

Technical Report

A Navigation Algorithm Using a System of Stationary Pseudolites

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Abstract— A navigation algorithmⁱ, which utilizes a system of stationary pseudolites is derived, analysed, discussed, and simulated in this article. Although this algorithm uses the same Doppler measurements as a conventional least square algorithm (LSA) (which produces user velocity and user clock drift) or Kalman based filter (which obtains user position/velocity etc.), in essence the Doppler measurements are applied differently and therefore this is the reason why this is an innovative algorithm. The solution accuracy obtained using this algorithm is compared with solution accuracy from the conventional pseudorange LSA. The result of this comparison is presented in the end of the article.

Index Terms—Pseudorange, Doppler, accumulated carrier phase, measurements, navigation, pseudolites, Kalman, least square algorithm, ionospheric propagation delay, indoor terrestrial applications.

1 Introduction

The Doppler effect, also known as the Doppler shift, is a fundamental concept in astronomy, physics and navigation science [2]-[4].

In the 1920s, astronomers found something most peculiar about the spectra of stars of distant galaxies (see Hawking, 1988 [2] pp 38-39.). Their spectra were shifted by a constant amount in the red spectrum compared with the spectra of stars of our galaxy. Of course, this implication can be explained

easily with the Doppler effect [2], which is the relationship between the signal frequency and the speed of the transmitting and/or receiving device (or electro mechanic equipment).

Moreover, when the Doppler effect was neglected, scientists led to erroneous conclusions in early optical physics' conclusions (The experiment of Einstein and Rupp) that were not supported by classical mechanics (see Heisenberg [3] pp 79-80).

In navigation science, during the investigation of Transit system, the Doppler shift data were recorded at one site during a single pass of the Sputnik 1 to determine its entire orbit (See

Parkinson et al. [4], p. 112) assuming a non-changing satellite orbit. Although this technique had world coverage and periodic updates, it had still limited accuracy, required post-processing time, and voluminous equipment from the user's point of view.

According to Parkinson (see Parkinson et al. [4], p. 112) the use of Doppler for navigation is not intuitive. The task to convince uncommitted sceptics back in 1958 was a difficult one.

Doppler measurements were used to determine the earth gravitational field, which led to accurately determining and predicting the Transit satellite's orbit (see Parkinson et al. [4], p. 113).

The new idea starting with the Timation concept (in 1964) measures range rather than Doppler and to use these measurements for navigation, which is the concept we are most familiar with [4]. A study conducted by the Radio Corporation of Americaⁱⁱ (RCA [5]) Astro-Electronics Division in Princeton, NJ, in June 1969 seems to suggest that a combination of range and Doppler measurements provides near-instantaneous positioning with only a single satellite (see Parkinson et al. [4], p. 125). Dual-phase Doppler measurements were also seen as a means of determining the ionospheric propagation delay [4].

Doppler measurements together with range and accumulated carrier phase are widely used by recursive Kalman filters to accurately determine user's velocity along with their position whether for air, outdoor terrestrial or indoor terrestrial applications [5]-[7].

Prior to my paper in 2001 (see Proгри et al. [1]) the Doppler measurements were never used for pseudolite (or stationary transmitter) applications, which is the reason why this was an innovative algorithm.

While the conventional LSA is depicted in [8] [9], here we revisit the basic GPS positioning problem from the Doppler measurement point of view as an innovative means of positioning and navigationⁱⁱⁱ.

First, we explore the idea of navigating with Doppler in a noiseless environment; and second, in a noisy media.

For simulation purposes, we have investigated the stationary and moving user scenarios. The error on the Doppler measurement was estimated from recorded Doppler measurements at the Satellite Navigation Lab at WPI in 2000-2001.

This paper is organized as follows: In Sect. 2 the algorithm description is given. Simulation of both stationary and moving receivers is provided in Sect. 3. Conclusion is summarized in Sect. 4 followed by acknowledgment in Sect. 5 and references

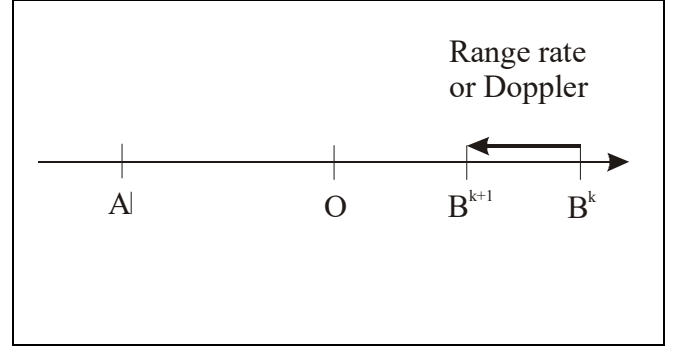


Fig. 1. 1-D system description.

in Sect. 6.

2 Algorithm Description

The algorithm description section contains the one-dimensional (1-D), two-dimensional (2-D), three-dimensional (3-D), and multi-D cases.

2.1 One Dimensional Applications

Assume that one stationary transmitter A and one receiver B are positioned in a 1-D axis as shown in Fig. 1.

We further assume that the receiver B is perfectly synchronized with transmitter A and that the initial location of receiver B is known. Let d_{AB}^k denote the geometric distance between the transmitter A and the receiver B at discrete time k . Under the assumption that the receiver B moves toward transmitter A by the amount of $\bar{v}_{AB}^k$, which is analytically determined from:

$$\bar{v}_{AB}^k \cong d_{AB}^{k+1} - d_{AB}^k \quad (1)$$

Assuming that the observations are performed at a 1 Hz rate, relationship (1) gives the formula for approximated average velocity during 1 sec interval. We can rewrite (1) assuming that A is stationary,

$$\bar{v}_{AB}^k \cong x_B^{k+1} - x_B^k \quad (2)$$

This expression enables us to form the navigation equation for the one-dimensional case in accordance with,

$$x_B^{k+1} \cong x_B^k + \bar{v}_B^k \quad (3)$$

It is evident in this case that by knowing the Doppler at any epoch we can determine the location of the receiver B with perfect accuracy in the one-dimensional case.

