

Research Article

Description of VBOC2 GMGM Special Cases Waveforms ACF—Theory, Computation, Simulations, and Animation

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Abstract—The main purpose of this research is the continuation of the theoretical analysis, modelling, and simulation framework of futuristic generalized multidimensional geolocation modulation (GMGM) waveforms.

It is already been established that GNSS waveforms that have been selected by the GNSS community have suboptimal performance in terms of the autocorrelation function (ACF) pure signal optimization (PSO) theoretical analysis, modelling, and simulation.

This paper examines the description of VBOC2 GMGM special cases waveforms ACF—theory, computation, simulations, and animation.

There are a number of reasons why the extension of this work is important.

First, it is not really obvious or intuitive the connection that exists for all these special cases. Second, there appears to be at least two formulations of VBOC1 and of VBOC2 as special cases of VBOC2. Third, these special cases serve now as building block. One should be able to easily generalize this work. Fourth, for the first time we have produced animation of the computation of the ACF as a result of the direct integration.

The inspection of the VBOC2 signals by means of the ACF demonstrates that we have produced one *hundred percent more efficient* than the corresponding ACFs of VBOC1 signals.ⁱ

Index Terms—Pulse generation, pulse amplitude modulation, pulse width modulation, multidimensional sequences, signal design, signal analysis, generalized functions, time analysis, animation, development, execution.

1 Introduction

The main objective of this paper is the description of VBOC2(α_1, α_2) GMGM special cases waveforms ACF—theory, computation, simulations, and animation.

VBOC is a GMGM waveform of a Binary Offset Carrier (BOC) modulation, which is a GM of a Binary Phase-Shift Keying (or BPSK) modulation.

BPSK (also sometimes called PRK, phase reversal keying, or 2PSK) is the simplest form of Phase Shift Keying (PSK).

PSK was originally invented back in the 1920s in telegraphy (J. Bell, 1920, [1]) and by its inventor (Nyquistⁱⁱ 1925, [2], [3]). I believe that PSK was invented at the Bellⁱⁱⁱ Telephone Company (later known as AT&T) [5] in the United States of America (USA) [1]–[5].

In this paper the description of the BPSK, BOC, and of the special cases of the VBOC2(α_1, α_2) GMGM by means of the *auto-correlation function* (ACF) is done by means of analytical derivations, computation, simulation, and animation.

The very first appearance of the term ACF comes from (Kendall^{iv}, 1945 [7]) and (Bartlett^v 1946, [8], [9]). According to (Bartlett 1946, [8]), (Kendall, 1945 [7]) has employed the term “*autocorrelation*” to denote a true value of which the observed value is a *serial correlation*. I [(Bartlett 1946, [8])] shall use what to be a more logical terminology—viz. *serial correlation* for any correlation of one time-series with another, and *autocorrelation* for the particular series correlation with itself.”

Bayley and Hammersley gave precise numerical evaluation of an exponentially decaying ACF in 1946 in their journal paper “The “effective” number of independent observations in an autocorrelated time series” for anti-aircraft radar application [10]. They were able to define, qualitatively and quantitatively, tabulate, evaluate, and graph the ACF [10]. According to Bayley and Hammersley, the ACF was one for zero lag and less than one for any non-zero lag (see (36) pg. 188, [10]).

In its earliest definition and usage, the **ACF** is used as a means of separating (or filtering) the signal from other disturbances such as thermal noise, cross-talk, tracking errors, including other transmitting and receiver errors (Levinson, 1946 [11], 1947 [12], 2019 [13]. According to Levinson^{vi} (1946 [11]), “Winner^{vii}’s work appeared in a book of limited circulation [*It was a classified document according to* [14]] in February 1942. An independent and similar but by no means identical work by Kolmogoroff^{viii} [15] had already appeared in Bull. Acad. Sciences USSR pp. 3-14, 1941.”

Since 1946, there have been thousands of publications that have employed the definition, computation, evaluation, plotting, and implementation and various BPSK waveforms and the corresponding ACF.

Nevertheless, I believe there have been a lot fewer publications focusing on the animation of the derived ACF of such waveforms. Therefore, I believe that animation plays a very important role in the description of the ACF of the VBOC modulation. One might ask two important questions: What is animation? What role has animation played in the description of various ACFs?

Definition: Animation is defined as the technique, or the execution of an algorithm by means of a computer simulation, of photographing successive drawings, pictures, illustrations, representation, portrayal, delineation, depiction, composition, diagrams, study, outline, graphs, design, plans, plots, charts, or snap shots, or models to create an illusion of movement when the movie is shown as a sequence.

One of the earliest descriptions of animation in the computation of the ACF comes from physical chemistry (Berne, Forster 1971, [17]). According to (Berne, Forster 1971, [17]) “*The computer output may be regarded as a dynamical movie of the manybody system and can be converted into a movie via computer animation techniques. This has been done by several investigators (Harp, Berne, Paskin, Rahman, and Fehder). Such sources are a particularly convenient way to present the enormous data so that the viewer can get some insight into the dynamical behavior of molecules in condensed media.*” Therefore, one should draw the conclusion that the earliest description of animation in the ACF was probably known by a very limited number of researchers, scholars, or scientists in the late 1960s.

One of the earliest descriptions of animation and BPSK signals is by Peter Banks of Stanford University in a National Administration Space Agency (NASA) report, (Banks 1986 pg. 2, [18]) “*in the area of video research, considerable progress has been made in the development of interfaces between the state-of-the-art Bosch FGS 4000 video graphics and animation system and the Evans and Sutherland PS 300 and the IRIS 2400 CAD/CAE systems.*” and (Banks 1986 pg. 40, [18]) “*The SM200A filter performance reduces this channel spacing to 7 times the symbol rate for versions using QPSK and 1.4 times the symbol rate for versions using BPSK. This can mean lower operating costs in many situations.*”

From the Naval Postgraduate School 1990 report of that

contains the compilation of abstract thesis (Anon. 1990, [19]) we have three records of animation: (1) E.L. Pagenkopf, Lieutenant (LT), United States Navy (USN) “dynamic stall analysis utilizing *interactive computer graphics*,” Mar. 1988; (2) F.E. Harris, LT, USN, “Preliminary work on the command and control workstation of the future,” June 1988; (3) T.W. Meier, LT, USN “Investigation into the use of texturing for real-time computer animation,” Dec. 1987. According to Pagenkopf, “*Flow field solutions in the form of pressure coefficient and stream function contour plots about an airfoil experiencing dynamic stall are plotted utilizing an IRIS 3000-series workstation and Graphical Animation System (GAS) software, developed by Sterling Software for NASA.*” According to Harris “*this initial display uses three resolutions to display large areas of Defense Mapping Agency Digital Terrain Elevation Data with near real time **animation**.*” And according to Meier “*... in this study an investigation into the use of texturing on the Silicon Graphics, Inc. IRIS for real-time computer **animation**.*”

In Zalesac, Huba, Mulbrandon, 1988 [20], [21] “3D dynamics of ionospheric plasma clouds” “*A combination of analytical models, 3D numerical simulations, and computer graphics/**animation** have enabled us to make significant progress in the understanding of ionospheric structuring processes important in the space-based tethered array antenna reliable operation of military systems.*” Also in Bracco, Davis, (1988, pg. 170, [20]) “Visualizing electronic warfare simulations,” “*This operator control extends to all aspects of the display including the size of the viewed area, the **animation** speed, and the highlighting or thinning of selected platforms.*”

I did find from Wikipedia’s “Convolution” article [22], [23] some very helpful *animations* and images which I considered very much consistent with my imagination of the description of the VBOC ACF and very helpful to develop the content of this masterpiece journal paper.

I am not surprised that I found no references from the Navigation Journal of the Institute of Navigation and other ION proceedings that explicitly or implicitly discuss animation of the ACF of BPSK, QPSK, BOC, or VBOC signals or waveforms [24]-[30] because animation is not been exploited as an analytical tool due to its complexity in computation and simulation.

Therefore, the case can be made that if one is looking for a very high-quality deceptive tool then animation is the tool that needs to be exploited, explained, developed, implemented, and

written about it.

However, these are not the only reasons why I decided to develop the description of the ACF of the VBOC modulation.

In 2007, I produced the very first investigation on the VBOC modulation in (Proгри et al. 2007, [31]) which was revised in (Proгри et al. 2017, [32]). I believe that this paper lacks significant details on the integration steps that are required to derive the ACF of GMGM.

In 2012, I published an excellent conference paper on GMGM (Proгри et al. 2012, [33]) that also lacks the same.

In 2014, I created another excellent conference paper on VBOC1(α) GMGM (Proгри 2014, [34]) that was later revised in (Proгри 2015, [35], [36]). There are a few significant problems of this publication [34]-[36]. First, the ACF (as given by (9)-(22) [35]) somehow depends of the integration offset time, t_0 . Second, it does not show the steps that are required to derive the ACF even in the simplest case. Third, it does not show whether or not the definition of VBOC1(α) is unique; i.e., is the definition of VBOC1(α) [34]-[36] unique or are other definitions of VBOC1(α) GMGM?

In 2015, I expanded the investigation of VBOC1(α) into VBOC2(α, β_-) in (Proгри, 2015 [37]-[39]). Amazingly enough these publications also lack exactly the same significant issues. First, the ACF (as given by (10)-(33) [37]) somehow depend of the integration offset time, t_0 . Second, it does not show the steps that are required to derive the ACF even in the simplest case. Third, it does not show whether or not the definition of VBOC2(α, β_-) is unique; i.e., is this the only definition of VBOC2(α, β_-) or are any other definitions of VBOC2(α, β_-) GMGM?

