Giftet Inc.
Journal of Geolocation, Geo-information, and Geo-intelligence
Volume 2020, article ID 2020071601, 12 pages
https://doi.org/10.18610/JG3.2020.071601

Research Article

The Computation of a 2F2 Hypergeometric Function

Ilir F. Progri[1]

[1]*Giftet Inc., 5 Euclid Ave. #3, Worcester, MA 01610, USA*

ORCID: 0000-0001-5197-1278

Correspondence should be addressed to Ilir Progri; iprogri@giftet.com

Received March 26, 2019; Revised April 1-June 30, 2020, Accepted July 16, 2020; Published November 1, 2020.

Scientific Editor-in-Chief/Editor: Ilir F. Progri

The computation of a 2F2 generalized hypergeometric function for a particular set of parameters is discussed in this paper. The Gauss power series is perhaps the most common form of direct computation of a 2F2 generalized hypergeometric function. For a particular set of parameters we can take advantage of various transformations such as Kummer transformation that produces other means of the computation of a 2F2 generalized hypergeometric function that leads to the creation of new entries such as the fundamental 2F2 in the Table of Integrals, Series, and Products. The fundamental 2F2 can be obtained by means of the difference of two Kampé de Fériet functions (or double hypergeometric series). This result is further expanded to a more generalized hypergeometric function. This paper is based almost entirely on the creation of original analytical derivation, but as far as numerical results special cases may be considered as such.

Index Terms—Hypergeometric functions, Fundamental 2F2, Kampé de Fériet function, partial derivatives, Euler integral, Gauss power series, Kummer transformation, Table of Integrals, Series, Products.

1 Introduction

The computation of a 2F2 hypergeometric function partial derivatives is a very important problem that has occurred and may occur more frequently than we think in the future.

For the first time I came across with the computation of a 2F2 generalized hypergeometric function partial derivatives as I was preparing Chap. 8 of my pioneer publication on *Indoor Geolocation Systems—Theory and Applications* [1] in 2013.

In 2016 as I was producing the "Generalized Bessel function distributions" (see Progri 2016, [3]) I needed to compute the integer case of the Generalized Bessel function distributions which I was successful to do so in 2019, see (Progri, 2019 [5]).

While the computation of the Generalized Bessel function distributions is complete (Progri, 2016, 2018, and 2019 [3]-[5]) the closed form expression of all the equations may not be unique. This paper exploits some of the relationship between a 2F2 generalized hypergeometric function and the Kampé de Fériet function.

The work discussed in this paper is entirely original, novel, and innovative. It is not based on any similar work presented in the literature. But, from 2015 until 2020 I have produced an enormous library that I continue to improve and publish the results of my investigation of these improvements. Therefore, this paper is really a continuation of the computational work that I have been doing for the last six years.

The main theme in this paper is to present several methods that discuss the computation of a 2F2 generalized hypergeometric function for a particular set of parameters and show the connections among each method. In particular there are two 2F2 generalized hypergeometric functions of interest: one of them is called the fundamental 2F2 generalized hypergeometric function and the other a 2F2 generalized hypergeometric function. For these two examples of the 2F2 generalized hypergeometric functions I was able to employ the Kummer second transformation [7] and produce a closed form expression of these 2F2 generalized hypergeometric functions by means of a difference of two Kampé de Fériet functions [1]-[5], [13]. The Kummer second transformation that is utilized is this paper is fundamentally different from the one used in [14], [15], and [23].

In this paper I was able to utilize several publications [6], [8], [9]-[11],[16]-[18], [20], [24] for well know computational functions and algorithms.

In this numerical results Sect., I show that in fact the computation of the 2F2 generalized hypergeometric functions for those set of parameters is in fact identical to the computation of the difference of the two Kampé de Fériet functions as produced in this paper. This result is another validation of the Kummer second transformation [7].

This paper is organized as follows: in Sect. 2 the 2F2 generalized hypergeometric function computation is discussed. In Sect. 3, the continued fraction of a 2F2 hypergeometric function is presented. Section 4 contains the computation of the fundamental 2F2 generalized hypergeometric function. In Sect. 5, the computation of a 2F2 generalized hypergeometric function is treated. Section 6 contains the partial derivatives of a 2F2 hypergeometric function. Section 7 contains the computation of special cases. In Sect. 8 several numerical examples are considered. Conclusion is provided in Sect. 9 along with a list of references.

2 The ${}_2F_2$ Generalized Hypergeometric Function

The ${}_2F_2$ generalized hypergeometric series is formally defined as a power series [9].

Letting ${}_2F_2\left[\begin{matrix}a,b; \\ c,d;\end{matrix} z\right]$ denote a ${}_2F_2$ hypergeometric function which can be written as:

$$
\begin{aligned}
{}_2F_2\left[\begin{matrix}a,b; \\ c,d;\end{matrix} z\right] &\equiv {}_2F_2\left[\begin{matrix}a,b \\ c,d\end{matrix}; z\right] \\
&\equiv {}_2F_2\left[\begin{matrix}a,b \\ c,d\end{matrix}\middle| z\right] \\
&\equiv {}_2F_2[a,b;c,d;z] \\
&\equiv \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(d)_n}\frac{z^n}{n!}
\end{aligned} \tag{1}
$$

where a, b, c, and d are real numbers; hence, $(a)_n$, $(b)_n$, $(c)_n$, and $(d)_n$ are the Pochhammer symbol [9].

Convergence: The ratio of coefficients tends to zero if and only if or iff $a,b,c,d \neq 0$. This implies that the series converges for any finite value of z and thus defines an entire function of z [9].

Since, the main focus of this paper is the continued fraction of the ${}_2F_2$ hypergeometric function is a generalized hypergeometric function which is given in multiple references and it is also used to compute is some cases of a hypergeometric function of two variables (or a Kampé de Fériet function [3]-[5]) [9]. Therefore, the question is as follows: Is it possible to reduce the continued fraction of the ${}_2F_2$ hypergeometric function to a single summation or finite number of ${}_1F_1$ functions and a fundamental function of ${}_2F_2$ for a particular set of coefficients a, b, c, and d? Is there a connection between ${}_2F_2$ and a pair of the Kampé de Fériet functions? The answer to these questions is surprisingly yes.

However, for a particular set of coefficients a, b, c, and d of the ${}_2F_2$ we may be able to reduce the computation of ${}_2F_2$ to a continued fraction of a finite number of ${}_1F_1$ and a fundamental ${}_2F_2$. Moreover, for another particular set of coefficients a, b, c, and d we may be able to produce the computation of ${}_2F_2$ to the computation of a pair of the Kampé de Fériet functions.

3 Continued Fraction of a ${}_2F_2$ Hypergeometric Function

The ${}_2F_2$ hypergeometric function is defined via Euler Integral, since Euler was the first to have studies its integral representation [9] for all values of $0 < z < \infty$.

Definition via Euler Integral: Let us consider the more general case; hence, $0 < z < \infty$, the integral representation of the hypergeometric function [9] is

$$ {}_2F_2\left[\begin{matrix} a, b; \\ c, d; \end{matrix} z\right] = \frac{\Gamma(d)\int_0^1 t^{b-1}(1-t)^{d-b-1}{}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix} tz\right]dt}{\Gamma(b)\Gamma(d-b)} \tag{2} $$

Where ${}_1F_1$ is defined as the confluent hypergeometric function [9].