Note: The same expression can be obtained when the receiver B moves away from the transmitter A with the only modification of (3): the plus sign, (+), now becomes minus, (-).

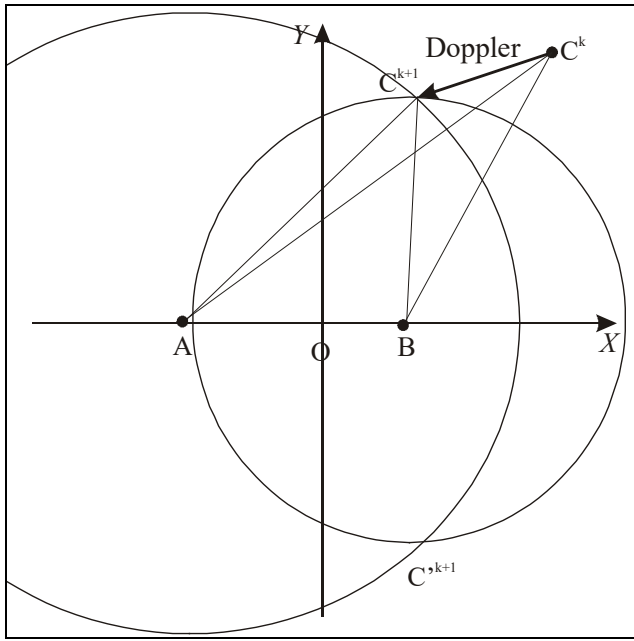


Fig. 2. 2-D system description.

2.2 Two-Dimensional Applications

At least two transmitters are required to determine the correct location of a single receiver in a two-dimensional plane.

Therefore, we assume that the transmitters A and B are in the same plane with the moving receiver C as shown in Fig. 2.

We will again assume that the two transmitters are stationary with the coordinate system under consideration. We will also assume that the initial location of the receiver B is known. Before going on, it can be shown analytically that for A and B stationary and for known distances AC and BC there are two solutions in which C can be located. Both points are symmetrical with respect to the line that passes between A and B. With these assumptions in mind, we move now to the derivation of the navigation equations.

At the $(k + 1)^{\text{th}}$ epoch the receiver C is found at the C^{k+1} location as shown in Fig. 2. The new ranges between the receiver C and the transmitters A and B are respectively d_{AC}^{k+1} and d_{BC}^{k+1} . Like the one-dimensional case the receiver C moves toward the transmitters A and B by the amount of,

$$\bar{v}_{AC}^k \cong d_{AC}^{k+1} - d_{AC}^k \quad (4)$$

$$\bar{v}_{BC}^k \cong d_{BC}^{k+1} - d_{BC}^k \quad (5)$$

The new point C^{k+1} is found from the intersection of two circles—one with its center at the transmitter A and with radius given by (6) and the other circle with its center at the transmitter B and with radius given by (7), as follows,

$$d_{AC}^{k+1} \cong d_{AC}^k + \bar{v}_{AC}^k \quad (6)$$

$$d_{BC}^{k+1} \cong d_{BC}^k + \bar{v}_{BC}^k \quad (7)$$

Without showing all the work the solution for the new location of the receiver C is determined from:

$$x_C^{k+1} = \frac{x_A + x_B + \frac{d_{AC}^k - d_{BC}^k + \bar{v}_{AC}^k - \bar{v}_{BC}^k}{x_B - x_A}}{2} \quad (8)$$

$$y_C^{k+1} = \pm \sqrt{d_{AC}^k + \bar{v}_{AC}^k - (x_C^{k+1} - x_A)^2} \quad (9)$$

Since the real receiver is mounted in a physical object which contains physical mass and inertia; therefore, it cannot be located at two different points at the same time, as it is against any law of classical mechanics. (Note: This may not be true for phenomena approximated by quantum mechanics law or physics; however, the scenario we are dealing with can be accurately described by the classical mechanics dynamics.)

Next in continuation to the argument stated right above, we determine which y_C^{k+1} should be chosen from the two solutions of (9).

According to the dynamics of the problem the new y_C^{k+1} will be inside the circle with radius the sum of the Doppler measurements and with its center the previous location of the receiver, C. Denote with r_{cr} the radius of the circle inside of which the new location of the receiver is found, which can be written as,

$$r_{cr}^k = \bar{v}_{AC}^k + \bar{v}_{BC}^k \quad (10)$$

Denote the distance between the current position and the new correct position of receiver by d_{CC}^{k+1} , such as,

$$d_{CC}^{k+1} = \sqrt{(x_C^{k+1} - x_C^k)^2 + (y_C^{k+1} - y_C^k)^2} \quad (11)$$

Similarly, the distance between the current position of the receiver and the mirror image of the new correct position of the receiver, denoted by $d_{CC'}^{k+1}$, can be written as,

$$d_{CC'}^{k+1} = \sqrt{(x_{C'}^{k+1} - x_C^k)^2 + (y_{C'}^{k+1} - y_C^k)^2} \quad (12)$$

Theorem 1: Relations (10), (11), and (12) fulfill the following inequality:

$$d_{CC}^{k+1} \leq r_{cr}^k \leq d_{CC'}^{k+1} \quad (13)$$

The analytical proof of this theorem is straightforward and therefore it is not presented here; hence, we would like to emphasize that (13) serves as the test criterion for choosing the correct y coordinate of the new receiver location.