In 2018, I expanded the investigation of VBOC1(α) in (Proгри, 2018 [40], [41]). In 2018 ([40], pg. 3), however, I raised the issue that [“The proof of theorem 2 is straightforward. However, *one always wanders how the ACF of VBOC1(α) was produced. The ACF is the result of integration...*”] the steps of the ACF integration need to be explained in great detail.

Why it is that I have not been able to clarify these issues? First, I believe that these issues were in my mind (or in my head) and I was not aware at the time how important these issues really were. Second, I was looking for a MATLAB simulation to produce the *animation* of the ACF so that would be able to verify my analytical expressions. Third, these issues were never brought up to me by readers, reviewers, editors, etc. Fourth, I never found the details of the discussion in this paper anywhere else.

I believe that the case can be made that *animation* may be seen as a very high qualitative and quantitative tool to bring into light insights, understanding, knowledge, verification of the ACF of the $VBOC2(\alpha_1, \alpha_2)$ GMGM that a researcher, scientist, scholar, developer, engineer, expert may not be able to derive by other means.

Nevertheless, as seen in the content of this paper, *development of animation* in MATLAB requires significant amount of work, creativity, originality, and novelty to produce the *execution of animation* via a GIF file of the special cases on the $VBOC2(\alpha_1, \alpha_2)$ GMGM.

This paper is entirely focused on the detailed investigation, explanation, and derivations of the steps that are required to produce the direct integration of the ACF of several special cases $VBOC2(\alpha_1, \alpha_2)$ GMGM by means of animation. The main focus of this work is to make the content as practical, visible (or graphical), understandable, as possible. After reading this paper, a reader should be able to understand in great depth as to how the direct integration process works. It should be no more mysterious, ambiguous, in my head etc. etc. I have attempted to collect enough information to make this content as clear and as user friendly as I possibly can.

This paper is organized as follows: The animation of the $VBOC1(\alpha)$ signal design is discussed next. The animation of the $VBOC2(\alpha_1, \alpha_2)$ signal design is discussed afterwards. Conclusion is given next and the paper is concluded with a list of references.

2 VBOC1(α) Signal Design Animation

In order to arrive at the animation of $VBOC2_{(m,n,\alpha)}(t)$ ACF we will start our discussion with the simplest animation: the animation of the BPSK signal ACF.

2.1 Description of the BPSK Signal and ACF Animation

PSK is a digital modulation process which conveys data by changing (modulating) the phase of a constant frequency reference signal (the carrier wave). The modulation is accomplished by varying the sine and cosine inputs at a precise time [3].

It uses two phases which are separated by 180° and so can also be termed 2-PSK. It does not particularly matter exactly where the constellation points are positioned, and in Fig. i they are shown on the real axis, at 0° and 180° . Therefore, it handles

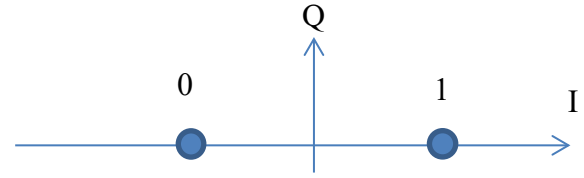


Fig. i Description of the I, Q (or constellation) diagram of BPSK signal.

the highest noise level or distortion before the demodulator reaches an incorrect decision. That makes it the most robust of all the PSKs. It is, however, only able to modulate at 1 bit/symbol (as seen in the Fig. i) and so is unsuitable for high data-rate applications [3].

Definition 1: $f(t)$ or the BPSK waveform is a periodic function with period $2T_s$, (T_s : the subcarrier period), for all values of $t - \infty < t < \infty$.

$$f(t) = f(t \pm 2T_s) \quad (1)$$

and the relation of T_s is given by

$$2T_s = T_c = T \quad (2)$$

where T_c is the defined as the chipping period and T is defined as the integration period.

The first period of the BPSK signal, $f(t)$, is given by

$$f(t) = \Pi(t) = \begin{cases} 0 & t_0 - 2T_s \leq t < t_0 - T_s \\ 1 & t_0 - T_s \leq t < t_0 \end{cases} \quad (3)$$

and t_0 as the time offset whose significance is explained later in the end of the section.

Definition 2: The ACF, $r(t)$, of this waveform is simply defined as the continuous cross-correlation integral of

$$\begin{aligned} r(t) &\equiv f(t) \otimes g(t) \\ &\equiv \int_{-\infty}^{\infty} f(\tau)g(\tau + t)d\tau \\ &\equiv \int_{-\infty}^{\infty} f(\tau + t)g(\tau)d\tau \end{aligned} \quad (4)$$

where $g(t)$ is the shifted moving waveform and $f(t)$ is the stationary waveform; which is equivalent with the convolution of the function with a flipped version of itself as

$$\begin{aligned} r(t) &= f(t) * g(-t) \\ &\equiv \int_{-\infty}^{\infty} f(\tau)g[t - (-\tau)]d\tau \\ &= f(-t) * g(t) \\ &\equiv \int_{-\infty}^{\infty} f[t - (-\tau)]g(\tau)d\tau \end{aligned} \quad (5)$$

It is obvious that (5) and (4) are identical integrals.

According to Wikipedia “The convolution of f and g is written $f * g$ [or as $f \otimes g$]; it is defined as *the integral of the product of the two functions after one is reversed and shifted*” [22], [23].

According to (Brown, 2010, pg. 15, [22]) “*The autocorrelation (correlation of a function with itself) is*

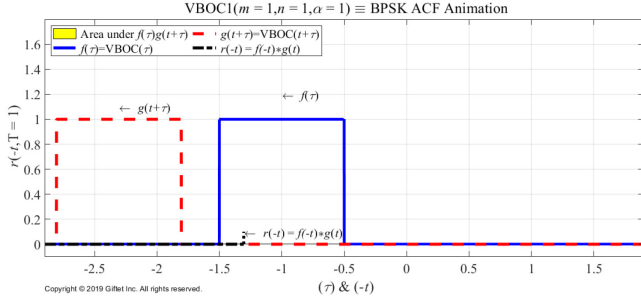


Fig. 1. BPSK ACF Case 1: no intersection between $f(\tau) = \text{VBOC1}_{(1,1,1)}(\tau)$ and $g(t + \tau) = \text{VBOC1}_{(1,1,1)}(t + \tau)$; $t_0 = -0.5$, $T = 1$.

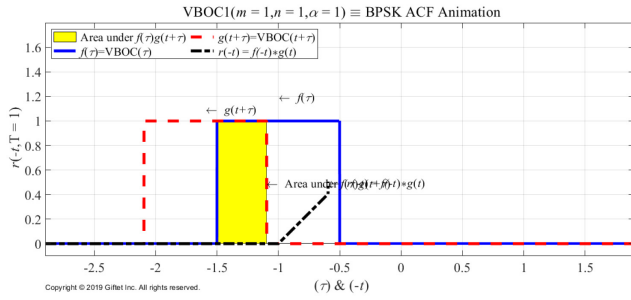


Fig. 2. BPSK ACF Case 2: The first intersection between $f(\tau) = \text{VBOC1}_{(1,1,1)}(\tau)$ and $g(t + \tau) = \text{VBOC1}_{(1,1,1)}(t + \tau)$; $t_0 = -0.5$, $T = 1$.

maximized at zero offset, and for aperiodic functions tends to drop off with distance.” Neither Brown nor Wikipedia “convolution” article make any statements whether the ACF is variant or invariant of distance.

Let us see how ACF $r(t)$ is obtained in animation. Since the animation is defined as the technique of successful drawings then the drawings considered for the purposes of deriving the analytical expression of the ACF should show the continuity of a state or transition or a case or a line or a segment; hence, the definition of case. *Case* is defined as a continuity of a flow of a state or a line or a segment of the ACF until a transition occurs to another case or line or a segment.

The ACF of the BPSK signal contains four cases, lines or transitions.

Case 1: In Fig. 1 the function $f(\tau)$ is stationary shown with solid blue and $g(t + \tau)$ is depicted with dashed red that slides (or moves) towards $f(\tau)$, which is given by

$$g(t + \tau) = \Pi(t + \tau)$$

$$= \begin{cases} 0 & t_0 - 2T_s \leq t + \tau < t_0 - T_s \\ 1 & t_0 - T_s \leq t + \tau < t_0 \end{cases}$$

$$= \begin{cases} 0 & t_0 - 2T_s - t \leq \tau < t_0 - T_s - t \\ 1 & t_0 - T_s - t \leq \tau < t_0 - t \end{cases} \quad (6)$$

In *Case 1*, the autocorrelation function $r(t)$ is zero if and

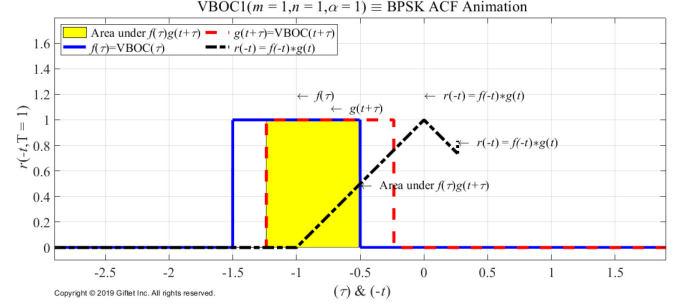


Fig. 3. BPSK ACF Case 3: The second intersection between $f(\tau) = \text{VBOC1}_{(1,1,1)}(\tau)$ and $g(t + \tau) = \text{VBOC1}_{(1,1,1)}(t + \tau)$; $t_0 = -0.5$, $T = 1$.