The integral representation of the confluent hypergeometric function ${}_1F_1$ is given by [6]

$$ {}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix} z\right] = \frac{\Gamma(c)\int_0^1 e^{zu}u^{a-1}(1-u)^{c-a-1}du}{\Gamma(a)\Gamma(c-a)} $$

$$ = \frac{\Gamma(c)\int_0^1 e^{tzu}u^{a-1}(1-u)^{c-a-1}du}{\Gamma(a)\Gamma(c-a)} \tag{3} $$

If we were to make the substitution, $t = e^{-v}$, then we obtain another identity of the integral representation of the hypergeometric function [8]

$$ {}_2F_2\left[\begin{matrix} a, b; \\ c, d; \end{matrix} z\right] = \frac{\Gamma(d)\int_0^\infty e^{-bv}(1-e^{-v})^{d-b-1}{}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix} e^{-v}z\right]dv}{\Gamma(b)\Gamma(d-b)} \tag{4} $$

Let us consider what happens when the coefficients b and d are given by

$$ d - b = 1 \Rightarrow d - b - 1 = 0 \Rightarrow \tag{5} $$

$$ \Gamma(d-b) = 1 \Rightarrow \Gamma(d) = b\Gamma(b) \tag{6} $$

Hence, substituting (6) into (2) yields

$$ {}_2F_2\left[\begin{matrix} a, b; \\ c, b+1; \end{matrix} z\right] = b\int_0^1 t^{b-1}{}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix} tz\right]dt $$

$$ = \frac{\Gamma(c)b\int_0^1 t^{b-1}\int_0^1 e^{ztu}u^{a-1}(1-u)^{c-a-1}du\,dt}{\Gamma(a)\Gamma(c-a)} $$

$$ = b\int_0^\infty e^{-bv}{}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix} e^{-v}z\right]dv \tag{7} $$

Let us consider what happens when the coefficients a and c are given by

$$ c - a = 1 \Rightarrow c - a - 1 = 0 \Rightarrow \tag{8} $$

$$ \Gamma(c-a) = 1 \Rightarrow \Gamma(c) = a\Gamma(a) \tag{9} $$

Substituting (9) into (7) yields,

$$ {}_2F_2\left[\begin{matrix} a, b; \\ a+1, b+1; \end{matrix} z\right] = b\int_0^1 \frac{{}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix} tz\right]}{t^{1-b}}dt $$

$$ = ab\int_0^1\int_0^1 \frac{e^{ztu}u^{a-1}}{t^{1-b}}du\,dt \tag{10} $$

Finally, if we assume that $a = b$ then we have

$$ {}_2F_2\left[\begin{matrix} a, a; \\ a+1, a+1; \end{matrix} z\right] = a\int_0^1 \frac{{}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix} tz\right]}{t^{1-a}}dt $$

$$ = a^2\int_0^1\int_0^1 e^{ztu}(tu)^{a-1}du\,dt \tag{11} $$

Next, we substitute $z = -z$ we have

$$ {}_2F_2\left[\begin{matrix} a, a; \\ a+1, a+1; \end{matrix} (-z)\right] = a^2\int_0^1 t^{a-1}\int_0^1 e^{-ztu}u^{a-1}du\,dt $$

$$ = \frac{a^2}{z^a}\int_0^1 t^{-1}\int_0^{zt} e^{-x}x^{a-1}dx\,dt $$

$$ = \frac{a^2}{z^a}\int_0^1 t^{-1}\gamma(a, zt)dt \tag{12} $$

I initially came across for this result (12) back in 2016 when I published (Progri 2016, [3]) and I attempted to develop the integer case without success in 2016; however, I was able to do so in (Progri 2019 [5], (145)).

Differentiation: The generalized hypergeometric function ${}_2F_2$ satisfies the following differentiation relation [9]

$$ \frac{d\,{}_2F_2\left[\begin{matrix} a, b; \\ c, d; \end{matrix} z\right]}{dz} = \frac{ab}{cd}\,{}_2F_2\left[\begin{matrix} a+1, b+1; \\ c+1, d+1; \end{matrix} z\right] \tag{13} $$

Similarly, the differentiation of the confluent hypergeometric function ${}_1F_1$ satisfies the following differentiation relation [6]

$$ \frac{d\,{}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix} z\right]}{dz} = \frac{a}{c}\,{}_1F_1\left[\begin{matrix} a+1; \\ c+1; \end{matrix} z\right] \tag{14} $$

Let us go back to (10) again and then integrating by parts we have

$$ \int_0^1 \frac{{}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix} tz\right]}{t^{1-b}}dt = \left|\begin{matrix} \left.\dfrac{\int {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix} uz\right]du}{t^{1-b}}\right|_0^1 - \\ (b-1)\int_0^1 \dfrac{\left[\int {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix} uz\right]du\right]}{t^{2-b}}dt \end{matrix}\right. \tag{15} $$

From the main differentiation theorem we have

$$ \frac{\int {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix} uz\right]dzu}{z} = \frac{a\,{}_1F_1\left[\begin{matrix} a-1; \\ a; \end{matrix} z\right]}{z(a-1)} \tag{16} $$

Substituting (16) into (15) yields

$$ \int_0^1 \frac{{}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix} tz\right]}{t^{1-b}}dt = \frac{a\,{}_1F_1\left[\begin{matrix} a-1; \\ a; \end{matrix} tz\right]\Big|_0^1}{t^{1-b}z(a-1)} - \frac{a(b-1)\int_0^1 \frac{{}_1F_1\left[\begin{matrix} a-1; \\ a; \end{matrix} tz\right]}{t^{2-b}}dt}{z(a-1)} $$

$$= \frac{a\,{}_1F_1\left[{a-1; \atop a;}z\right]}{z(a-1)} - \frac{a(b-1)\int_0^1 \frac{{}_1F_1\left[{a-1; \atop a;}tz\right]}{t^{2-b}}dt}{z(a-1)} \tag{17}$$

Next, we evaluate the integral in (17) by parts in much the same way as in (16) by substituting $b \to b-1$ and $a \to a-1$

$$\int_0^1 \frac{{}_1F_1\left[{a-1; \atop a;}tz\right]}{t^{2-b}}dt = \left|\begin{array}{l} \frac{(a-1)\,{}_1F_1\left[{a-2; \atop a-1;}z\right]}{z(a-2)} - \\ \frac{(a-1)(b-2)\int_0^1 \frac{{}_1F_1\left[{a-2; \atop a-1;}tz\right]}{t^{3-b}}dt}{z(a-2)} \end{array}\right. \tag{18}$$

Substituting (18) into (17) yields

$$\int_0^1 \frac{{}_1F_1\left[{a; \atop a+1;}tz\right]}{t^{1-b}}dt = a\left|\begin{array}{l} \frac{{}_1F_1\left[{a-1; \atop a;}z\right]}{z(a-1)} - \frac{(b-1)\,{}_1F_1\left[{a-2; \atop a-1;}z\right]}{z^2(a-2)} + \\ \frac{(b-1)(b-2)\int_0^1 \frac{{}_1F_1\left[{a-2; \atop a-1;}tz\right]}{t^{3-b}}dt}{z^2(a-2)} \end{array}\right. \tag{19}$$

Equation (19) can be written as a summation as follows

$$\int_0^1 \frac{{}_1F_1\left[{a; \atop a+1;}tz\right]}{t^{1-b}}dt = a\left|\begin{array}{l} \sum_{k=0}^{n-1} \frac{(-1)^k \begin{bmatrix} 1 & k=0 \\ (b-k) & k>0 \end{bmatrix} {}_1F_1\left[{a-1-k; \atop a-k;}z\right]}{z^{k+1}(a-1-k)} \\ + \frac{(-1)^n \prod_{k=1}^n (b-k) \int_0^1 \frac{{}_1F_1\left[{a-n; \atop a+1-n;}tz\right]}{t^{n+1-b}}dt}{z^n(a-n)} \end{array}\right. \tag{20}$$

If we assume that b is an integer and $b = p+1$ an integer; hence, the term that multiplies the integral simplifies; therefore, (20) becomes

$$\int_0^1 \frac{{}_1F_1\left[{a; \atop a+1;}tz\right]}{t^{1-a}}dt = a\left|\begin{array}{l} \sum_{k=0}^{n-1} \frac{(-1)^k \begin{bmatrix} 1 & k=0 \\ (a-k) & k>0 \end{bmatrix} {}_1F_1\left[{a-1-k; \atop a-k;}z\right]}{z^{k+1}(a-1-k)} + \\ \frac{\prod_{k=1}^n (p+1-k) \int_0^1 t^{p-n}\,{}_1F_1\left[{a-n; \atop a+1-n;}tz\right]dt}{(-1)^{-n} z^n(a-n)} \end{array}\right.$$

$$= a\left|\begin{array}{l} \sum_{k=0}^{n-1} \frac{(-1)^k \begin{bmatrix} 1 & k=0 \\ (a-k) & k>0 \end{bmatrix} {}_1F_1\left[{a-1-k; \atop a-k;}z\right]}{z^{k+1}(a-1-k)} + \\ \frac{\prod_{k=1}^n (p+1-k) \int_0^1 t^{p-n} \sum_{k=0}^{\infty} \frac{(a-n)_k}{(a+1-n)_k}\frac{(tz)^k}{k!}dt}{(-1)^{-n} z^n(a-n)} \end{array}\right.$$

$$= a\left|\begin{array}{l} \sum_{k=0}^{n-1} \frac{(-1)^k \begin{bmatrix} 1 & k=0 \\ (a-k) & k>0 \end{bmatrix} {}_1F_1\left[{a-1-k; \atop a-k;}z\right]}{z^{k+1}(a-1-k)} + \\ \frac{\prod_{k=1}^n (p+1-k) \sum_{k=0}^{\infty} \frac{(a-n)_k}{(a+1-n)_k}\frac{z^k}{k!} \int_0^1 t^{p-n+k}dt}{(-1)^{-n} z^n(a-n)} \end{array}\right.$$