2.3 Three-Dimensional Applications

In a noise and error free environment, the correct location of

the receiver D in the three-dimensional case can be determined with help of the three transmitters (A, B, and C), which are not in the same line as shown in figure 3. Let D^{k+1} determine the location of the receiver D at the $(k+1)^{\text{th}}$ epoch (see Fig. 3). It is clear that the new ranges between receiver D and transmitters A, B and C are respectively d_{AD}^{k+1} , d_{BD}^{k+1} , and d_{CD}^{k+1} . Like the one-dimensional and two-dimensional cases, the receiver D moves toward transmitters A, B, and C by the amount of,

$$\bar{v}_{AD}^k \cong d_{AD}^{k+1} - d_{AD}^k \quad (14)$$

$$\bar{v}_{BD}^k \cong d_{BD}^{k+1} - d_{BD}^k \quad (15)$$

$$\bar{v}_{CD}^k \cong d_{CD}^{k+1} - d_{CD}^k \quad (16)$$

The new point D^{k+1} is found from the intersection of three spherical surfaces—one with center at the transmitter A and with radius given by (17), one circle with center at the transmitter B and with radius given by (18), and the final circle with center at the transmitter C and with radius given by (19), as follows,

$$d_{AD}^{k+1} \cong d_{AD}^k + \bar{v}_{AD}^k \quad (17)$$

$$d_{BD}^{k+1} \cong d_{BD}^k + \bar{v}_{BD}^k \quad (18)$$

$$d_{CD}^{k+1} \cong d_{CD}^k + \bar{v}_{CD}^k \quad (19)$$

Without showing all the work the solution for the new location of the receiver D is determined from:

$$x_D^{k+1} = \frac{A_1^k(y_B - y_C) - A_2^k(y_B - y_A)}{(x_B - x_A)(y_B - y_C) + (x_B - x_C)(y_B - y_A)} \quad (20)$$

$$y_D^{k+1} = \frac{A_1^k(x_B - x_A) - A_2^k(x_B - x_C)}{(x_B - x_A)(y_B - y_C) + (x_B - x_C)(y_B - y_A)} \quad (21)$$

$$z_D^{k+1} = \pm \sqrt{d_{CD}^k + \bar{v}_{CD}^k - (x_D^{k+1} - x_C)^2 - (y_D^{k+1} - y_C)^2} \quad (22)$$

In expressions (20) and (21) the parameters A_1^k and A_2^k correspond to:

$$A_1^k = \frac{d_{AD}^k - d_{BD}^k + \bar{v}_{AD}^k - \bar{v}_{BD}^k + x_B^2 - x_A^2 + y_B^2 - y_A^2}{2} \quad (23)$$

$$A_2^k = \frac{d_{BD}^k - d_{CD}^k + \bar{v}_{BD}^k - \bar{v}_{CD}^k + x_B^2 - x_C^2 + y_B^2 - y_C^2}{2} \quad (24)$$

A criterion similar to theorem 1 for the two-dimensional case can be utilized here to determine the correct z_D^{k+1} for the three-dimensional case.

2.4 Multidimensional noisy Applications

Consider the applications now when more than four stationary transmitters (or pseudolites) are used to determine the location

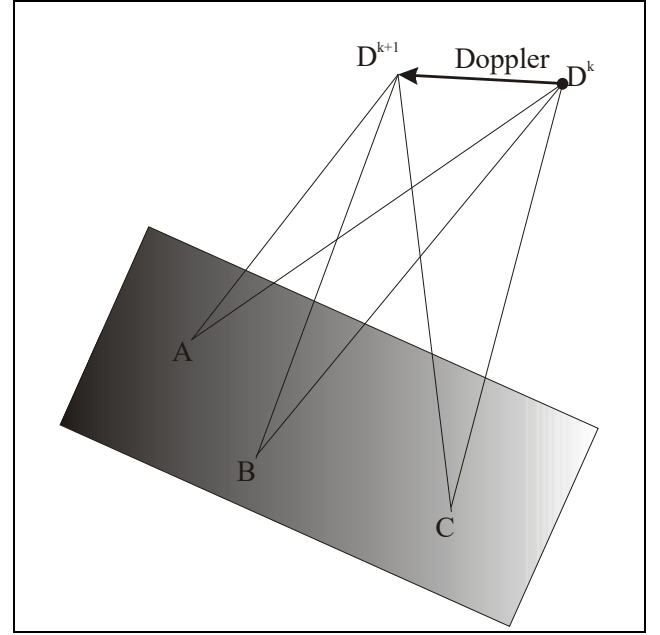


Fig. 3. 3-D system description.

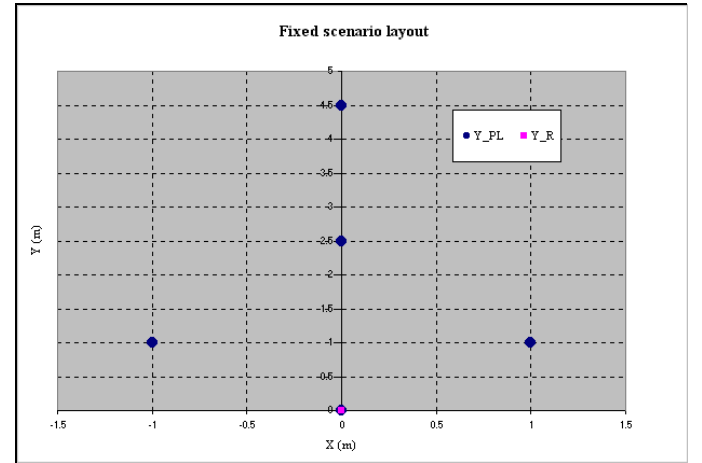


Fig. 4. Pseudolite, stationary receiver description.

of a single receiver. Suppose that the Doppler measurement vector provided from the receiver at the k^{th} epoch can be written as,

$$\dot{\boldsymbol{\phi}}^k = \dot{\mathbf{d}}^k + c(\Delta \mathbf{f}^k - \Delta \mathbf{F}^k) + \mathbf{e}_{\dot{\phi}} \quad (25)$$

where, $\dot{\mathbf{d}}^k$ is the true range-rate vector, $\Delta \mathbf{f}^k$ is the receiver clock drift vector, $\Delta \mathbf{F}^k$ is the transmitter's clock drift vector, $\mathbf{e}_{\dot{\phi}}$ is the Doppler measurement noise vector.