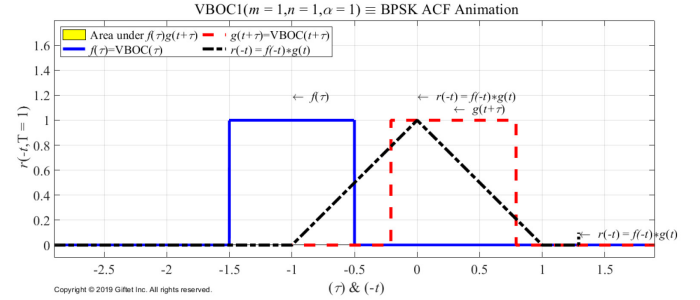


Fig. 4. BPSK ACF Case 4: No intersection after the integration between $f(\tau) = \text{VBOC1}_{(1,1,1)}(\tau)$ and $g(t + \tau) = \text{VBOC1}_{(1,1,1)}(t + \tau)$; $t_0 = -0.5$, $T = 1$.

only if (of iff)

$$r(t) = f(t) * g(-t)$$

$$= |0 \quad t_0 - t < t_0 - T_s$$

$$= |0 \quad t > T_s$$

$$= |0 \quad t > 1 \quad (7)$$

In order to understand why this is the answer, we need to visualize that $f(\tau)$ is centered at $t = 0$ and $\tau = -1$; therefore, this function needs to be shifted to the left by more than one unit so as not to have any overlaps with the function $f(\tau)$.

Case 2: In Fig. 2 in addition to the function shown in Fig. 1, we show the intersection or the area of the non-zero product of $f(\tau)g(t + \tau)$ with yellow and with the solid dot line the integral or the ACF, $r(t)$. Since the product is equal to one then $r(t)$ is the same as

$$r(t) = \int_{t_0 - T_s}^{t_0 - t} d\tau \quad T_s \leq t < 0$$

$$= |T_s - t \quad T_s \leq t < 0 \quad (8)$$

If we substitute our numerical values we get, $r(t) = 1$, $t = 0$.

Case 3: In Fig. 3, the intersection or the yellow area is between left edge of the $g(t + \tau)$ and the right edge of the

$f(\tau)$. Hence, we have

$$\begin{aligned} r(t) &= \int_{t_0-T_s-t}^{t_0} d\tau \quad -T_s \leq t < 0 \\ &= |T_s + t| \quad -T_s \leq t < 0 \end{aligned} \quad (9)$$

If we substitute our numerical values we get, $r(t) = 0$, $t = -1$.

Case 4: In Fig. 4 it is shown the case with the function $f(t)$ is left behind the function $g(t)$. This occurs when

$$\begin{aligned} r(t) &= f(t) * g(-t) \\ &= |0| \quad t < -T_s \end{aligned} \quad (10)$$

Finally, if we combine (7) through (10) we obtain

$$r(t) = \begin{cases} 0 & t > T_s \\ T_s - t & 0 \leq t < T_s \\ T_s + t & -T_s \leq t < 0 \\ 0 & t < -T_s \end{cases} \quad (11)$$

The maximum value of the $r(t)$ occurs at $t = 0$ which is the same as T_s . So, if we divide $r(t)$ by T_s we obtain the normalized autocorrelation function as follows

$$\hat{r}(t) = \begin{cases} 1 - \frac{|t|}{T_s} & |t| \leq T_s \\ 0 & |t| > T_s \end{cases} \quad (12)$$

The normalized ACF $\hat{r}(t)$ has its maximum peak equal to one at the values of $t = 0$. *Since the normalized ACF $\hat{r}(t)$ is invariant of t_0 ; hence, the ACF peak by itself does not give any distance information; it is only useful to decode the signal.*

This concludes the discussion on the BPSK signal $f(t)$ and the computation and animation of its ACF waveform $r(t)$ and $\hat{r}(t)$.

2.2 Description of BOC(1,1) Signal and ACF Animation

BOC modulation was developed by John Betz, Ph.D., in order to allow interoperability of satellite navigation systems. It is currently used in the US GPS system, Indian IRNSS system and in Galileo and is a square sub-carrier modulation, where a signal is multiplied by a rectangular sub-carrier of frequency f_{sc} equal or higher to the chip rate. Following this sub-carrier multiplication, the spectrum of the signal is divided into two parts, therefore BOC modulation is also known as a split-spectrum modulation [16], [24]-[26].

The main idea behind BOC modulation is to reduce the interference with BPSK-modulated signal, which has a sinc function shaped spectrum. Therefore, BPSK-modulated signals such as C/A GPS codes have most of their spectral energy

concentrated around the carrier frequency, while BOC-modulated signals (used in Galileo system) have low energy around the carrier frequency and two main spectral lobes further away from the carrier (thus, the name of split-spectrum) [16], [24]-[26].

The main issue with the BOC signals is that its ACF optimization performance is sub-optimal according to Progni 2007, [31]-Progni [41].

Nevertheless, since the BOC signal is a special case of the VBOC signal, the study of the animation of the BOC signal is very important.

The BOC(1,1) signal satisfies the *definition 1* of the BPSK signal whose first period of the BOC(1,1) signal, $f(t)$, is given by

$$f(t) = \begin{cases} 1 & t_0 - T_s \leq t < t_0 - 0.5T_s \\ -1 & t_0 - 0.5T_s \leq t < t_0 \end{cases} \quad (13)$$

and t_0 as the time offset whose significance is explained later in the end of the section.

The ACF of the BOC(1,1) signal, $f(t)$, is defined exactly the same as in the case of the BPSK signal given by *definition 2*.

In contrast to the BPSK signal ACF which contained only four cases, the animation computation of the ACF of the BOC(1,1) signal, $f(t)$, contains six cases.

Case 1: In Fig. 5 the function $f(\tau)$ is stationary shown with solid blue and $g(t + \tau)$ is depicted with dashed red that slides (or moves) towards $f(\tau)$, which is given by

$$\begin{aligned} g(t + \tau) &= \begin{cases} 1 & t_0 - T_s \leq t + \tau < t_0 - 0.5T_s \\ -1 & t_0 - 0.5T_s \leq t + \tau < t_0 \end{cases} \\ &= \begin{cases} 1 & t_0 - T_s - t \leq \tau < t_0 - 0.5T_s - t \\ -1 & t_0 - 0.5T_s - t \leq \tau < t_0 - t \end{cases} \end{aligned} \quad (14)$$

In Case 1, the autocorrelation function $r(t)$ is zero if and only if

$$\begin{aligned} r(t) &= f(t) * g(-t) \\ &= |0| \quad t_0 - T_s > t_0 - t \\ &= |0| \quad t > T_s \\ &= |0| \quad t > 1 \end{aligned} \quad (15)$$

In order to understand why this is the answer, we need to visualize that $f(\tau)$ is centered at $t = 0$ and $\tau = -1$; $f(t + \tau)$ is centered at $t = t$ and $\tau = -1$. So this function needs to be shifted to the left by more than one unit so as not to have any overlaps with the function $f(\tau)$.

Case 2: In Fig. 6 in addition to the function shown in Fig. 5,

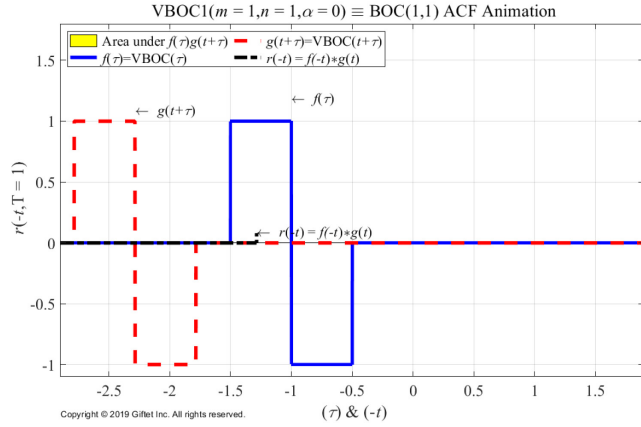


Fig. 5. BOC(1,1) ACF Case 1: no intersection between $f(\tau) = \text{VBOC1}_{(1,1,0)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

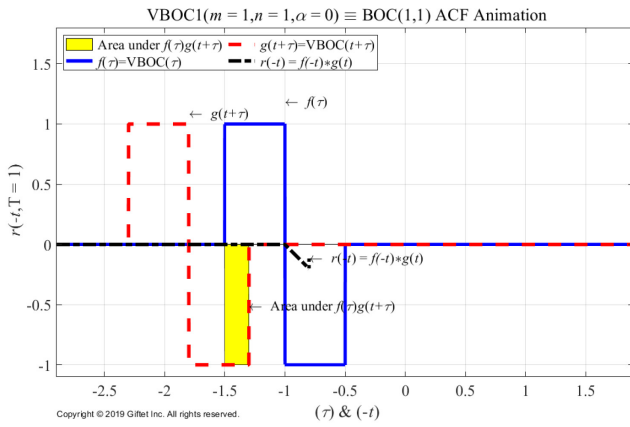


Fig. 6. BOC(1,1) ACF Case 2: The first intersection between $f(\tau) = \text{VBOC1}_{(1,1,0)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

we show the intersection or the area of the non-zero product of $f(\tau)g(t+\tau)$ with yellow and with the solid dot line the integral or the ACF, $r(t)$. Since the product is equal to one then $r(t)$ is the same as

$$r(t) = \left| -\int_{t_0-T_s}^{t_0-t} d\tau \quad 0.5T_s \leq t < T_s \right. \\ = |t - T_s| \quad 0.5T_s \leq t < T_s \quad (16)$$

If we substitute our numerical values we get, $r(t) = 0$, $t = 1$.