$$= a\left|\begin{array}{l} \sum_{k=0}^{n-1} \frac{(-1)^k \begin{bmatrix} 1 & k=0 \\ (a-k) & k>0 \end{bmatrix} {}_1F_1\left[{a-1-k; \atop a-k;}z\right]}{z^{k+1}(a-1-k)} + \\ \frac{\prod_{k=1}^n (p+1-k) \sum_{k=0}^{\infty} \frac{(a-n)_k(1)_k}{(a+1-n)_k(2)_k}\frac{z^k}{k!}}{(-1)^{-n} z^n(a-n)} \end{array}\right.$$

$$= a\left|\begin{array}{l} \sum_{k=0}^{n-1} \frac{(-1)^k \begin{bmatrix} 1 & k=0 \\ (a-k) & k>0 \end{bmatrix} {}_1F_1\left[{a-1-k; \atop a-k;}z\right]}{z^{k+1}(a-1-k)} + \\ \frac{\prod_{k=1}^n (p+1-k)\,{}_2F_2\left[{a-n,1; \atop a+1-n,2;}z\right]}{(-1)^{-n} z^n(a-n)} \end{array}\right. \tag{21}$$

Again, going back to (10) if we assume that we have $a = n+1$ an integer then

$${}_2F_2\left[{a,b; \atop a+1,b+1;}z\right] = b\int_0^1 \frac{{}_1F_1\left[{a; \atop a+1;}tz\right]}{t^{1-a}}dt$$

$$= ab\left|\begin{array}{l} \sum_{k=0}^{n-1} \frac{\begin{bmatrix} 1 & k=0 \\ (b-k) & k>0 \end{bmatrix} {}_1F_1\left[{a-1-k; \atop a-k;}z\right]}{(-1)^{-k}z^{k+1}(a-1-k)} \\ + \frac{\prod_{k=1}^n (p+1-k)\,{}_2F_2\left[{1,1; \atop 2,2;}z\right]}{(-1)^{-n} z^n} \end{array}\right. \tag{22}$$

Hence, for this special case we have reduced the computation of ${}_2F_2$ into a finite number of computations of ${}_1F_1$ and one computation of a fundamental ${}_2F_2[1,1;2,2;z]$.

Now, we can apply the continued fraction expansion of ${}_1F_1$ we obtain the following [6]

$${}_2F_2\left[{a,b; \atop a+1,b+1;}z\right] = ab\left|\begin{array}{l} e^z \sum_{k=0}^{n-1} \frac{\begin{bmatrix} 1 & k=0 \\ (b-k) & k>0 \end{bmatrix} {}_1F_1\left[{1; \atop a-k;}-z\right]}{(-1)^{-k}z^{k+1}(a-1-k)} \\ + \frac{\prod_{k=1}^n (p+1-k)\,{}_2F_2\left[{1,1; \atop 2,2;}z\right]}{(-1)^{-n} z^n(a-n)} \end{array}\right. \tag{23}$$

If we compare and contrast (23) with identity 4, in [9] we see that we have taken advantage of the differentiation and integration properties to reduce the efficient and accurate computation of ${}_2F_2$ via continued fraction to a finite sum of the efficient computation of ${}_1F_1$ and the computation of the fundamental of ${}_2F_2$. The identity 4, in [9] cannot be employed in this case; nevertheless, because it does not satisfy the conditions of the parameters a and b.

So, if also $b = a = n+1 = m$ we have

$${}_2F_2\left[{m,m; \atop m+1,m+1;}z\right] = m^2\left|\begin{array}{l} \frac{\sum_{k=0}^{n-1} \frac{\begin{bmatrix} 1 & k=0 \\ (m-k) & k>0 \end{bmatrix} {}_1F_1\left[{1; \atop m-k;}-z\right]}{(-1)^{-k}z^{k+1}(n-k)}}{e^{-z}} \\ + \frac{\prod_{k=1}^{n-1} (m-k)\,{}_2F_2\left[{1,1; \atop 2,2;}z\right]}{(-1)^{n} z^n} \end{array}\right. \tag{24}$$

The question is as follows: why is the computation of ${}_2F_2[m,m;m+1,m+1;z]$ so important? Why cannot we just use the direct computation via the following

$${}_2F_2\left[{m,m; \atop m+1,m+1;}z\right] = \sum_{k=0}^{\infty} \frac{(m)_k(m)_k}{(m+1)_k(m+1)_k}\frac{z^k}{k!} \tag{25}$$

The main reason why (24) is preferred instead of (25) is because in certain situations a simplified expression like (24) is preferred instead of (25).

Since in (24) the computation of ${}_2F_2[1,1;2,2;z]$ is required; we need to explore this computation a bit more in great details.

This concludes the derivations of the continued fraction of a special case of ${}_2F_2$ hypergeometric function.

4 The Computation of the Fundamental ${}_2F_2$ Generalized Hypergeometric Function

In this section we discuss the computation of the fundamental ${}_2F_2[1,1;2,2;z]$ hypergeometric function.

First the integral definition of the fundamental ${}_2F_2[1,1;2,2;z]$ hypergeometric function is as follows:

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \frac{\Gamma(2)\int_0^1 t^{1-1}(1-t)^{1-1}\,{}_1F_1\left[\begin{matrix}1;\\2;\end{matrix}tz\right]dt}{\Gamma(1)\Gamma(1)} $$

$$ = \int_0^1 {}_1F_1\left[\begin{matrix}1;\\2;\end{matrix}tz\right]dt \tag{26} $$

If we substitute the integral expression of ${}_1F_1$ we obtain the following

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \int_0^1\int_0^1 e^{ztu}du\,dt \tag{27} $$

Next, substituting the series expansion of the exponential function we obtain

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \int_0^1\int_0^1 \sum_{k=0}^{\infty}\frac{(ztu)^k}{k!}du\,dt \tag{28} $$

Changing the order of summation and integration yields

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \sum_{k=0}^{\infty}\frac{z^k}{k!}\int_0^1 t^k dt\int_0^1 u^k du \tag{29} $$

Hence, the final answer is as follows

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \sum_{k=0}^{\infty}\frac{z^k}{k!}\frac{t^{k+1}\big|_0^1 u^{k+1}\big|_0^1}{(k+1)(k+1)} \tag{30} $$

Which can be written in the computational form that we are most familiar with the help of the Pochhammer symbol [10], gamma function [11], or in a recursive form

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \sum_{k=0}^{\infty}\frac{(1)_k(1)_k}{(2)_k(2)_k}\frac{z^k}{k!} $$

$$ = \sum_{k=0}^{\infty}\frac{\Gamma(2)\Gamma(2)\Gamma(1+k)\Gamma(1+k)}{\Gamma(1)\Gamma(1)\Gamma(2+k)\Gamma(2+k)}\frac{z^k}{k!} $$

$$ = \sum_{k=0}^{\infty}\frac{1}{(k+1)^2}\frac{z^k}{k!} \tag{31} $$

Luckily there is another way to compute ${}_2F_2[1,1;2,2;z]$. Going back to (26) and using the identity of the confluent hypergeometric function [6] we have

$$ {}_1F_1\left[\begin{matrix}1;\\2;\end{matrix}z\right] = e^{\frac{z}{2}}\,{}_0F_1\left[\begin{matrix}-;\\\frac{3}{2};\end{matrix}\frac{z^2}{16}\right] = e^{\frac{z}{2}}\left(\frac{z}{4}\right)^{-\frac{1}{2}}\Gamma\left(\frac{3}{2}\right)I_{\frac{1}{2}}\left(\frac{z}{2}\right) \tag{32} $$

Using the series expansion of $I_{1/2}(z/2)$ (see Gradshteyn, Ryzhik, 2007 [19] pg. 919 ex. 8.445) we have

$$ I_{\frac{1}{2}}\left(\frac{z}{2}\right) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma\left(\frac{3}{2}+k\right)}\frac{z^{\frac{1}{2}+2k}}{2^{1+4k}} \tag{33} $$

Next, substituting (33) into (32) yields

$$ {}_1F_1\left[\begin{matrix}1;\\2;\end{matrix}z\right] = e^{\frac{z}{2}}\Gamma\left(\frac{3}{2}\right)\sum_{k=0}^{\infty}\frac{1}{k!\Gamma\left(\frac{3}{2}+k\right)}\frac{z^{2k}}{2^{4k}} \tag{34} $$