Under the assumption that the location of the receiver at the k^{th} epoch is known, we seek to determine its location at the next epoch. Therefore, instead of the true range between the next receiver location and the stationary transmitters, we will have the Doppler Derived (DD) pseudorange (this is different from the true pseudorange, which is provided from the receiver). The analytical expression of the DD pseudorange vector is given by,

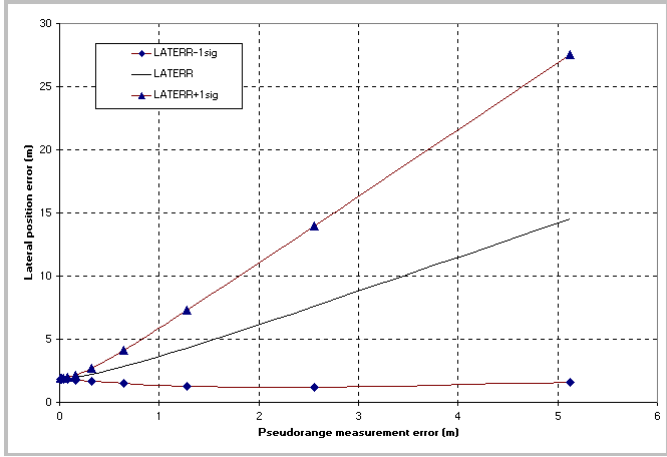


Fig. 5. Lateral position error vs. pseudorange measurement noise CLSA.

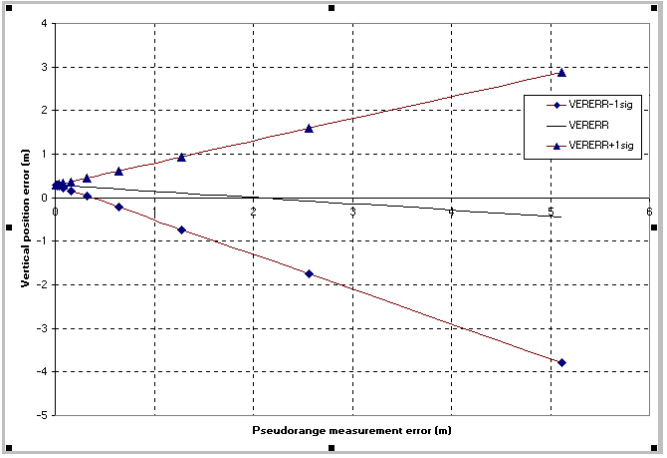


Fig. 6. Vertical position error vs. pseudorange measurement noise CLSA.

$$\hat{\mathbf{p}}^{k+1} = \mathbf{d}^k + \boldsymbol{\phi}^k \quad (26)$$

Denote the state vector by,

$$\mathbf{s}^k = [x^k \ y^k \ z^k \ \Delta f^k]^T \quad (27)$$

For reasons explained later in the paper, we form the residual vector expressed as,

$$\mathbf{r}^{k+1} = \hat{\mathbf{p}}^{k+1} - \mathbf{H}^{k+1} \cdot \mathbf{s}^{k+1} \quad (28)$$

We will seek a solution for the new state vector, which minimizes the norm of the residual vector as follows,

$$\arg \min \|\mathbf{r}^{k+1}\| = \arg \min \mathbf{r}^{k+1T} \mathbf{r}^{k+1} \quad (29)$$

Provided that $\mathbf{H}^{k+1T} \cdot \mathbf{H}^{k+1}$ is nonsingular that the solution of (29) is determined from,

$$\hat{\mathbf{s}}^{k+1} = (\mathbf{H}^{k+1T} \cdot \mathbf{H}^{k+1})^{-1} \mathbf{H}^{k+1T} \cdot \hat{\mathbf{p}}^{k+1} \quad (30)$$

Along with solution (30) another valid solution, when the measurement error covariance matrix, $\mathbf{R}$, is provided, reads,

$$\hat{\mathbf{s}}^{k+1} = (\mathbf{H}^{k+1T} \mathbf{R}^{-1} \cdot \mathbf{H}^{k+1})^{-1} \mathbf{H}^{k+1T} \mathbf{R}^{-1} \cdot \hat{\mathbf{p}}^{k+1} \quad (31)$$

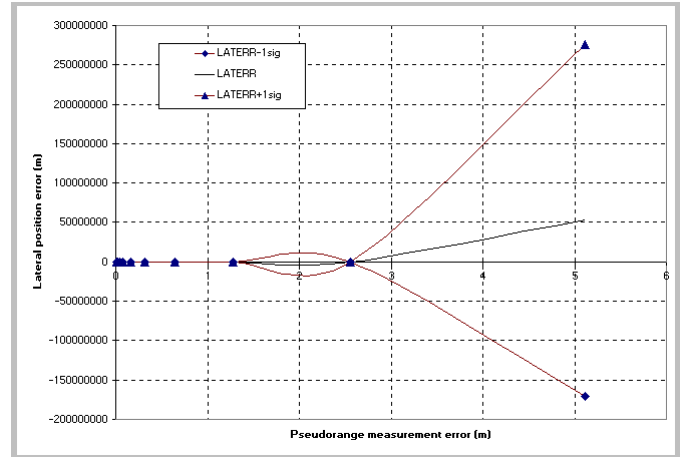


Fig. 7. Lateral position error vs. pseudorange measurement noise MLSA.

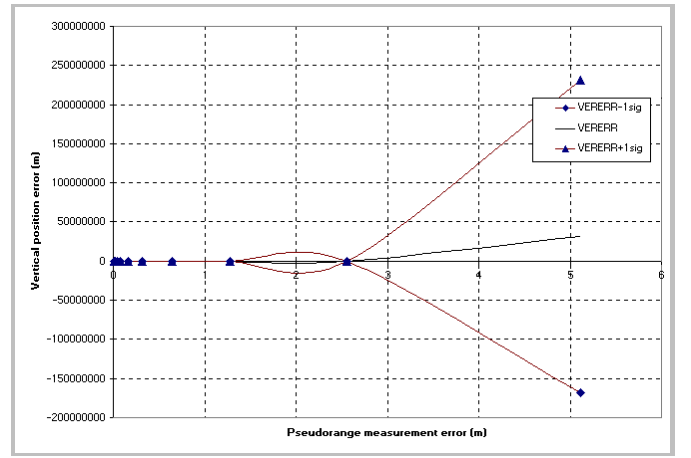


Fig. 8. Vertical position error vs. pseudorange measurement noise MLSA

which is based on the criteria $\arg \min \mathbf{r}^{k+1T} \mathbf{R}^{-1} \mathbf{r}^{k+1}$.