Case 3: In Fig. 7, the intersection or the yellow area is between left edge of $f(\tau)$ and the right edge of $g(t+\tau)$, which is split into three integrals. Hence, we have

$$r(t) = \int_{t_0-T_s}^{t_0-0.5T_s-t} d\tau - \int_{t_0-0.5T_s-t}^{t_0-0.5T_s} d\tau + \int_{t_0-0.5T_s}^{t_0-t} d\tau \\ = |-t + 0.5T_s - t - t + 0.5T_s| \quad 0 \leq t < 0.5T_s \\ = |-3t + T_s| \quad 0 \leq t < 0.5T_s \quad (17)$$

If we substitute our numerical values we get, $r(t) = T_s$, $t =$

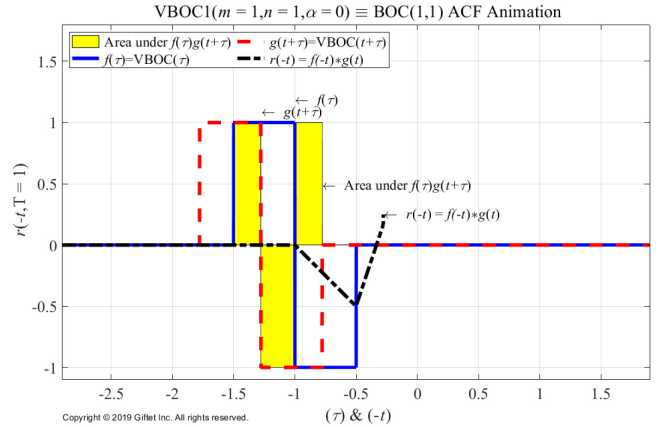


Fig. 7. BOC(1,1) ACF Case 1: no intersection between $f(\tau) = \text{VBOC1}_{(1,1,0)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

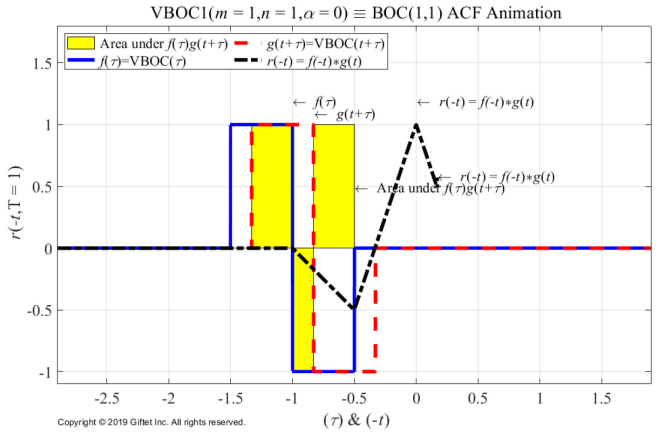


Fig. 8. BOC(1,1) ACF Case 2: The first intersection between $f(\tau) = \text{VBOC1}_{(1,1,0)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

0.

Case 4: In Fig. 8, the intersection or the yellow area is between left edge of the $g(t+\tau)$ and the right edge of the $f(\tau)$. Hence, we have

$$r(t) = \int_{t_0-T_s-t}^{t_0-0.5T_s} d\tau - \int_{t_0-0.5T_s-t}^{t_0-0.5T_s} d\tau + \int_{t_0-0.5T_s}^{t_0} d\tau \\ = |t + 0.5T_s + t + t + 0.5T_s| \quad -0.5T_s \leq t < 0 \\ = |3t + T_s| \quad -0.5T_s \leq t < 0 \quad (18)$$

If we substitute our numerical values we get, $r(t) = -0.5T_s$, $t = -0.5T_s$.

Case 5: In Fig. 9, the intersection or the yellow area is between left edge of the $g(t+\tau)$ and the right edge of the $f(\tau)$. Hence, we have

$$r(t) = \int_{t_0-T_s-t}^{t_0} d\tau \\ = |t + T_s| \quad -T_s \leq t < -0.5T_s \quad (19)$$

The same answer as in Case 4 can be obtained in Case 5

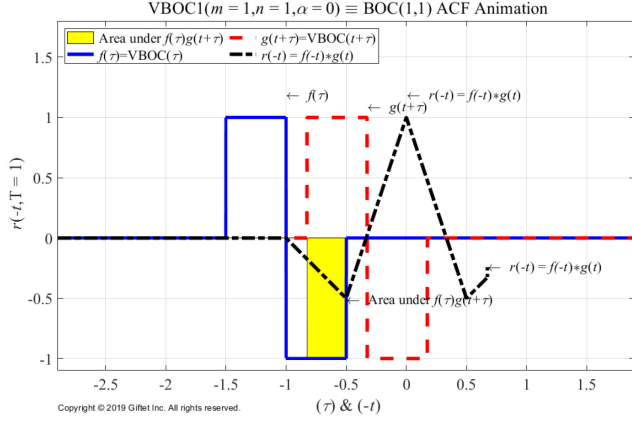


Fig. 9. BOC(1,1) ACF Case 5: The fourth intersection between $f(\tau) = \text{VBOC1}_{(1,1,0)}(\tau)$ and $g(t + \tau) = \text{VBOC1}_{(1,1,0)}(t + \tau)$; $t_0 = -0.5$, $T = 1$.

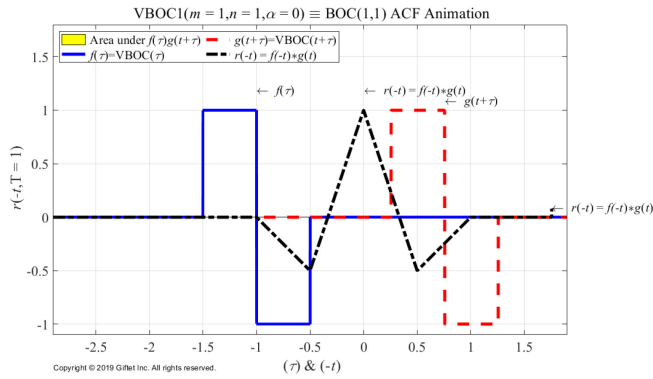


Fig. 10. BOC(1,1) ACF Case 6: No intersection after the integration between $f(\tau) = \text{VBOC1}_{(1,1,0)}(\tau)$ and $g(t + \tau) = \text{VBOC1}_{(1,1,0)}(t + \tau)$; $t_0 = -0.5$, $T = 1$.

when $r(t) = -0.5T_s$, $t = -0.5T_s$.

Case 6: In Fig. 10 it is shown the case with the function $f(t)$ is left behind the function $g(t)$. This occurs when $r(t) = f(t) * g(-t)$

$$= |0 \quad t < -T_s \quad (20)$$

Finally, if we combine (15) through (20) we obtain

$$r(t) = \begin{cases} 0 & t > T_s \\ T_s - t & 0.5T_s \leq t < T_s \\ T_s - 3t & 0 \leq t < 0.5T_s \\ T_s + 3t & -0.5T_s \leq t < 0 \\ T_s + t & -T_s \leq t < -0.5T_s \\ 0 & t < -T_s \end{cases} \quad (21)$$

The maximum value of the $r(t)$ occurs at $t = 0$ which is the same as T_s . So, if we divide $r(t)$ by T_s we obtain the normalized autocorrelation function as follows

$$\hat{r}(t) = \begin{cases} 1 - \frac{3|t|}{T_s} & |t| \leq 0.5T_s \\ 1 - \frac{|t|}{T_s} & 0.5T_s \leq |t| \leq T_s \\ 0 & |t| > T_s \end{cases} \quad (22)$$

The normalized ACF $\hat{r}(t)$ has its maximum peak equal to one at the values of $t = 0$. Since the normalized ACF $\hat{r}(t)$ is invariant of t_0 ; hence, the ACF peak by itself does not give any distance information; it is only useful to decode the signal.

This concludes the discussion on the $\text{BOC}(1,1)$ signal $f(t)$ and the computation and animation of its ACF waveform $r(t)$ and $\hat{r}(t)$.

2.3 Description of $\text{VBOC1}(1,1,\alpha)$ Signal and ACF Animation

The $\text{VBOC1}_{(1,1,\alpha)}(t)$ signal satisfies the definition 1 of the BPSK signal whose first period of the $\text{VBOC1}_{(1,1,\alpha)}(t)$ signal, $f(t)$, is given by

$$f(t) = \begin{cases} 1 & t_0 - T_s \leq t < t_0 - 0.5\beta_-T_s \\ -1 & t_0 - 0.5\beta_-T_s \leq t < t_0 \end{cases} \quad (23)$$

where α is a variable parameter that determines the asymmetrical offset of the $\text{VBOC1}_{(1,1,\alpha)}(t)$ waveform

$$0 \leq \alpha \leq 1 \quad (24)$$

And β_- is also a variable parameter that is related to α as follows

$$\beta_- \equiv 1 - \alpha \quad (25)$$

Hence, it follows that

$$0 \leq \beta_- \leq 1 \quad (26)$$

The ACF of the $\text{VBOC1}_{(1,1,\alpha)}(t)$ signal, $f(t)$, is defined exactly the same as in the case of the BPSK signal given by definition 2.