Next, substituting (34) into (26) produces

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \Gamma\left(\frac{3}{2}\right)\int_0^1 e^{\frac{tz}{2}}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma\left(\frac{3}{2}+k\right)}\frac{z^{2k}t^{2k}}{2^{4k}}dt \tag{35} $$

Changing the order of summation and integration produces

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \Gamma\left(\frac{3}{2}\right)\sum_{k=0}^{\infty}\frac{1}{k!\Gamma\left(\frac{3}{2}+k\right)}\frac{z^{2k}}{2^{4k}}\int_0^1 t^{2k}e^{\frac{tz}{2}}dt \tag{36} $$

Using the example in (see Gradshteyn, Ryzhik, 2007 [19] pg. 340 ex. 3.351 1.) we have

$$ \int_0^1 t^{2k}e^{\frac{tz}{2}}dt = \left(-\frac{z}{2}\right)^{-2k-1}\gamma\left(2k+1,-\frac{z}{2}\right) \tag{37} $$

Next, substituting (37) into (36) we obtain the following

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \Gamma\left(\frac{3}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^{-2k-1}}{k!\Gamma\left(\frac{3}{2}+k\right)}\frac{z^{-1}\gamma\left(2k+1,-\frac{z}{2}\right)}{2^{2k-1}} \tag{38} $$

Next, using the series expansion of the incomplete gamma function based on (see Progri 2016 [3] pg. 28 (139)) we have

$$ \gamma\left(2k+1,-\frac{z}{2}\right) = \frac{e^{\frac{z}{2}}\left[\frac{\sum_{m=0}^{\infty}\frac{{z_2}^m}{\left(k+\frac{3}{2}\right)_m(k+1)_m}}{2\left(k+\frac{1}{2}\right)}+\frac{-\frac{z}{2}\sum_{m=0}^{\infty}\frac{{z_2}^m}{(k+2)_m\left(k+\frac{3}{2}\right)_m}}{4\left(k+\frac{1}{2}\right)(k+1)}\right]}{\left(-\frac{z}{2}\right)^{-2k-1}} \tag{39} $$

Where

$$ z_2 = \frac{\left(-\frac{z}{2}\right)^2}{4} = \frac{z^2}{16} \tag{40} $$

Next, substituting (39) into (38) produces

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \frac{\Gamma\left(\frac{3}{2}\right)\left|\begin{matrix}\sum_{k,m=0}^{\infty}\frac{\frac{(1)_m}{\left(k+\frac{3}{2}\right)_m(k+1)_m}\frac{z^{2k}}{4^{2k}}{z_2}^m}{\left(k+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}+k\right)\;\;k!\;\;m!}\\-\frac{z}{4}\sum_{k,m=0}^{\infty}\frac{\frac{(1)_m}{(k+2)_m\left(k+\frac{3}{2}\right)_m}\frac{z^{2k}}{4^{2k}}{z_2}^m}{\left(k+\frac{1}{2}\right)(k+1)\Gamma\left(\frac{3}{2}+k\right)\;\;k!\;\;m!}\end{matrix}\right.}{2e^{-\frac{z}{2}}} \tag{41} $$

Next, from the definition of the Pochhammer symbol (the rising factorials) [10] we have $\Gamma(3/2+k)$ as follows

$$\left(\frac{3}{2}\right)_k = \frac{\Gamma\left(\frac{3}{2}+k\right)}{\Gamma\left(\frac{3}{2}\right)} \tag{42}$$

Substituting (42) into (41) yields

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \frac{e^{\frac{z}{2}}}{2}\left|\begin{matrix}\sum_{k,m=0}^{\infty}\frac{\frac{(1)_m}{\left(k+\frac{3}{2}\right)_m(k+1)_m}}{\left(k+\frac{1}{2}\right)\left(\frac{3}{2}\right)_k}\frac{{z_1}^k}{k!}\frac{{z_2}^m}{m!}-\\ \frac{z}{4}\sum_{k,m=0}^{\infty}\frac{\frac{(1)_m}{(k+2)_m\left(k+\frac{3}{2}\right)_m}}{\left(k+\frac{1}{2}\right)(k+1)\left(\frac{3}{2}\right)_k}\frac{{z_1}^k}{k!}\frac{{z_2}^m}{m!}\end{matrix}\right. \tag{43}$$

Next, from the definition of the Pochhammer symbol (the rising factorials) [10] we have $(1/2+k)$ as follows

$$\left(\frac{1}{2}+k\right) = \frac{\frac{1}{2}\left(\frac{3}{2}\right)_k}{\left(\frac{1}{2}\right)_k} \tag{44}$$

And

$$(1+k) = \frac{(2)_k}{(1)_k} \tag{45}$$

Next, substituting (44) and (45) into (43) yields

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = e^{\frac{z}{2}}\left|\begin{matrix}\sum_{k,m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k(1)_m}{\left(\frac{3}{2}\right)_k\left(\frac{3}{2}\right)_k\left(k+\frac{3}{2}\right)_m(k+1)_m}\frac{{z_1}^k}{k!}\frac{{z_2}^m}{m!}-\\ \frac{z}{4}\sum_{k,m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k(1)_k(1)_m}{\left(\frac{3}{2}\right)_k\left(\frac{3}{2}\right)_k(2)_k(k+2)_m\left(k+\frac{3}{2}\right)_m}\frac{{z_1}^k}{k!}\frac{{z_2}^m}{m!}\end{matrix}\right. \tag{46}$$

Next, from the definition of the Pochhammer symbol (the rising factorials) [10] we have as follows

$$(a)_k(k+a)_m = (a)_{k+m} \tag{47}$$

Next, substituting (47) into (46) we obtain the following

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = e^{\frac{z}{2}}\left|\begin{matrix}\sum_{k,m=0}^{\infty}\frac{(1)_k\left(\frac{1}{2}\right)_k(1)_m}{\left(\frac{3}{2}\right)_{k+m}(1)_{k+m}\left(\frac{3}{2}\right)_k}\frac{{z_1}^k}{k!}\frac{{z_2}^m}{m!}-\\ \frac{z}{4}\sum_{k,m=0}^{\infty}\frac{(1)_k\left(\frac{1}{2}\right)_k(1)_m}{(2)_{k+m}\left(\frac{3}{2}\right)_{k+m}\left(\frac{3}{2}\right)_k}\frac{{z_1}^k}{k!}\frac{{z_2}^m}{m!}\end{matrix}\right. \tag{48}$$

We can easily see that the fundamental ${}_2F_2[1,1;2,2;z]$ can be obtained by the means of the difference of two Kampé de Fériet function (or double hypergeometric series) [3]-[5]

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] = \frac{F^{0:2;1}_{2:1;0}\left[\begin{matrix}-:1,\frac{1}{2};1;\\ \frac{3}{2},1;\frac{3}{2};-;\end{matrix}z_1,z_2\right]-\frac{z}{4}F^{0:2;1}_{2:1;0}\left[\begin{matrix}-:1,\frac{1}{2};1;\\ 2,\frac{3}{2};\frac{3}{2};-;\end{matrix}z_1,z_2\right]}{e^{-\frac{z}{2}}} \tag{49}$$

Where

$$z_1 \equiv z_2 = \left(\frac{z}{4}\right)^2 = \frac{z^2}{16} \tag{50}$$

We cannot really compute the ${}_2F_2[1,1;2,2;z]$ any other way than either (31) or (49) or it will lead to singularities.