Note 1: The advantage of this algorithm compared to the ordinary least square^v solution [9], which uses only true pseudoranges, consists in the small diagonal values of the covariance matrix formed with DD pseudorange. This occurs for the reasons explained below. Let $\mathbf{e}_s^{k+1}$ denote the navigation state error vector. The Doppler measurement noise vector, $\mathbf{e}_{\dot{\phi}}^{k+1}$, can be translated into the state error vector, $\mathbf{e}_{\dot{\phi}}^{k+1}$, in accordance with,

$$\mathbf{e}_s^{k+1} = (\mathbf{H}^{k+1T} \cdot \mathbf{H}^{k+1})^{-1} \mathbf{H}^{k+1T} \mathbf{e}_{\dot{\phi}}^{k+1} \quad (32)$$

Assuming that for a set of four or more pseudolites and for a single receiver we estimate coherently and in parallel the receiver position using the conventional LSA (or CLSA) on the one hand and the modified LSA algorithm (or MLSA) on the other. Without considering the effect of the geometry, there is an improvement of the navigation accuracy utilizing the Doppler MLSA versus the pseudorange CLSA by the ration of $\sigma_{\rho}/\sigma_{\dot{\phi}}$. For typical GPS receivers this number would be greater than 50.

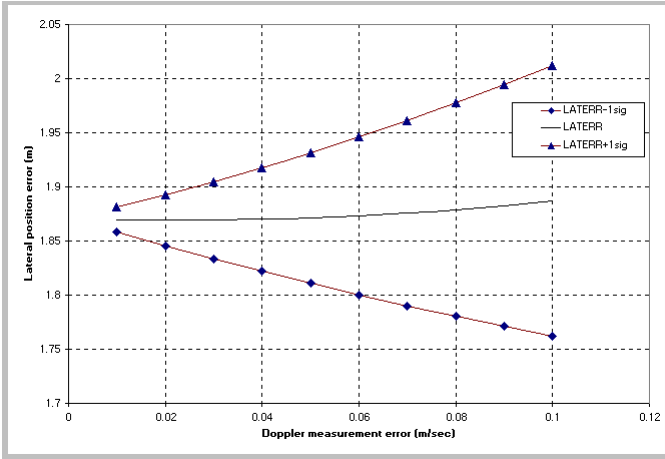


Fig. 9. Lateral position error vs. Doppler measurement noise CLSA

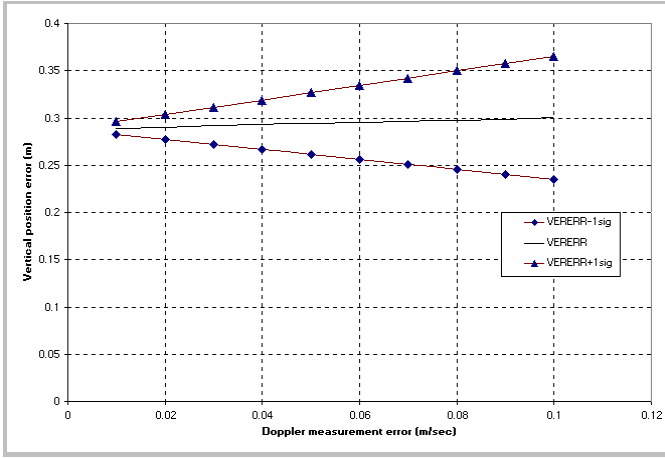


Fig. 10. Vertical position error vs. Doppler measurement noise CLSA

Note 2: For the application when only a set of 4 pseudolites is available for navigation then the solution is reduced to:

$$\hat{\mathbf{s}}^{k+1} = \mathbf{H}^{k+1^{-1}} \cdot \hat{\mathbf{p}}^{k+1} \quad (33)$$

Note 3: Quite often a pretty good estimate of the starting point is provided. For this application the challenge is to maintain the same accuracy during the navigation. We can modify the solution given by (30), (31), and (33) as follows,

$$\Delta \hat{\mathbf{s}}^{k+1} = (\mathbf{H}^{k+1^T} \cdot \mathbf{H}^{k+1})^{-1} \mathbf{H}^{k+1^T} \cdot \mathbf{r}^{k+1} \quad (34)$$

$$\Delta \hat{\mathbf{s}}^{k+1} = (\mathbf{H}^{k+1^T} \mathbf{R}^{-1} \mathbf{H}^{k+1})^{-1} \mathbf{H}^{k+1^T} \mathbf{R}^{-1} \mathbf{r}^{k+1} \quad (35)$$

$$\Delta \hat{\mathbf{s}}^{k+1} = \mathbf{H}^{k+1^{-1}} \cdot \mathbf{r}^{k+1} \quad (36)$$

Once the solution estimate increment is determined by either (34), (35), or (36) then the absolute solution estimate is easily computed from:

$$\hat{\mathbf{s}}^{k+1} = \hat{\mathbf{s}}^k + \Delta \hat{\mathbf{s}}^{k+1} \quad (37)$$

The benefit of using (34-37) versus (30), (31), and (33) will be demonstrated from the simulation results.