Corollary 1: BPSK signal is simply obtained for $\alpha = 1$; i.e., $\text{BPSK}(t) \equiv \text{VBOC1}_{(1,1,\alpha=1)}(t)$.

Corollary 2: BOC(1,1) signal is simply obtained for $\alpha = 0$; i.e., $\text{BOC}(1,1) \equiv \text{VBOC1}_{(1,1,\alpha=0)}(t)$.

In contrast to the BPSK signal ACF which contained only four cases, and the animation computation of the ACF of the BOC(1,1) signal, which contained *six* cases, the animation computation of the ACF of the $\text{VBOC1}_{(1,1,\alpha)}(t)$ signal, $f(t)$, contains *eight* cases.

Case 1: In Fig. 11 the function $f(\tau)$ is stationary shown with solid blue and $g(t + \tau)$ is depicted with dashed red that slides (or moves) towards $f(\tau)$, which is given by

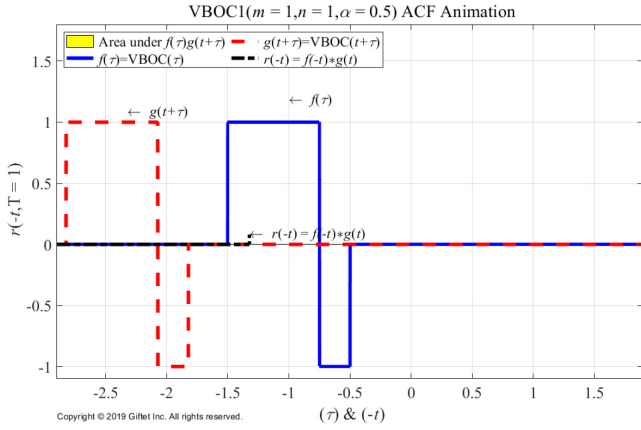


Fig. 11. VBOC1_(1,1,0.5) ACF Case 1: no intersection between $f(\tau) = \text{VBOC1}_{(1,1,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

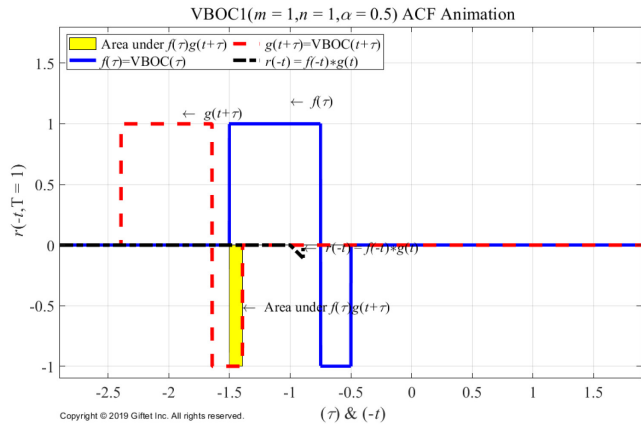


Fig. 12. VBOC1_(1,1,0.5) ACF Case 2: The first intersection between $f(\tau) = \text{VBOC1}_{(1,1,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

$$g(t+\tau) = \begin{cases} 1 & t_0 - T_s \leq t+\tau < t_0 - 0.5\beta_- T_s \\ -1 & t_0 - 0.5\beta_- T_s \leq t+\tau < t_0 \end{cases}$$

$$= \begin{cases} 1 & t_0 - T_s - t \leq \tau < t_0 - \beta_- T_s/2 - t \\ -1 & t_0 - \beta_- T_s/2 - t \leq \tau < t_0 - t \end{cases} \quad (27)$$

In Case 1, ACF $r(t)$ is zero if and only if

$$r(t) = f(t) * g(-t)$$

$$= |0 \quad t_0 - T_s > t_0 - t$$

$$= |0 \quad t > T_s$$

$$= |0 \quad t > 1 \quad (28)$$

Case 2: In Fig. 12 in addition to the function shown in Fig. 11, we show the intersection or the area of the non-zero product of $f(\tau)g(t+\tau)$ with yellow and with the solid dot line the integral or the ACF, $r(t)$. Since the product is equal to one then $r(t)$ is the same as

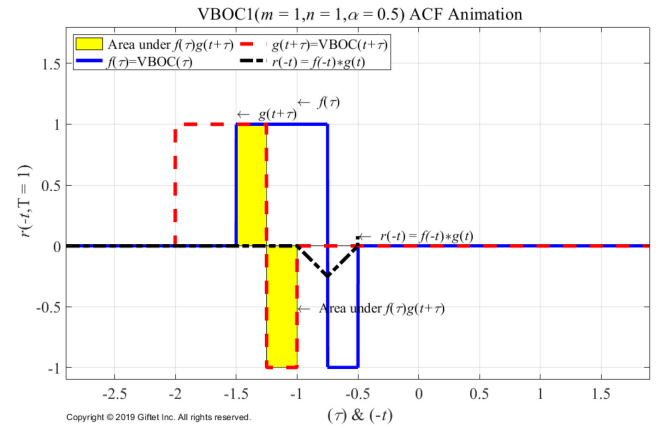


Fig. 13. VBOC1_(1,1,0.5) ACF Case 3: The second intersection between $f(\tau) = \text{VBOC1}_{(1,1,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

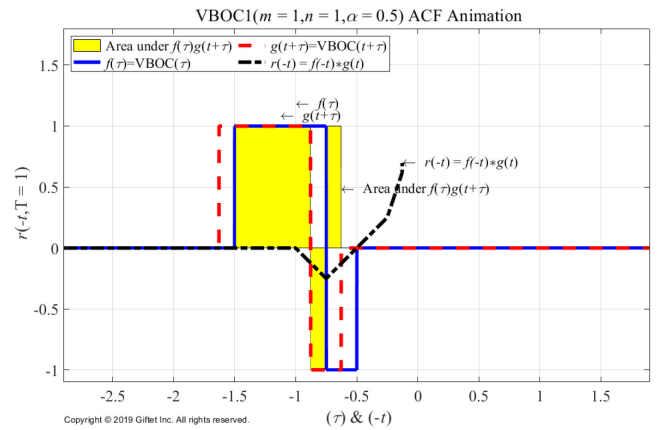


Fig. 14. VBOC1_(1,1,0.5) ACF Case 4: The third intersection between $f(\tau) = \text{VBOC1}_{(1,1,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

$$r(t) = \begin{cases} -\int_{t_0-T_s}^{t_0-t} d\tau & 0.5\beta_+ T_s \leq t < T_s \\ = |t - T_s & 0.5\beta_+ T_s \leq t < T_s \end{cases} \quad (29)$$

Where

$$\beta_+ \equiv 1 + \alpha \quad (30)$$

Hence, it follows that from (24) that

$$1 \leq \beta_+ \leq 2 \quad (31)$$

If we substitute our numerical values we get, $r(t) = 0$, $t = 1$.

Case 3: In Fig. 13, the intersection or the yellow area is between left edge of $f(\tau)$ and the right edge of $g(t+\tau)$ which is split into two integrals. Hence, we have

$$r(t) = \int_{t_0-T_s}^{t_0-0.5\beta_- T_s-t} d\tau - \int_{t_0-0.5\beta_- T_s-t}^{t_0-t} d\tau$$

$$= |\alpha T_s - t \quad 0.5\beta_- T_s \leq t < 0.5\beta_+ T_s \quad (32)$$

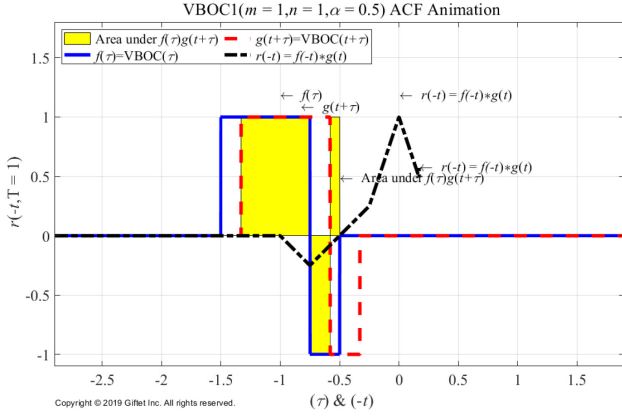


Fig. 15. VBOC1_(1,1,0.5) ACF Case 5: The fourth intersection between $f(\tau) = \text{VBOC1}_{(1,1,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

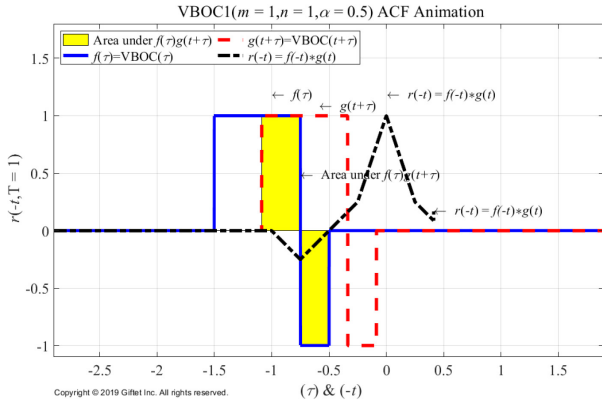


Fig. 16. VBOC1_(1,1,0.5) ACF Case 6: The fifth intersection between $f(\tau) = \text{VBOC1}_{(1,1,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