5 The Computation of a ${}_2F_2$ Generalized Hypergeometric Function

In this section we consider the computation of another ${}_2F_2$ generalized hypergeometric function [9] of the form

$$ {}_2F_2\left[\begin{matrix}a,b;\\2a,d;\end{matrix}z\right] = \frac{\Gamma(d)\int_0^1 t^{b-1}(1-t)^{d-b-1}{}_1F_1\left[\begin{matrix}a;\\2a;\end{matrix}tz\right]dt}{\Gamma(b)\Gamma(d-b)} \tag{51}$$

Applying the conditions (5) and (6) in (51) we obtain

$$ {}_2F_2\left[\begin{matrix}a,b;\\2a,b+1;\end{matrix}z\right] = b\int_0^1 t^{b-1}{}_1F_1\left[\begin{matrix}a;\\2a;\end{matrix}tz\right]dt \tag{52}$$

Next, employing the identity of the confluent hypergeometric function [6] we have

$$ {}_1F_1\left[\begin{matrix}a;\\2a;\end{matrix}z\right] = e^{\frac{z}{2}}{}_0F_1\left[\begin{matrix}-;\\a+\frac{1}{2};\end{matrix}\frac{z^2}{16}\right] = e^{\frac{z}{2}}\frac{\Gamma\left(a+\frac{1}{2}\right)I_{a-\frac{1}{2}}\left(\frac{z}{2}\right)}{\left(\frac{z}{4}\right)^{a-\frac{1}{2}}} \text{ii} \tag{53}$$

Using the series expansion of $I_{a-1/2}(z/2)$ (see Gradshteyn, Ryzhik, 2007 [19] pg. 919 ex. 8.445) we have

$$I_{a-\frac{1}{2}}\left(\frac{z}{2}\right) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma\left(a+\frac{1}{2}+k\right)}\frac{z^{a-\frac{1}{2}+2k}}{2^{2a-1+4k}} \tag{54}$$

Next, substituting (54) into (53) yields

$$ {}_1F_1\left[\begin{matrix}a;\\2a;\end{matrix}z\right] = e^{\frac{z}{2}}\sum_{k=0}^{\infty}\frac{1}{k!\frac{\Gamma\left(a+\frac{1}{2}+k\right)}{\Gamma\left(a+\frac{1}{2}\right)}}\frac{z^{2k}}{2^{4k}} \tag{55}$$

Next, substituting (55) into (52) produces

$$ {}_2F_2\left[\begin{matrix}a,b;\\2a,b+1;\end{matrix}z\right] = b\int_0^1 t^{b-1}e^{\frac{tz}{2}}\sum_{k=0}^{\infty}\frac{z^{2k}}{k!\frac{\Gamma\left(a+\frac{1}{2}+k\right)}{\Gamma\left(a+\frac{1}{2}\right)}}\frac{t^{2k}}{2^{4k}}dt \tag{56}$$

Changing the order of summation and integration produces

$$ {}_2F_2\left[\begin{matrix}a,b;\\2a,b+1;\end{matrix}z\right] = b\sum_{k=0}^{\infty}\frac{\frac{z^{2k}}{2^{4k}}\int_0^1 e^{\frac{tz}{2}}t^{b+2k-1}dt}{k!\frac{\Gamma\left(a+\frac{1}{2}+k\right)}{\Gamma\left(a+\frac{1}{2}\right)}} \tag{57}$$

Using the example in (see Gradshteyn, Ryzhik, 2007 [19] pg. 340 ex. 3.351 1.) we have

$$\int_0^1 e^{\frac{tz}{2}}t^{b+2k-1}dt = \left(-\frac{z}{2}\right)^{-b-2k}\gamma\left(b+2k,-\frac{z}{2}\right) \tag{58}$$

Next, substituting (58) into (57) we obtain the following

$$ {}_2F_2\left[\begin{matrix}a,b;\\2a,b+1;\end{matrix}z\right] = b\sum_{k=0}^{\infty}\frac{\frac{z^{2k}}{2^{4k}}\left(-\frac{z}{2}\right)^{-b-2k}\gamma\left(b+2k,-\frac{z}{2}\right)}{k!\frac{\Gamma\left(a+\frac{1}{2}+k\right)}{\Gamma\left(a+\frac{1}{2}\right)}} \tag{59}$$

Next, substituting (19) of (Progri, 2018 [4]) and (47) into (59) produces

$$ {}_2F_2\left[\begin{matrix} a,b; \\ 2a,b+1; \end{matrix} z\right] = e^{\frac{z}{2}} \left| \begin{array}{l} \sum_{k,m=0}^{\infty} \frac{\left(\frac{b+1}{2}\right)_k \left(\frac{b}{2}\right)_k (1)_m \frac{z_1{}^k}{k!}\frac{z_2{}^m}{m!}}{\left(\frac{b+2}{2}\right)_{k+m}\left(\frac{b+1}{2}\right)_{k+m}\left(a+\frac{1}{2}\right)_k} - \\ \frac{z\sum_{k,m=0}^{\infty} \frac{\left(\frac{b+1}{2}\right)_k \left(\frac{b}{2}\right)_k (1)_m \frac{z_1{}^k}{k!}\frac{z_2{}^m}{m!}}{\left(\frac{b+3}{2}\right)_{k+m}\left(\frac{b+2}{2}\right)_{k+m}\left(a+\frac{1}{2}\right)_k}}{2(b+1)} \end{array}\right. \quad (60) $$

We can easily see that the ${}_2F_2[a,b;2a,b+1;z]$ can be obtained by the means of the difference of two Kampé de Fériet function (or double hypergeometric series) [3]-[5]

$$ {}_2F_2\left[\begin{matrix} a,b; \\ 2a,c; \end{matrix} z\right] = \frac{F^{0:2;1}_{2:1;0}\left[\begin{matrix} -:b_1,b_0;1; \\ b_2,b_1;a_1;-; \end{matrix} z_1,z_2\right] - \frac{z}{4b_1}F^{0:2;1}_{2:1;0}\left[\begin{matrix} -:b_1 b_0;1; \\ b_3,b_2;a_1;-; \end{matrix} z_1,z_2\right]}{e^{-\frac{z}{2}}} \quad (61) $$

where

$$ b_0 = \frac{b}{2} \quad (62) $$

$$ b_1 = \frac{b+1}{2} \quad (63) $$

$$ b_2 = \frac{b+2}{2} \quad (64) $$

$$ b_3 = \frac{b+3}{2} \quad (65) $$

$$ a_1 = a + \frac{1}{2} \quad (66) $$

and where z_1, z_2 are given by (50) and (40) respectively and $c - b = 1$.

One can easily show that for $a = 1$, and $b = 1$ (61) reduces to (49); hence, this concludes the derivations of the computation of a ${}_2F_2$ generalized hypergeometric function.

6 Partial Derivatives of a ${}_2F_2$ Hypergeometric Function

There may be applications that require the computation of the partial derivatives of ${}_2F_2$. In this section we derive all the steps that are required to complete such a computation.

Next, let us take the partial derivative of (11) with respect to a

$$ \frac{\partial\, {}_2F_2\left[\begin{matrix} a,a; \\ a+1,a+1; \end{matrix}(-z)\right]}{\partial a} = \frac{\partial\left[a\int_0^1 t^{a-1}\, {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix}(-tz)\right]dt\right]}{\partial a} $$

$$ = \frac{\partial\left[a^2\int_0^1\int_0^1 e^{-ztu}(tu)^{a-1}du\,dt\right]}{\partial a} \quad (67) $$

Using the main differentiation property

$$ \frac{\partial\, {}_2F_2\left[\begin{matrix} a,a; \\ a+1,a+1; \end{matrix}(-z)\right]}{\partial a} = \left| \begin{array}{l} \int_0^1 t^{a-1}\, {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix}(-tz)\right]dt + \\ a\frac{\partial\int_0^1 t^{a-1}\, {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix}(-tz)\right]dt}{\partial a} \end{array}\right. $$

$$ = \left| \begin{array}{l} 2a\int_0^1\int_0^1 e^{-ztu}(tu)^{a-1}du\,dt + \\ a^2\frac{\partial\int_0^1\int_0^1 e^{-ztu}(tu)^{a-1}du\,dt}{\partial a} \end{array}\right. \quad (68) $$

From where we find from [16]-[18]

$$ \frac{\partial\left[\int_0^1 t^{a-1}\, {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix}(-tz)\right]dt\right]}{\partial a} = \int_0^1 \frac{\partial\left[t^{a-1}\, {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix}(-tz)\right]\right]}{\partial a}dt $$

$$ = \left| \begin{array}{l} \int_0^1\int_0^1 e^{-ztu}(tu)^{a-1}du\,dt + \\ a\int_0^1\int_0^1 e^{-ztu}\frac{\partial\left[(tu)^{a-1}\right]}{\partial a}du\,dt \end{array}\right. \quad (69) $$

From where we find that

$$ \frac{\partial\left[a\int_0^1\int_0^1 e^{-ztu}(tu)^{a-1}du\,dt\right]}{\partial a} = \left| \begin{array}{l} \int_0^1\int_0^1 \frac{(tu)^{a-1}}{e^{ztu}}du\,dt + \\ a\int_0^1\int_0^1 \frac{(tu)^{a-1}\log(tu)}{e^{ztu}}du\,dt \end{array}\right. \quad (70) $$

Substituting (70) into (69) yields

$$ \frac{\partial\left[\int_0^1 t^{a-1}\, {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix}(-tz)\right]dt\right]}{\partial a} = \left| \begin{array}{l} \frac{\int_0^1 t^{a-1}\, {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix}(-tz)\right]dt}{a} + \\ a\int_0^1\int_0^1 \frac{(tu)^{a-1}\log(tu)}{e^{ztu}}du\,dt \end{array}\right. \quad (71) $$