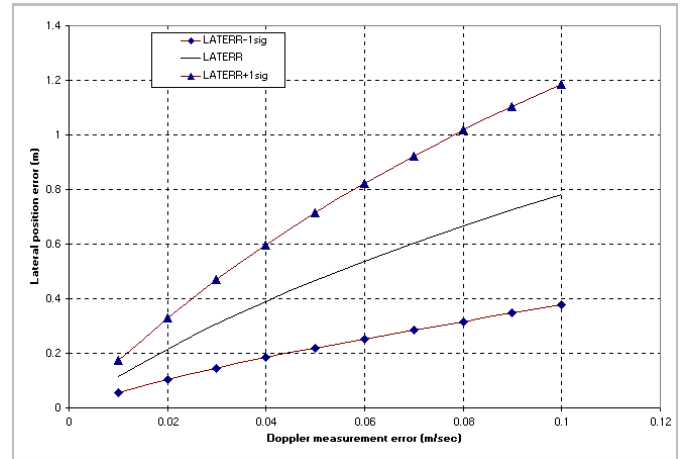


Fig. 11. Lateral position error vs. Doppler measurement noise MLCA.

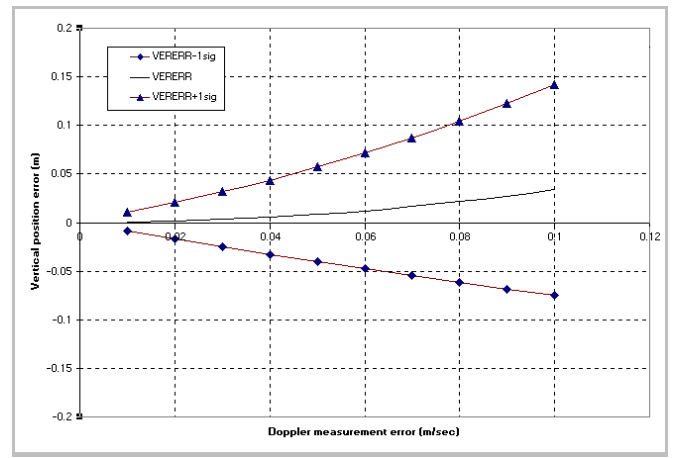


Fig. 12. Vertical position error vs. pseudorange measurement noise MLCA

3 Simulation

Simulation results are provided for one stationary and one moving receivers.

3.1 Stationary Receiver

This is the scenario in which the receiver is assumed stationary. This scenario is important because it is by far the easiest and the simplest one.

The pseudolite and the receiver layout are shown in Fig. 4. The receiver is located 4 m above the plane of the pseudolites. Considering our interest in the indoor geo-location applications, we have depicted this scenario because it can represent any of the indoor-geolocation environment such private home, school, business or commercial building, gym, shopping mall, gas station, etc.

1. Raw pseudorange CLSA

First, we process raw pseudoranges using the CLSA algorithm and obtain the lateral and vertical position error for pseudorange measurement error (1 sigma) ranging from 0.01 to

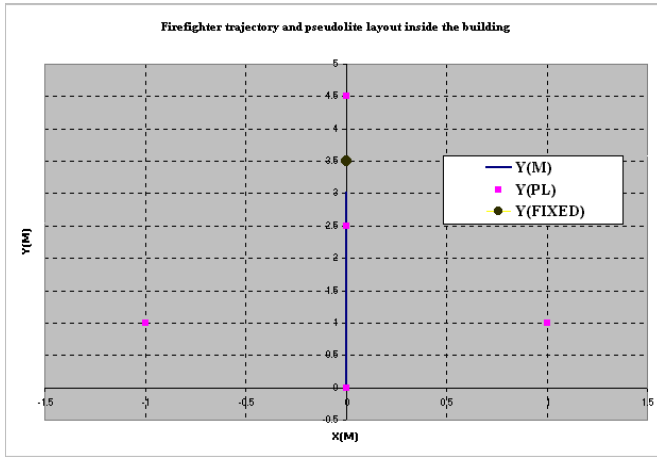


Fig. 13. Pseudolite, moving receiver description.

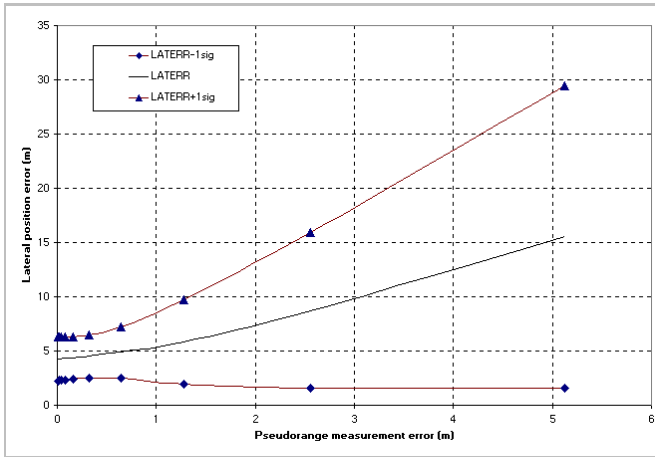


Fig. 14. Lateral position error vs. pseudorange measurement noise CLSA.

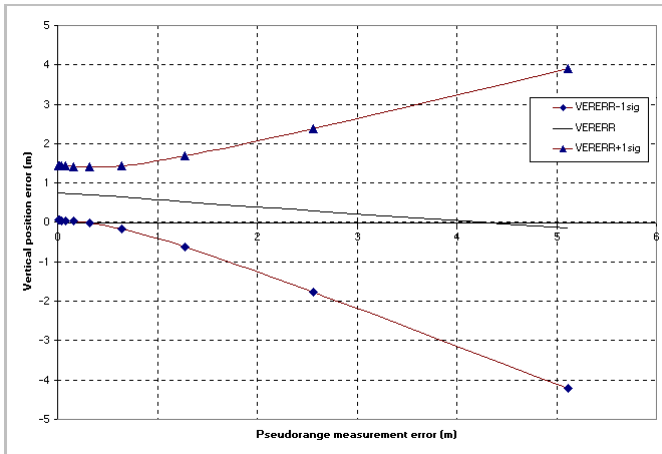


Fig. 15. Vertical position error vs. pseudorange measurement noise CLSA.