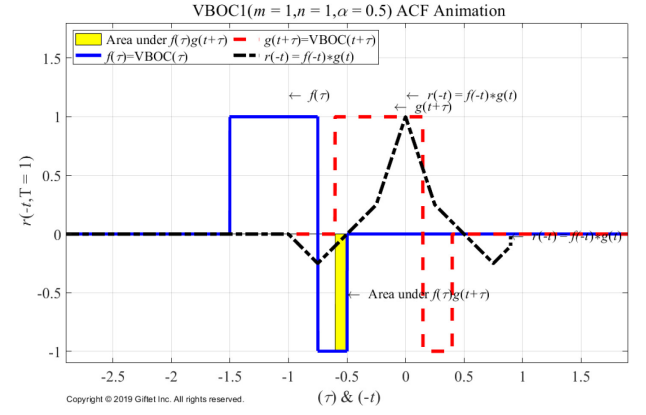


Fig. 17. VBOC1_(1,1,0.5) ACF Case 7: The sixth intersection between $f(\tau) = \text{VBOC1}_{(1,1,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

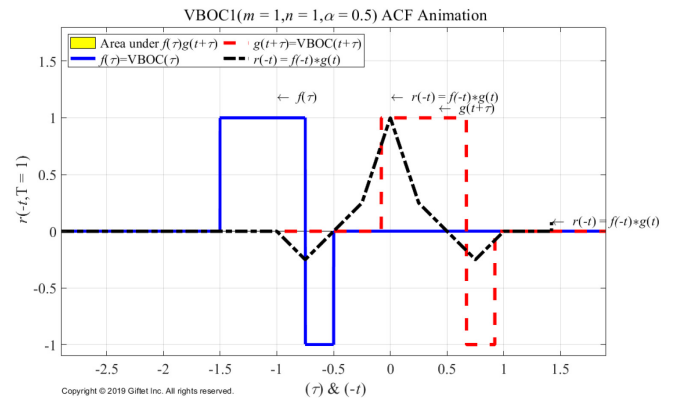


Fig. 18. VBOC1_(1,1,0.5) ACF Case 8: No intersection after the integration between $f(\tau) = \text{VBOC1}_{(1,1,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC1}_{(1,1,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 1$.

If we substitute our numerical values we get, $r(t) = \alpha T_s - 0.5\beta_+ T_s = -0.5\beta_- T_s$, $t = 0.5\beta_+ T_s$.

Case 4: In Fig. 14, the intersection or the yellow area is between left edge of $f(\tau)$ and the right edge of $g(t+\tau)$, which is split into three integrals. Hence, we have

$$r(t) = \int_{t_0 - \frac{\beta_- T_s}{2} - t}^{t_0 - \frac{\beta_- T_s}{2}} d\tau - \int_{t_0 - \frac{\beta_- T_s}{2}}^{t_0 - \frac{\beta_- T_s}{2} - t} d\tau + \int_{t_0 - \frac{\beta_- T_s}{2}}^{t_0 - t} d\tau$$

$$= |T_s - 3t| \quad 0.5\beta_- T_s \leq t < 0 \quad (33)$$

If we substitute our numerical values we get, $r(t) = T_s - 1.5\beta_+ T_s \equiv 0.5(3\alpha - 1)T_s$, at $t = 0.5\beta_- T_s$.

Case 5: In Fig. 15, the intersection or the yellow area is between left edge of the $g(t+\tau)$ and the right edge of the $f(\tau)$, which is split into *three* integrals. Hence, we have

$$r(t) = \int_{t_0 - T_s - t}^{t_0 - \frac{\beta_- T_s}{2}} d\tau - \int_{t_0 - \frac{\beta_- T_s}{2}}^{t_0 - \frac{\beta_- T_s}{2} - t} d\tau + \int_{t_0 - \frac{\beta_- T_s}{2}}^{t_0 - \frac{\beta_- T_s}{2} - t} d\tau$$

$$= |T_s + 3t| \quad -0.5\beta_- T_s \leq t < 0 \quad (34)$$

The same answer as in *Case 4* can be obtained in *Case 5* when $r(t) = 0.5(3\alpha - 1)T_s$, $t = -0.5\beta_- T_s$.

Case 6: In Fig. 16, the intersection or the yellow area is between left edge of the $g(t+\tau)$ and the right edge of the $f(\tau)$, which is split into *two* integrals. Hence, we have

$$r(t) = \int_{t_0 - T_s - t}^{t_0 - \frac{\beta_- T_s}{2}} d\tau - \int_{t_0 - \frac{\beta_- T_s}{2}}^{t_0 - \frac{\beta_- T_s}{2} - t} d\tau$$

$$= |\alpha T_s + t| \quad -0.5\beta_+ T_s \leq t < -0.5\beta_- T_s \quad (35)$$

The same answer as in *Case 3* can be obtained when $r(t) = \alpha T_s - 0.5\beta_+ T_s \equiv -0.5\beta_- T_s$, at $t = -0.5\beta_+ T_s$.

Case 7: In Fig. 17, the intersection or the yellow area is between left edge of the $g(t+\tau)$ and the right edge of the $f(\tau)$. Hence, we have

$$r(t) = \int_{t_0 - T_s - t}^{t_0} d\tau$$

$$= |T_s + t| \quad -T_s \leq t < -0.5\beta_+ T_s \quad (36)$$

The same answer as in *Case 2* can be obtained when $r(t) = 0$, at $t = -T_s$.

Case 8: In Fig. 18 it is shown the case with the function $f(t)$ is left behind the function $g(t)$. This occurs when

$$\begin{aligned}
r(t) &= f(t) * g(-t) \\
&= |0 \quad t < -T_s|
\end{aligned} \quad (37)$$

Finally, if we combine (28), (29), (32) through (37) we obtain

$$r(t) = \begin{cases} 0 & t > T_s \\ T_s - t & 0.5\beta_+ T_s \leq t < T_s \\ \alpha T_s - t & 0.5\beta_- T_s \leq t < 0.5\beta_+ T_s \\ T_s - 3t & 0 \leq t < 0.5\beta_- T_s \\ T_s + 3t & -0.5\beta_- T_s \leq t < 0 \\ \alpha T_s + t & -0.5\beta_+ T_s \leq t < -0.5\beta_- T_s \\ T_s + t & -T_s \leq t < -0.5\beta_+ T_s \\ 0 & t < -T_s \end{cases} \quad (38)$$

The maximum value of the $r(t)$ occurs at $t = 0$ which is the same as T_s . So, if we divide $r(t)$ by T_s we obtain the normalized autocorrelation function as follows

$$\hat{r}(t) = \begin{cases} 1 - \frac{3|t|}{T_s} & 0 \leq |t| \leq 0.5\beta_- T_s \\ \alpha - \frac{|t|}{T_s} & 0.5\beta_- T_s \leq |t| \leq 0.5\beta_+ T_s \\ 1 - \frac{|t|}{T_s} & 0.5\beta_+ T_s \leq |t| \leq T_s \\ 0 & |t| > T_s \end{cases} \quad (39)$$

The normalized ACF $\hat{r}(t)$ has its maximum peak equal to one at the values of $t = 0$. Since the normalized ACF $\hat{r}(t)$ is invariant of t_0 ; hence, the ACF peak by itself does not give any distance information; it is only useful to decode the signal.

This concludes the discussion on the $\text{VBOC1}_{(1,1,\alpha)}(t)$ signal $f(t)$ and the computation and animation of its ACF waveform $r(t)$ and $\hat{r}(t)$.

3 The Animation of the $\text{VBOC2}(\alpha_1, \alpha_2)$ Signal Design

Before we discuss the animation of the $\text{VBOC2}(\alpha_1, \alpha_2)$ signal design let us first define the $\text{VBOC2}_{(2,1,\alpha_1,\alpha_2)}(t)$ signal.

Definition 3: The $\text{VBOC2}_{(2,1,\alpha_1,\alpha_2)}(t)$ signal satisfies the definition 1 of the BPSK signal whose first period of the $\text{VBOC2}_{(2,1,\alpha_1,\alpha_2)}(t)$ signal, $f(t)$, is given by

$$f(t) = \begin{cases} 1 & t_0 - 2T_s \leq t < t_0 - \gamma_{1-}T_s \\ -1 & t_0 - \gamma_{1-}T_s \leq t < t_0 - T_s \\ 1 & t_0 - T_s \leq t < t_0 - 0.5\beta_{2-}T_s \\ -1 & t_0 - 0.5\beta_{2-}T_s \leq t < t_0 \end{cases} \quad (40)$$

where α , β_{2-} , and γ_{1-} are variable parameters that determine the asymmetrical offset of the $\text{VBOC2}_{(2,1,\alpha_1,\alpha_2)}(t)$ waveform

$$0 \leq \alpha_1, \alpha_2 \leq 1 \quad (41)$$

$$\gamma_{1-} \equiv 2 - \alpha_1; \Rightarrow \quad (42)$$

$$2 \leq \gamma_{1-} \leq 3 \quad (43)$$

$$\beta_{2-} \equiv 1 - \alpha_2; \Rightarrow \quad (44)$$

$$1 \leq \beta_{2-} \leq 2 \quad (45)$$

and t_0 as the time offset whose significance is explained later in the end of the section.

This is a new, improved, original, expanded, and novel definition of $\text{VBOC2}_{(2,1,\alpha_1,\alpha_2)}(t)$ not published anywhere else and it is a different definition from the one employed in Progrid 2015 [38], [39].

The natural questions are as follows: what special cases can be obtained from definition 3 of $\text{VBOC2}_{(2,1,\alpha_1,\alpha_2)}(t)$ and how many are they?