Next, substituting (71) into (68) produces

$$ \frac{\partial\, {}_2F_2\left[\begin{matrix} a,a; \\ a+1,a+1; \end{matrix}(-z)\right]}{\partial a} = \left| \begin{array}{l} 2\int_0^1 t^{a-1}\, {}_1F_1\left[\begin{matrix} a; \\ a+1; \end{matrix}(-tz)\right]dt + \\ a^2\int_0^1\int_0^1 \frac{(tu)^{a-1}\log(tu)}{e^{ztu}}du\,dt \end{array}\right. \quad (72) $$

Next, substituting (10) into (72) we obtain

$$ \frac{\partial\, {}_2F_2\left[\begin{matrix} a,a; \\ a+1,a+1; \end{matrix}(-z)\right]}{\partial a} = \left| \begin{array}{l} \frac{2\,{}_2F_2\left[\begin{matrix} a,a; \\ a+1,a+1; \end{matrix}(-z)\right]}{a} + \\ a^2\int_0^1\int_0^1 \frac{\log(tu)(tu)^{a-1}}{e^{ztu}}du\,dt \end{array}\right. \quad (73) $$

Let us evaluate the inside integral of (73) by parts

$$ \int_0^1 \frac{\log(tu)(tu)^{a-1}}{e^{ztu}}du = \left.\frac{\log(tu)\gamma(a,ztu)}{z^a}\right|_0^1 + \frac{\int_0^1 u^{-1}\gamma(a,zu)du}{z^a} $$

$$ = \frac{\log(t)\gamma(a,zt)}{z^a} + \frac{\int_0^1 u^{-1}\gamma(a,zu)du}{z^a} \text{ iii} \quad (74) $$

Substituting (74) into (73) yields

$$ \frac{\partial\, {}_2F_2\left[\begin{matrix} a,a; \\ a+1,a+1; \end{matrix}(-z)\right]}{\partial a} = \left| \begin{array}{l} \frac{2\,{}_2F_2\left[\begin{matrix} a,a; \\ a+1,a+1; \end{matrix}(-z)\right]}{a} + \frac{a^2\int_0^1 \log(t)\gamma(a,zt)dt}{z^a} \\ + \frac{a^2\int_0^1\int_0^1 u^{-1}\gamma(a,zu)du\,dt}{z^a} \end{array}\right. \quad (75) $$

Next, substituting (12) into (75) produces

$$\frac{\partial\, {}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]}{\partial a}=\left|\begin{array}{l}\frac{2\,{}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]}{a}+\frac{a^2\int_0^1\log(t)\gamma(a,zt)dt}{z^a}\\+{}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]\int_0^1 dt\end{array}\right. \tag{76}$$

Next, we evaluate the last integral of (76)

$$\int_0^1\log(t)\gamma(a,zt)dt=\left|\begin{array}{l}\log(t)\int\gamma(a,zu)du\big|_0^1-\\\int_0^1 t^{-1}\int\gamma(a,zu)du\,dt\end{array}\right. \tag{77}$$

From the definition of the incomplete gamma function we have

$$\gamma(a,zu)=\frac{(zu)^a\,{}_1F_1(a,1+a;-zu)}{a} \tag{78}$$

Substituting (78) into the integral of $\int\gamma(a,zu)du$ we obtain

$$\int\gamma(a,zu)du=\int\frac{(zu)^a\,{}_1F_1(a,1+a;-zu)}{a}du$$

$$=\frac{z^a\int u^a\,{}_1F_1(a,1+a;-tu)du}{a} \tag{79}$$

Next, substituting (11) into (79) yields

$$\int\frac{{}_1F_1(a,1+a;-zu)du}{u^{-a}}={}_2F_2\left[\begin{matrix}a,a+1;\\a+1,a+2;\end{matrix}(-zt)\right]-1 \tag{80}$$

Substituting (80) into (79) produces

$$\int\gamma(a,zu)du=\frac{z^a\left[{}_2F_2\left[\begin{matrix}a,a+1;\\a+1,a+2;\end{matrix}(-zt)\right]-1\right]}{a} \tag{81}$$

Substituting (81) into (77) yields

$$\int_0^1\log(t)\gamma(a,zt)dt=\left|\begin{array}{l}\frac{\log(t)z^a\left[{}_2F_2\left[\begin{matrix}a,a+1;\\a+1,a+2;\end{matrix}(-zt)\right]-1\right]}{a}\Bigg|_0^1-\\\int_0^1 t^{-1}\int\gamma(a,zu)du\,dt\end{array}\right.$$

$$=\left|\begin{array}{l}\frac{\log(t)z^a\left[{}_2F_2\left[\begin{matrix}a,a+1;\\a+1,a+2;\end{matrix}(-zt)\right]-1\right]}{a}\Bigg|_0^1-\\\int_0^1 t^{-1}\frac{z^{\left[{}_2F_2\left[\begin{matrix}a,a+1;\\a+1,a+2;\end{matrix}(-zt)\right]-1\right]a}}{a}dt\end{array}\right.$$

$$=\left|\begin{array}{l}\frac{z^a\log(t)\,{}_2F_2\left[\begin{matrix}a,a+1;\\a+1,a+2;\end{matrix}(-zt)\right]}{a}\Bigg|_{t=1}-\\\frac{z^a\int_0^1 t^{-1}\,{}_2F_2\left[\begin{matrix}a,a+1;\\a+1,a+2;\end{matrix}(-zt)\right]dt}{a}\end{array}\right.$$

$$=-\frac{z^a\int_0^1 t^{-1}\,{}_2F_2\left[\begin{matrix}a,a+1;\\a+1,a+2;\end{matrix}(-zt)\right]dt}{a} \tag{82}$$

Substituting (82) into (76) yields

$$\frac{\partial\, {}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]}{\partial a}=\left|\begin{array}{l}\frac{(2+a)\,{}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]}{a}-\\a\int_0^1\frac{{}_2F_2\left[\begin{matrix}a,a+1;\\a+1,a+2;\end{matrix}(-zt)\right]}{t}dt\end{array}\right. \tag{83}$$

Next, from the definition of ${}_3F_3$ we have [9]

$${}_3F_3\left[\begin{matrix}a,b,h;\\c,d,g;\end{matrix}(-z)\right]=\frac{\Gamma(g)\int_0^1 t^{h-1}(1-t)^{g-h-1}\,{}_2F_2\left[\begin{matrix}a,b;\\c,d;\end{matrix}(-tz)\right]dt}{\Gamma(h)\Gamma(g-h)} \tag{84}$$

If we set the following conditions

$$g-h=1\ \Rightarrow\ g=h+1\ \Rightarrow \tag{85}$$

$$\Gamma(g-h)=1\ \Rightarrow\ \Gamma(g)=h\Gamma(h) \tag{86}$$

Substituting (86) into (84) yields

$${}_3F_3\left[\begin{matrix}a,b,h;\\c,d,h+1;\end{matrix}(-z)\right]=h\int_0^1 t^{h-1}\,{}_2F_2\left[\begin{matrix}a,b;\\c,d;\end{matrix}(-tz)\right]dt \tag{87}$$

Taking the partial derivative of ${}_3F_3$ with respect to h yields

$$\frac{\partial\,{}_3F_3\left[\begin{matrix}a,b,h;\\c,d,h+1;\end{matrix}(-z)\right]}{\partial h}=\left|\begin{array}{l}\int_0^1 t^{h-1}\,{}_2F_2\left[\begin{matrix}a,b;\\c,d;\end{matrix}(-tz)\right]dt+\\h(h-1)\int_0^1 t^{h-2}\,{}_2F_2\left[\begin{matrix}a,b;\\c,d;\end{matrix}(-tz)\right]dt\end{array}\right. \tag{88}$$

We evaluate (88) this at $h=0$ produces the desired result

$$\frac{\partial\,{}_3F_3\left[\begin{matrix}a,b,h;\\c,d,h+1;\end{matrix}(-z)\right]}{\partial h}\Bigg|_{h=0}=\int_0^1 t^{-1}\,{}_2F_2\left[\begin{matrix}a,b;\\c,d;\end{matrix}(-tz)\right]dt \tag{89}$$

Hence, our desired integral is as follows

$$\int_0^1\frac{{}_2F_2\left[\begin{matrix}a,a+1;\\a+1,a+2;\end{matrix}(-zt)\right]}{t}dt=\frac{\partial\,{}_3F_3\left[\begin{matrix}a,a+1,h;\\a+1,a+2,h+1;\end{matrix}(-z)\right]}{\partial h}\Bigg|_{h=0} \tag{90}$$

Finally, substituting (90) into (83) produces

$$\frac{\partial\, {}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]}{\partial a}=\left|\begin{array}{l}\frac{(2+a)\,{}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]}{a}-\\a\frac{\partial\,{}_3F_3\left[\begin{matrix}a,a+1,h;\\a+1,a+2,h+1;\end{matrix}(-z)\right]}{\partial h}\Bigg|_{h=0}\end{array}\right. \tag{91}$$

This is as far as we can go here. This concludes the derivations of the partial derivative of a ${}_2F_2$ generalized hypergeometric function.