5.12 m (see Figs. 5 and 6).

2. Raw pseudorange MLSA

Next, we repeat the previous test utilizing the MLSA algorithm (see Figs. 7 and 8).

3. DD pseudorange CLSA

Next, we process DD pseudoranges employing the CLSA and obtain the lateral and vertical position error for Doppler measurement error (1 sigma) ranging from 0.01 to 0.1 m/sec (see Figs. 9 and 10).

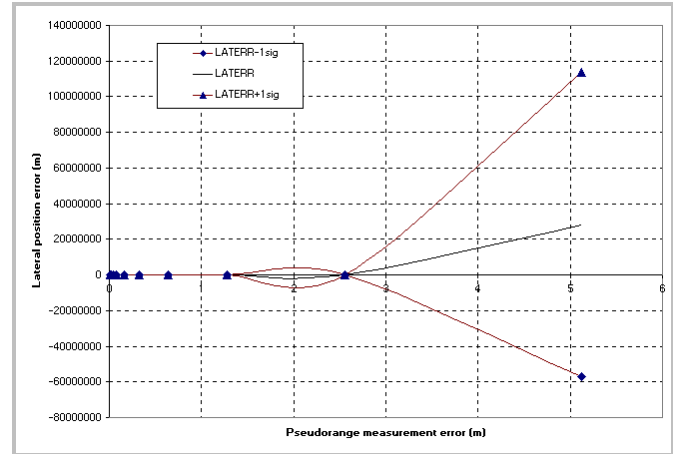


Fig. 16. Lateral position error vs. pseudorange measurement noise MLSA.

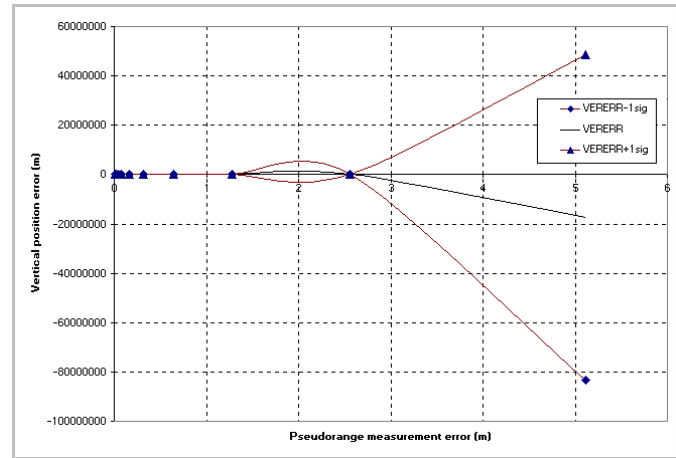


Fig. 17. Vertical position error vs. pseudorange measurement noise MLSA.

4. DD pseudorange MLSA

Next, we repeat the previous test utilizing the MLSA algorithm (see Figs. 11 and 12).

3.2 Moving Receiver

The moving scenario depicted in Fig. 9 was selected from one of the indoor geo-location applications for the same reasons as the stationary one was selected [7].

1. Raw pseudorange CLSA

First, we process raw pseudoranges using CLSA and obtain the lateral and vertical position error for pseudorange measurement error (1 sigma) ranging from 0.01 to 5 m (see Figs. 14 and 15).

2. Raw pseudorange MLSA

Next, we repeat the previous test utilizing the MLSA algorithm (see Figs. 16 and 17).

3. DD pseudorange CLSA

Next, we process DD pseudoranges employing the CLSA and obtain the lateral and vertical position error for Doppler measurement error (1 sigma) ranging from 0.01 to 0.1 m/sec (see Figs. 18 and 19).

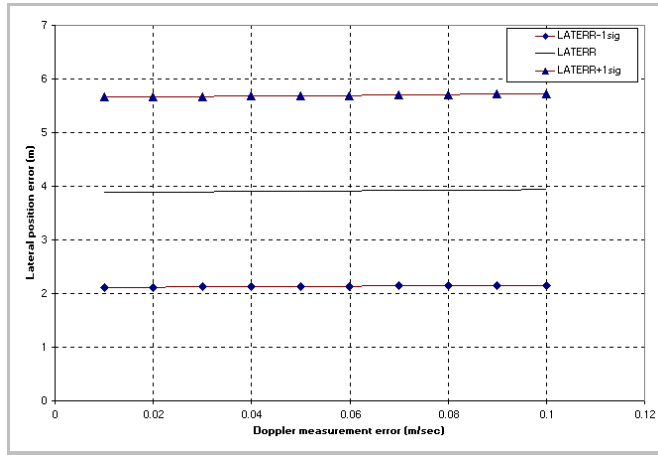


Fig. 18. Lateral position error vs. Doppler measurement noise CLSA.

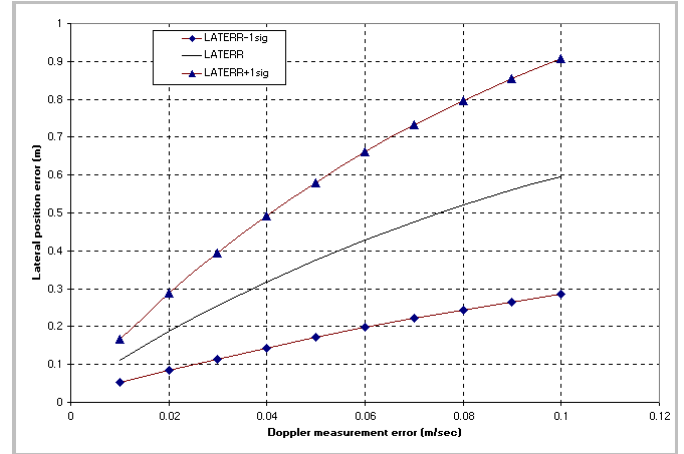


Fig. 20. Lateral position error vs. pseudorange measurement noise MLSA.