Corollary 3: BPSK signal is simply obtained for $\alpha_1 \equiv \alpha_2 = \{0,1\}$; i.e.,

$$\text{BPSK}(t) \equiv \text{VBOC2}_{(2,1,\alpha_1=\{0,1\},\alpha_2=\{0,1\})}(t)^{\text{xii}} \quad (46)$$

Corollary 4: BOC(1,1) signal is simply obtained for $\alpha_1 = \{0,1\}$, $\alpha_2 = \{1,0\}$; i.e.,

$$\text{BOC}(1,1)(t) \equiv \text{VBOC2}_{(2,1,\alpha_1=\{0,1\},\alpha_2=\{1,0\})}(t)^{\text{xiii}} \quad (47)$$

Corollary 5: $\text{VBOC1}(1,1,0.5)$ (the optimum VBOC1 waveform according to 2015 [34]-[37]) signal is simply obtained for $\alpha_1 = \{1,0.5\}$, $\alpha_2 = \{0.5,0\}$; i.e.,

$$\text{VBOC1}_{(1,1,0.5)}(t) \equiv \text{VBOC2}_{(2,1,\alpha_1=\{1,0.5\},\alpha_2=\{0.5,0\})}(t)^{\text{xiv}} \quad (48)$$

Corollary 6: BOC(2,1) signal is simply obtained for $\alpha_1 \equiv \alpha_2 = 0.5$; i.e.,

$$\text{BOC}(2,1)(t) \equiv \text{VBOC2}_{(2,1,\alpha_1=0.5,\alpha_2=0.5)}(t)^{\text{xv}} \quad (49)$$

Corollary 7: $\text{VBOC2}(2,1,0.4,0.6)$ (the optimum waveform according to Progrid 2015 [38], [39]) signal is simply obtained for $\alpha_1 = 0.7$, $\alpha_2 = 0.3$; i.e.,

$$\text{VBOC2}_{(2,1,0.4)}(t) \equiv \text{VBOC2}_{(2,1,\alpha_1=0.7,\alpha_2=0.3)}(t)^{\text{xvi}} \quad (50)$$

The relation between α_1 defined in 2019 and the one given in Progrid 2015 [38], [39] is as follows:

$$\alpha_{1,2019} \rightarrow 0.5 + \frac{\alpha_{1,2015}}{2} \quad (51)$$

where in the right hand side corresponds the value of α computed in Progrid 2015 [38], [39]; i.e.,

$$\alpha_{1,2019} \rightarrow 0.5 + \frac{\alpha_{1,2015}=0.4}{2} = 0.7 \quad (52)$$

Since for this particular case we also have

$$\alpha_{1,2019} = 1 - \alpha_{2,2019} \quad (53)$$

$$\alpha_{1,2015} = 1 - \alpha_{2,2015} \quad (54)$$

Hence, substituting (53) and (54) into (51) the following

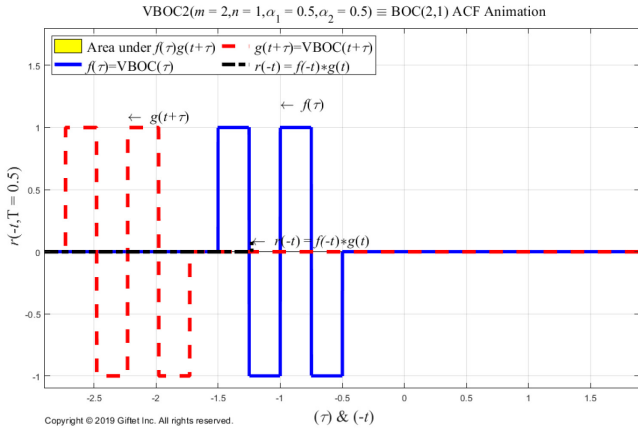


Fig. 19. BOC(2,1) ACF Case 1: no intersection between $f(\tau) = \text{VBOC}_{2(2,1,0.5,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC}_{2(2,1,0.5,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

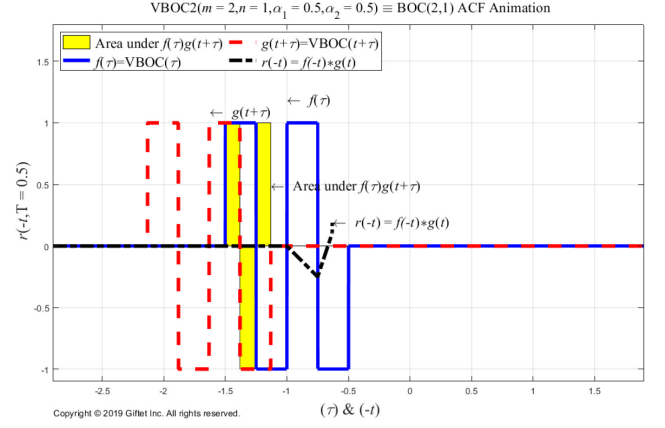


Fig. 21. BOC(2,1) ACF Case 3: the second intersection between $f(\tau) = \text{VBOC}_{2(2,1,0.5,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC}_{2(2,1,0.5,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

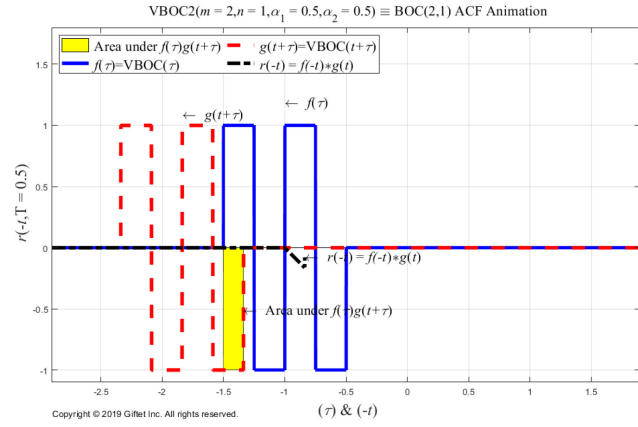


Fig. 20. BOC(2,1) ACF Case 2: the first intersection between $f(\tau) = \text{VBOC}_{2(2,1,0.5,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC}_{2(2,1,0.5,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

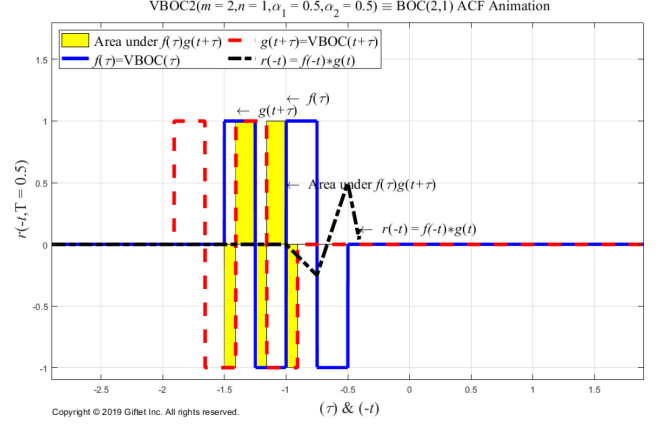


Fig. 22. BOC(2,1) ACF Case 4: the third intersection between $f(\tau) = \text{VBOC}_{2(2,1,0.5,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC}_{2(2,1,0.5,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

relations for α_2

$$1 - \alpha_{2,2019} \rightarrow 0.5 + \frac{1 - \alpha_{2,2015}}{2} \rightarrow 1 - \frac{\alpha_{2,2015}}{2} \quad (55)$$

Which is identical to

$$\alpha_{2,2019} \rightarrow \frac{\alpha_{2,2015}}{2} \quad (56)$$

Furthermore, $\text{VBOC}_{2(2,1,0.7,0.3)}(t)$ is identical to the $\text{VBOC}_{2(2,1,0.4,0.6)}(t)$ defined in Progrid 2015 [38], [39].

$$\underbrace{\text{VBOC}_{2(2,1,0.7,0.3)}(t)}_{2019} \equiv \underbrace{\text{VBOC}_{2(2,1,0.4,0.6)}(t)}_{2015} \quad (57)$$

Just by inspecting *Corollaries* 3-7 the case can be made that $\text{VBOC}_{2(2,1,\alpha_1,\alpha_2)}$ includes all the previously defined, discusses, optimized signal designs BPSK, BOC(1,1), BOC(2,1),

$\text{VBOC}_{1(1,1,\alpha)}$ and $\text{VBOC}_{2(1,1,\alpha,1-\alpha)}$ as a total of *eight special cases*.

Hence, the case can be made that $\text{VBOC}_{2(2,1,\alpha_1,\alpha_2)}$ is a better signal design than either or both $\text{VBOC}_{1(1,1,\alpha)}$ or/and $\text{VBOC}_{2(1,1,\alpha,1-\alpha)}$; i.e., it cannot be worse than any of the previously published signal designs in Progrid 2014, [34]-2018, [40]. The detailed discussion of $\text{VBOC}_{2(2,1,\alpha_1,\alpha_2)}$ is subject to a very special journal paper. This journal paper serves as a building block or setting the stage for the complete and detailed discussion of $\text{VBOC}_{2(2,1,\alpha_1,\alpha_2)}$.

Since the *eight special cases* of $\text{VBOC}_{2(2,1,\alpha_1,\alpha_2)}$ the animation of the ACF are already discussed in great detail in Sect. 2; hence, it remains to discuss the animation of the ACF $\text{VBOC}_{2(2,1,\alpha_1,\alpha_2)}$ of the last *two special cases*: BOC(2,1) corresponding to *Corollary* 6 and $\text{VBOC}_{2(2,1,0.4,0.6)}$ (Progrid 2015 [38], [39]) corresponding to *Corollary* 7.

