplete Gamma Functions. This case occurs when $b(t) - a(t)$ is a positive integer or $a(t)$ is a positive integer see (85) and Tab. IV.

It is clear as discussed in (85) through (95) that MATLAB fails to compute the correct answer for $z \rightarrow \pm\infty$ see Tab. IV.

The Error Functions. This case occurs when $a(t) = 1/2$ is and $b(t) = 3/2$ see (96) see Tab. V.

It is clear as discussed in (96) through (101) that MATLAB fails to compute the correct answer for $z \rightarrow \pm\infty$ see Tab. V.

Finally, we perform the computation and plot the results of the confluent hypergeometric function corresponding to the exponential function, underdamped, and overdamped as illustrated in Figs. 1 and 2. The exponential function a.k.a. the critically or naturally damped response corresponds to values of $b = a = 1$; the underdamped response corresponds to values of $b = 2a = 2$; and the overdamped response corresponds to values of $a = 2b = 2$.

In Fig. 1, the of the confluent hypergeometric function corresponding to the exponential function, underdamped, and overdamped cases are plotted for values of $-5 < z < 0$ and in Fig. 2 the same is done for $0 < z < 5$.

As shown in Fig. 1 when $-5 < z < 0$ the underdamped response decreases much slower than the exponential function as opposed to the overdamped response which decreases much faster than the exponential function but oscillates between positive and negative values.

In Fig. 2 however, when $0 < z < 5$ the underdamped response grows behind the exponential function as opposed to the overdamped response which grows much more rapidly than the exponential function; hence, validating our analysis in Sect. 3.

In Fig. 3 we have plotted the underdamped response for $b =$

$200a = 200$ and the $\log(z)$ when $0 < z < 200$. It is clear that the underdamped response is smaller than the $\log(z)$ for some values of $3 \leq z \leq 163$; however, once it surpasses the $\log(z)$ for $z > 163$ it grows to $+\infty$ as $z \rightarrow +\infty$.

Our analysis in Sects. 3 through 5 and validated with this very simple MATLAB computation and plot.

Here we conclude the discussion on several numerical examples in MATLAB and again emphasize that there is no reason why MATLAB cannot perform the proper computation of the singularities of the confluent hypergeometric function for the values of the parameters discussed in Sect. 2. Next, we summarize the conclusions of this work.

7 Conclusions

I believe that this is the first journal paper that offers great insights into the computation of the confluent hypergeometric function irregular singularities.

The computation of the confluent hypergeometric function was performed via the Gauss power series using the method of the derivative as beautifully discussed in Proгри (2016, [7]) and via the incomplete gamma function using the method of manoeuvring Proгри (2016, [7]).

The L'Hôpital's rule [12] was utilized to perform the computation of the irregular singularities of the confluent hypergeometric function in order to avoid the singularities.

The computation of the confluent hypergeometric function was finally performed in MATLAB and the summary of the computational work was given in the numerical examples section.

I strongly recommend that MATLAB modify the *hypergeom* and enable the correct computation of the irregular singularities of the confluent hypergeometric function for the values of the parameters discussed in Sect. 2.

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I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [6] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Proгри pioneer publication *Indoor Geolocation Systems—Theory and Applications. Vol. 1* (Not yet available in print) [15].

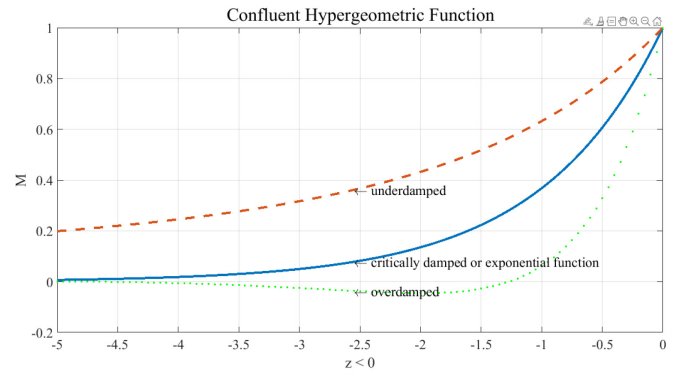


Fig. 1. The computation of the confluent hypergeometric function for $z < 0$.

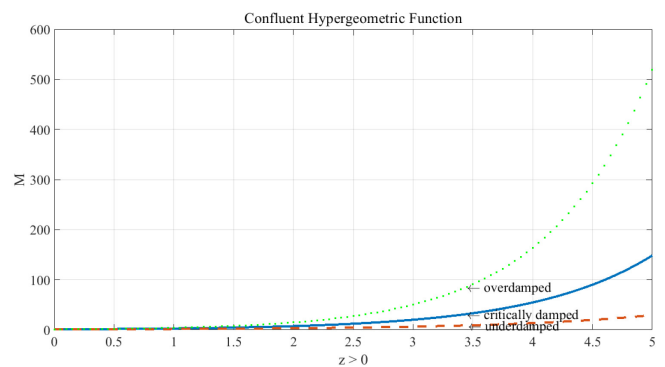


Fig. 2. The computation of the confluent hypergeometric function for $z > 0$.

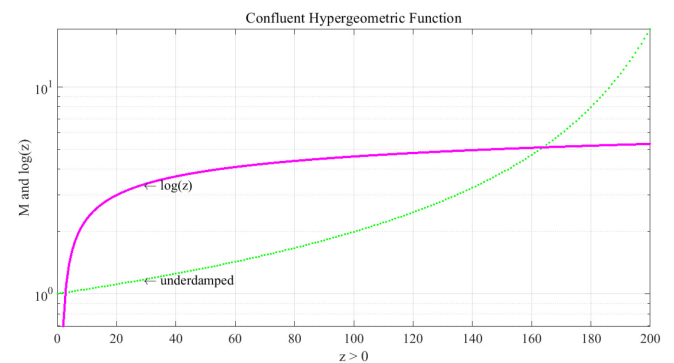


Fig. 3. The computation of the underdamped confluent hypergeometric function and $\log(z)$ for $z > 0$.

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Proгри, my dear father, Fiqiri Proгри, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Proгри.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless, to say I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus reminds me of my grandfather who I loved him very much.

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ⁱ There is an important assumption that is missing in the Muller expansion [8].

ⁱⁱ Where $P(b, z)$ is defined as the regularized incomplete gamma function [4] which is implemented in MATLAB via the `gammainc(z,b)` function [6].

ⁱⁱⁱ Ditto ii.

^{iv} Ditto ii. This implementation is not advisable because it leads to more computational errors.

^v Ditto ii.