7 Computation of Special Cases

There are several special cases that we can recognize in (24). Based on ([19], pg. 899, ex. 8.352 1.) we have

$$ {}_1F_1\left[\begin{matrix}1;\\ m-k;\end{matrix}-z\right]=\frac{(m-k-1)!\left[e^{-z}-\sum_{l=0}^{m-k-2}\frac{(-1)^l z^l}{l!}\right]}{(-1)^{m-k-1}z^{m-k-1}} \tag{92}$$ [iv]

Next, substituting (92) into (24) yields

$$ {}_2F_2\left[\begin{matrix}m,m;\\ m+1,m+1;\end{matrix}z\right]=m^2\left|\begin{matrix}\dfrac{\dfrac{(m-2)!\left[1-e^{z}\sum_{l=0}^{u}\frac{(-1)^l z^l}{l!}\right]}{(-1)^{m-1}}+\sum_{k=1}^{m-2}\dfrac{(m-k)!\left[1-e^{z}\sum_{l=0}^{m-2-k}\frac{(-1)^l z^l}{l!}\right]}{(-1)^{m-2k-1}(m-1-k)}}{z^m}\\ +\dfrac{\prod_{k=1}^{m-2}(m-k)\,{}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right]}{(-1)^{m-1}z^{m-1}}\end{matrix}\right. \tag{93}$$ [v]

Let us consider several special cases in order.

1. $a=m=0$. For this special case we *cannot* utilize (93) because it will vanish; however, for this special case we have

$$ {}_2F_2\left[\begin{matrix}0,0;\\1,1;\end{matrix}z\right]=1,\ \forall z=[0,+\infty) \tag{94}$$

2. $a=m=1$. For this special case we *can* utilize (93); hence, we have the fundamental ${}_2F_2[1,1;2,2;z]$

$$ {}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right]={}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right] \tag{95}$$

3. $a=m=2$. For this special case we *can* utilize (93); hence, we have

$$ {}_2F_2\left[\begin{matrix}2,2;\\3,3;\end{matrix}z\right]=4\frac{e^z-1}{z^2}-4\frac{{}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}z\right]}{z} \tag{96}$$ [vi]

The main disadvantage is that it only depends on particular values of z and it requires new computation (or it is laborious) as every time it requires new analytical and numerical computations depending on particular values of z.

8 Numerical Examples

Before we conclude this paper, we consider several numerical examples.

Example 1: The first numerical example considers the computation of a ${}_2F_2$ based on (1) and (11) for $a=\{2,3\}$ and $z=0.5$. The results of the direct computation are shown in Tab. I and of the absolute error of the direct computation are shown in Tab. II.

Example 2: The second numerical example considers the computation of a ${}_2F_2$ based on (1) and (12) for $a=\{2,3\}$ and $z=0.5$. The results of the direct computation are shown in Tab. III and of the absolute error of the direct computation are shown in Tab. IV.

TABLE I: DIRECT COMPUTATION

${}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]$	(1)	(11)
$a=2,\ z=0.5$	0.805963821286109	0.805963821286109
$a=3,\ z=0.5$	0.758985640539387	0.758985640539387

TABLE II: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

${}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]$	(11) – (1)
$a=2,\ z=0.5$	0
$a=3,\ z=0.5$	$-1.110223024625157\times10^{-16}$

TABLE III: DIRECT COMPUTATION

${}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]$	(1)	(12)
$a=2,\ z=0.5$	0.805963821286109	0.805963821286109
$a=3,\ z=0.5$	0.758985640539387	0.758985640539387

TABLE IV: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

${}_2F_2\left[\begin{matrix}a,a;\\a+1,a+1;\end{matrix}(-z)\right]$	(12) – (1)	(12) – (11)
$a=2,\ z=0.5$	0	0
$a=3,\ z=0.5$	$-1.11022302462515\times10^{-16}$	0

Example 3: The third numerical example considers the computation of a ${}_2F_2$ based on (1) and (24) for $a=\{2,3\}$ and $z=0.5$. The results of the direct computation are shown in Tab. V and of the absolute error of the direct computation are shown in Tab. VI.

For $a=2$ (22) and (24) become respectively equal to

$$ {}_2F_2\left[\begin{matrix}2,2;\\3,3;\end{matrix}-z\right]=\frac{4}{-z}\left[{}_1F_1\left[\begin{matrix}1;\\2;\end{matrix}-z\right]-{}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}-z\right]\right] \tag{97}$$

$$ {}_2F_2\left[\begin{matrix}2,2;\\3,3;\end{matrix}-z\right]=\frac{4}{-z}\left[e^{-z}\,{}_1F_1\left[\begin{matrix}1;\\2;\end{matrix}z\right]-{}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}-z\right]\right] \tag{98}$$

For $a=3$ (22) and (24) become respectively equal to

$$ {}_2F_2\left[\begin{matrix}3,3;\\4,4;\end{matrix}-z\right]=\frac{9\left[-\frac{z}{2}\,{}_1F_1\left[\begin{matrix}2;\\3;\end{matrix}-z\right]-2\,{}_1F_1\left[\begin{matrix}1;\\2;\end{matrix}-z\right]+2\,{}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}-z\right]\right]}{z^2} \tag{99}$$

$$ {}_2F_2\left[\begin{matrix}3,3;\\4,4;\end{matrix}-z\right]=\frac{9\left[-\frac{ze^{-z}}{2}\,{}_1F_1\left[\begin{matrix}1;\\3;\end{matrix}z\right]-2e^{-z}\,{}_1F_1\left[\begin{matrix}1;\\2;\end{matrix}z\right]+2\,{}_2F_2\left[\begin{matrix}1,1;\\2,2;\end{matrix}-z\right]\right]}{z^2} \tag{100}$$

As indicated by the numerical results in table VI, the absolute error due to continues fraction via (24) – (1) it is larger than when no continues fraction is applied.

Example 4: The fourth numerical example considers the computation of a ${}_2F_2$ based on (1) and (49) for $a=2$ and $z=0.5$. The results of the direct computation are shown in Tab. VII and of the absolute error of the direct computation are shown in Tab. VIII.

As indicated by the numerical results in table VIII, the absolute error due to the difference of two Kampé de Fériet functions is within numerical computational error 10^{-16}; hence, (1) and (49) are numerically equivalent.

Example 5: The fifth numerical example considers the computation of a ${}_2F_2$ based on (51) and (61) for $a = \{2,3\}$, $b = \{3,4\}$, and $z = 0.5$. The results of the direct computation are shown in Tab. IX and of the absolute error of the direct computation are shown in Tab. X.

As indicated by the numerical results in table X, the absolute error due to the difference of two Kampé de Fériet functions is within numerical computational error 10^{-16}; hence, (51) and (61) are numerically equivalent.

9 Conclusions

In conclusion, I have offered several techniques for the computation of a hypergeometric function ${}_2F_2$ either by means of direct computation or via the difference of Kampé de Fériet functions.

I have shown that a special case of ${}_2F_2$ may lead to partial fraction expansion via ${}_1F_1$ and the computation of the fundamental ${}_2F_2[1,1;2,2;z]$ as shown in (23). This is an original result never published before.

The fundamental ${}_2F_2[1,1;2,2;z]$ hypergeometric function is given by the means of the difference of two Kampé de Fériet functions as in (49). This is an original result never published before.

The ${}_2F_2[a,b;2a,b+1;z]$ generalized hypergeometric function can be obtained by the means of the difference of two Kampé de Fériet function (or double hypergeometric series) [3]-[5] as in (62). This is an original result never published before.

This paper is based almost entirely on the creation of original analytical derivations, but as far as numerical results special cases may be considered as such.

10 Acknowledgement

This work was supported by Giftet Inc. executive office.

I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [25] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications. Vol. I* (Not yet available in print) [1].

This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progri, my dear father, Fiqiri Progri, my father's first cousin Dr. Peter Demir, and Qazim Demir, the brother of my grandfather, Xhevdet Progri.

This journal paper is also dedicated to the Golden Bear, Jack Nicklaus, the greatest golfer of all time. Needless to say I have fallen in love with his masterpiece book, *Golf My Way*. Moreover, Jack Nicklaus [21] reminds me of my grandfather who I loved him very much.