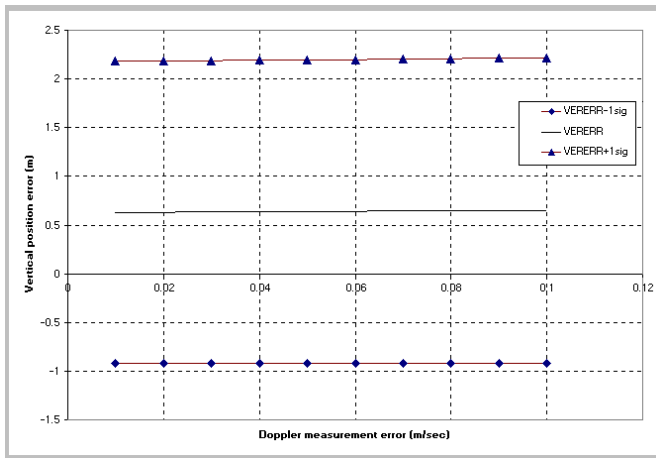


Fig. 19. Vertical position error vs. Doppler measurement noise CLSA.

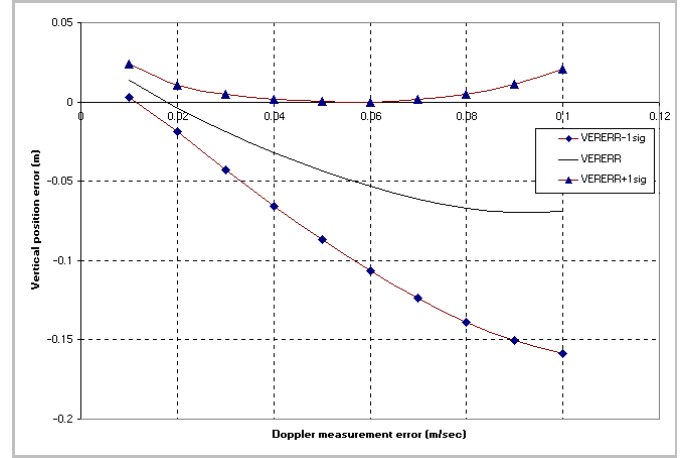


Fig. 21. Vertical position error vs. pseudorange measurement noise MLSA.

4. DD pseudorange MLSA

Next, we repeat the previous test utilizing the MLSA algorithm (see Figs. 20 and 21).

4 Conclusions

According to the analytical and simulation results of this work it appears that both positioning and navigation with Doppler measurements are possible even for indoor or outdoor pseudolite geolocation applications.

When Doppler measurements are used, the MLSA formulation (Note 4 in DD pseudorange MLSA) the navigation accuracy appears to be superior compared to the CLSA formulation (Note 3 in DD pseudorange CLSA).

For no data loss, navigating with Doppler using the MLSA algorithm appears to provide a far better accuracy than the pseudorange MLSA and MLSA algorithm.

For distances comparable to 1 sigma value, the CLSA algorithm provides a better accuracy than the MLSA algorithm.

5 Acknowledgement

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ⁱ *Editor's Note:* This article was originally published in 2001 [1]. The main results of this article are valid if only the data are assumed to be normally or Gaussian distributed. For other types of data distributions, I believe that a modification of the algorithm may be needed. The original publication contained a few typos and some minor language problems that I believe are fixed in this edition. A number of other clarifications are made in this publication and also the equation editor is greatly improved. Since, the original article did not contain a DOI and it is a copyright by the authors I believe that the publication of this article in the form of a technical report is a good way to preserve the originality and authenticity of the article and provide a much better understanding of the results of this work.

ⁱⁱ The RCA Corporation was a major American electronics company, which was founded as the Radio Corporation of America in 1919. It was initially a patent trust owned by General Electric (GE), Westinghouse, AT&T Corporation and United Fruit Company. In 1932, RCA became an independent company after the partners were required to divest their ownership as part of the settlement of a government antitrust suit. (see further in [5]).

ⁱⁱⁱ I do not take any credit for the feasibility of the idea that Doppler can be successfully used for positioning and/or navigation. My contribution consists primarily of extending that already known and well-tested feasibility and operational capability idea of using Doppler measurements in terrestrial indoor pseudolite geolocation applications.

^{iv} See IV.

^v From Financial Analysis we have the following definition of the LSA: The least squares criterion is a formula used to measure the accuracy of a straight line in depicting the data that was used to generate it. That is, the formula determines the line of best fit.

The least squares method is a statistical procedure to find the best fit for a set of data points by minimizing the sum of the offsets or residuals of points from the plotted curve.

The method of least squares is a standard approach in regression analysis to approximate the solution of overdetermined systems (sets of equations in which there are more equations than unknowns) by minimizing the sum of the squares of the residuals made in the results of every single equation [9].

This mathematical formula is used to predict the behavior of the dependent variables. The approach is also called the least squares regression line.

It is used to estimate the accuracy of a line in depicting the data that was used to create it.

The first clear and concise exposition of the method of least squares was published by Legendre in 1805. The technique is described as an algebraic procedure for fitting linear equations to data and Legendre demonstrates the new method by analyzing the same data as Laplace for the shape of the earth. The value of Legendre's method of least squares was immediately recognized by leading astronomers and geodesists of the time.

In 1809 Carl Friedrich Gauss published his method of calculating the orbits of celestial bodies. In that work he claimed to have been in possession of the method of least squares since 1795. This naturally led to a priority dispute with Legendre. However, to Gauss's credit, he went beyond Legendre and succeeded in connecting the method of least squares with the principles of probability and to the normal distribution. He had managed to complete Laplace's program of specifying a mathematical form of the probability density for the observations, depending on a finite number of unknown parameters, and define a method of estimation that minimizes the error of estimation. Gauss showed that the arithmetic mean is indeed the best estimate of the location parameter by changing both the probability density and the method of estimation. He then turned the problem around by asking what form the density should have and what method of estimation should be used to get the arithmetic mean as estimate of the location parameter. In this attempt, he invented the normal distribution (see more in [9]).