TABLE V: DIRECT COMPUTATION

${}_2F_2\left[{a,a; \atop a+1,a+1;}(-z)\right]$	(22)	(24)
$a=2,\ z=0.5$	0.805963821286109	0.805963821286109
$a=3,\ z=0.5$	0.758985640539388	0.758985640539404

TABLE VI: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

${}_2F_2\left[{a,a; \atop a+1,a+1;}(-z)\right]$	(22) − (1)	(24) − (1)
$a=2,\ z=0.5$	1.110223024625157 $\times 10^{-16}$	1.887379141862766 $\times 10^{-15}$
$a=3,\ z=0.5$	7.771561172376096 $\times 10^{-16}$	3.330669073875470 $\times 10^{-15}$

TABLE VII: DIRECT COMPUTATION

${}_2F_2\left[{a,a; \atop a+1,a+1;}(-z)\right]$	(1)	(49)
$a=2,\ z=0.5$	0.887684158235497	0.887684158235497

TABLE VIII: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

${}_2F_2\left[{a,a; \atop a+1,a+1;}(-z)\right]$	(49) − (1)
$a=2,\ z=0.5$	1.110223024625157e-16

TABLE IX: DIRECT COMPUTATION

${}_2F_2\left[{a,b; \atop 2a,b+1;}(-z)\right]$	(51)	(61)
$a=2,\ b=3,\ z=0.5$	0.833066215469258	0.833066215469257
$a=3,\ b=4,\ z=0.5$	0.821829539886384	0.821829539886384

TABLE X: ABSOLUTE ERROR OF THE DIRECT COMPUTATION

${}_2F_2\left[{a,b; \atop 2a,b+1;}(-z)\right]$	(61) − (51)
$a=2,\ b=3,\ z=0.5$	8.881784197001252e-16
$a=3,\ b=4,\ z=0.5$	3.330669073875470e-16

Finally, I am also dedicating this journal paper to our most beloved President Ronald Reagan, who reminds me of my grandfather, [22] who spoke from the heart, who spoke the truth, who was a staunch supporter of personal freedom, the greatest leader of the free world, and perhaps the greatest critic of the wasteful government control and spending[vii].

"*Government is not the answer, government is the problem*."—Ronald Reagan

I would like to express my deepest gratitude to my mother Lumturi Progri, my sister Adriana Dine, and Ms. Elizabeth Demir for their support, loyally, and dedication to me during some of the most difficult times in my life as a boy, man, and scholar.

11 References

[1] I.F. Progri, *Indoor Geolocation Systems—Theory and Applications. Vol. I*, 1st ed., Worcester, MA: Giftet Inc., ~800 pp., ~ 2020. (not yet available in print)

[2] I.F. Progri, "Hypergeometric function partial derivatives," vol. 2016, article ID 2016071604, 21 pg., Nov. 2016. DOI: https://doi.org/10.18610/JG3.2016.071604

[3] I.F. Progri, "Generalized Bessel function distributions," *J. Geol. Geoinfo. Geointel.,* vol. 2016, article ID 2016071602, 15 pg., Nov. 2016. DOI: https://doi.org/10.18610/JG3.2016.071602

[4] I.F. Progri, "Efficient computation of generalized Bessel function distributions," *J. Geol. Geoinfo. Geointel.,* vol. 2018, article ID 2018071604, 12 pg., Nov. 2018. DOI: https://doi.org/10.18610/JG3.2018.071605

[5] I.F. Progri, "Efficient computation of the special cases of the generalized Bessel function distributions," *J. Geol. Geoinfo. Geointel.,* vol. 2019, article ID 2019071601, 31 pg., Nov. 2019. DOI: https://doi.org/10.18610/JG3.2019.071601

[6] Anon., "Confluent hypergeometric function," *From Wikipedia, the free encyclopedia*, Apr. 2019. https://en.wikipedia.org/wiki/Confluent_hypergeometric_function

[7] Anon., "Ernst Kummer," *From Wikipedia, the free encyclopedia*, Apr. 2019. https://en.wikipedia.org/wiki/Ernst_Kummer

[8] Anon., "Hypergeometric function," *From Wikipedia, the free encyclopedia*, Apr. 2016. https://en.wikipedia.org/wiki/hypergeometric_function

[9] Anon., "Generalized hypergeometric function," *From Wikipedia, the free encyclopedia*, Apr. 2016. https://en.wikipedia.org/wiki/Generalized_hypergeometric_function

[10] Anon., "Falling and rising factorials," *From Wikipedia, the free encyclopedia*, May 2019. https://en.wikipedia.org/wiki/Falling_and_rising_factorials

[11] Anon., "Gamma function," *From Wikipedia, the free encyclopedia*, May 2019. https://en.wikipedia.org/wiki/Gamma_function

[12] I. Gessel, D. Stanton, "Strange evaluations of hypergeometric series," *SIAM J. Math. Anal.*, vol. 13, no. 2, pp. 295-308, 1982. DOI: https://doi.org/10.1137/0513021

[13] I. Gessel, "On the reducibility of the Kampé deFériet function," *J. Sym. Com.*, vol. 20, no 5-6, pp. 537-566, Nov. 1995. DOI: https://doi.org/10.1006/jsco.1995.1064

[14] R.B. Paris, "A Kummer-type transformation for a 2F2 hypergeometric function," *J. Comput. & Appl. Math.*, vol. 173, no. 2, pp. 379-382, 2005. DOI: https://doi.org/10.1016/j.cam.2004.05.005

[15] A.R. Miller, R.B. Paris, "Clausen's series 3F2(1) with integral parameter differences and transformations of the hypergeometric function 2F2(x)," *Integral Trans. & Special Func.*, vol. 23, no. 1, pp. 21-33, 2012. DOI: https://doi.org/10.1080/10652469.2011.552263

[16] Anon., "Leibniz integral rule," *From Wikipedia, the free encyclopedia*, Apr. 2016. https://en.wikipedia.org/wiki/Leibniz_integral_rule

[17] Anon., "Differentiation under the integral sign," *From Wikipedia, the free encyclopedia*, Apr. 2016. https://en.wikipedia.org/wiki/Differentiation_under_the_integral_sign

[18] Anon., "Reynolds transport theorem," *From Wikipedia, the free encyclopedia*, Apr. 2016. https://en.wikipedia.org/wiki/Reynolds_transport_theorem

[19] I.S. Gradshteyn, I.M. Ryzhik, (A. Jeffrey, D. Zwillinger, editors) *Table of Integrals, Series, and Products*, 7th ed., Burlington, MA: Academic Press, 1171 pg., 2007.

[20] Anon., "Beta function," *From Wikipedia, the free encyclopedia*, Apr. 2016. https://en.wikipedia.org/wiki/Beta_function

[21] Anon., "Jack Nicklaus," *Wikipedia, the free encyclopedia*, Oct. 2019. https://en.wikipedia.org/wiki/Jack_Nicklaus

[22] Anon., "Ronald Reagan," *Wikipedia, the free encyclopedia*, June. 2020. https://en.wikipedia.org/wiki/Ronald_Reagan

[23] J.E. Gottschalk, E.N. Maslen, "Reduction formulae for generalised hypergeometric functions of one variable," *J. Phys. A: Math. Gen.*, vol. 21 pp. 1983-1998, 1988. DOI: https://doi.org/10.1088/0305-4470/21/9/015

[24] G.B. Arfken, H.J. Weber, *Mathematical Methods for Physicists*, San Diego, CA: Academic Press, 1995.

[25] Anon, "MATLAB 2019b," *The MathWorks, Inc.*, Natick, MA, Copyright © 1994-2019. The MathWorks, Inc., http://mathworks.com/products

[i] This identity is sometimes referred to as the Kummer's [7] second transformation.

[ii] Ditto i.

[iii] See the discussion in Progri 2016 [3], 2018 [4], and 2019 [5] for a detailed discussion of the lower limit of $\log(tu)\frac{\gamma(a,ztu)}{z^a}\Big|_0^1$.

[iv] Corrected in 2023.

[v] Ditto iv.

[vi] Ditto iv.

[vii] There is a false impression that only the research that is funded by the government bureaucrats is worthy of financial support, awards, & recognition. This is not to say that President Ronald Reagan wanted to demonize the role of the government as an enterprise, or resource, or originator of many inventions. But, when the government spending is abused for wasteful spending of the taxpayers trillions of $ dollars, and moreover, when it controls and suppresses innovation and ignores or completely denies funding to innovators and incubators based on political beliefs, then said President Ronald Reagan this type of government is the problem.