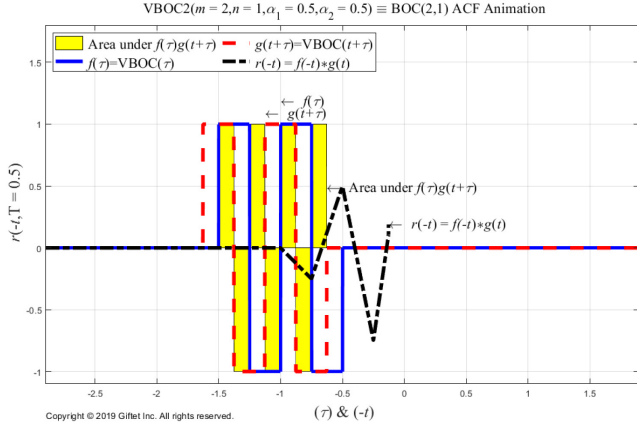


Fig. 23. BOC(2,1) ACF Case 5: the fourth intersection between $f(\tau) = \text{VBOC}_{(2,1,0.5,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC}_{(2,1,0.5,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

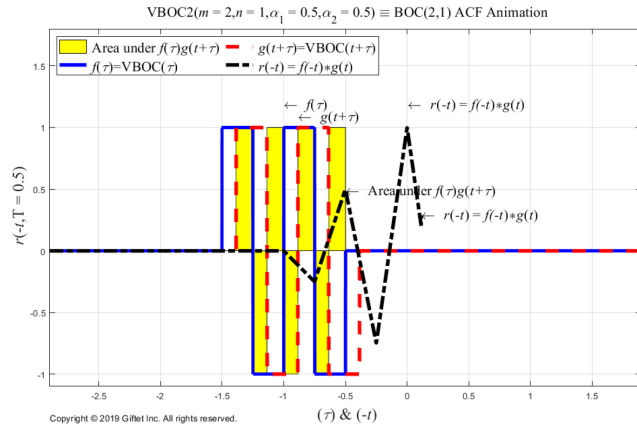


Fig. 24. BOC(2,1) ACF Case 6: the fifth intersection between $f(\tau) = \text{VBOC}_{(2,1,0.5,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC}_{(2,1,0.5,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

3.1 Description of BOC(2,1) Signal and ACF Animation

The BOC(2,1)(t) signal satisfies the *definition 1* of the BPSK signal whose first period of the BOC(2,1) signal, $f(t)$, is given by

$$f(t) = \begin{cases} 1 & t_0 - 2T_s \leq t < t_0 - 1.5T_s \\ -1 & t_0 - 1.5T_s \leq t < t_0 - T_s \\ 1 & t_0 - T_s \leq t < t_0 - 0.5T_s \\ -1 & t_0 - 0.5T_s \leq t < t_0 \end{cases} \quad (58)$$

where t_0 is the time offset whose significance is explained later in the end of the section.

The ACF of the BOC(2,1)(t) signal, $f(t)$, is defined exactly the same as in the case of the BPSK signal given by *definition 2*.

In contrast to the BOC(1,1) signal ACF which contained

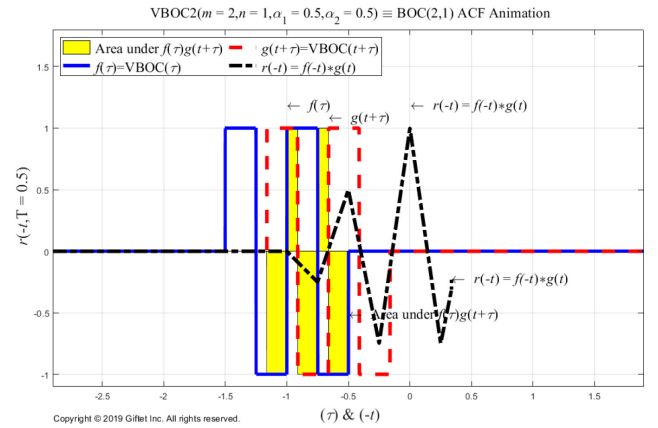


Fig. 25. BOC(2,1) ACF Case 7: the sixth intersection between $f(\tau) = \text{VBOC}_{(2,1,0.5,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC}_{(2,1,0.5,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

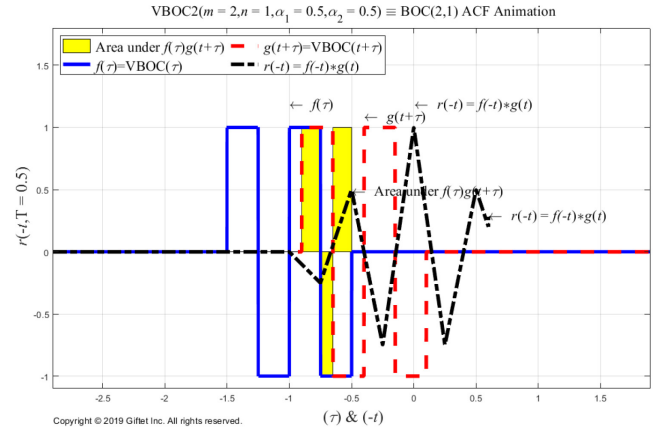


Fig. 26. BOC(2,1) ACF Case 8: the seventh intersection between $f(\tau) = \text{VBOC}_{(2,1,0.5,0.5)}(\tau)$ and $g(t+\tau) = \text{VBOC}_{(2,1,0.5,0.5)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

only *six* cases, the animation computation of the ACF of the BOC(2,1) signal, $f(t)$, contains *ten* cases.

Case 1: In Fig. 19 the function $f(\tau)$ is stationary shown with solid blue and $g(t+\tau)$ is depicted with dashed red that slides (or moves) towards $f(\tau)$, which is given by

$$g(t+\tau) = \begin{cases} 1 & t_0 - 2T_s \leq t+\tau < t_0 - 1.5T_s \\ -1 & t_0 - 1.5T_s \leq t+\tau < t_0 - T_s \\ 1 & t_0 - T_s \leq t+\tau < t_0 - 0.5T_s \\ -1 & t_0 - 0.5T_s \leq t+\tau < t_0 \end{cases}$$

$$= \begin{cases} 1 & t_0 - 2T_s - t \leq \tau < t_0 - 1.5T_s - t \\ -1 & t_0 - 1.5T_s - t \leq \tau < t_0 - T_s - t \\ 1 & t_0 - T_s - t \leq \tau < t_0 - 0.5T_s - t \\ -1 & t_0 - 0.5T_s - t \leq \tau < t_0 - t \end{cases} \quad (59)$$

In *Case 1*, the autocorrelation function $r(t)$ is zero if and only if

$$r(t) = f(t) * g(-t)$$

$$\begin{aligned}
&= |0 \quad t_0 - 2T_s > t_0 - t \\
&= |0 \quad t > 2T_s \\
&= |0 \quad t > 1
\end{aligned} \tag{60}$$

In order to understand why this is the answer, we need to visualize that $f(\tau)$ is centered at $t = 0$ and $\tau = -1$; whereas, $f(t + \tau)$ is centered at $t = t$ and $\tau = -1$. So the function, $f(t + \tau)$, needs to be shifted to the left by more than one unit so as not to have any overlaps with the function $f(\tau)$.

Case 2: In Fig. 20 in addition to the function shown in Fig. 19, we show the intersection or the area of the non-zero product of $f(\tau)g(t + \tau)$ with yellow and with the solid dot line the integral or the ACF, $r(t)$. Since the product is equal to one then $r(t)$ is the same as

$$\begin{aligned}
r(t) &= \left| -\int_{t_0-2T_s}^{t_0-t} d\tau \quad 1.5T_s \leq t < 2T_s \right. \\
&= |t - 2T_s \quad 1.5T_s \leq t < 2T_s
\end{aligned} \tag{61}$$

If we substitute our numerical values we get, $r(t) = 0$, $t = 1$.

Case 3: In Fig. 21, the intersection or the yellow area is between left edge of $f(\tau)$ and the right edge of $g(t + \tau)$, which is split into *three* integrals. Hence, we have

$$\begin{aligned}
r(t) &= \int_{t_0-2T_s}^{t_0-0.5T_s-t} d\tau - \int_{t_0-0.5T_s-t}^{t_0-1.5T_s} d\tau + \int_{t_0-1.5T_s}^{t_0-t} d\tau \\
&= |-3t + 4T_s \quad T_s \leq t < 1.5T_s
\end{aligned} \tag{62}$$

If we substitute our numerical values we get, $r(t) = T_s$, $t = T_s$.

Case 4: In Fig. 22, the intersection or the yellow area is between left edge of $f(\tau)$ and the right edge of $g(t + \tau)$, which is split into *five* integrals. Hence, we have

$$\begin{aligned}
r(t) &= \left| -\int_{t_0-2T_s}^{t_0-T_s-t} d\tau + \int_{t_0-T_s-t}^{t_0-1.5T_s} d\tau - \int_{t_0-1.5T_s}^{t_0-0.5T_s-t} d\tau \right. \\
&\quad \left. + \int_{t_0-0.5T_s-t}^{t_0-T_s} d\tau - \int_{t_0-T_s}^{t_0-t} d\tau \right. \\
&= |5t - 4T_s \quad 0.5T_s \leq t < T_s
\end{aligned} \tag{63}$$

If we substitute our numerical values we get, $r(t) = T_s$, $t = T_s$.

Case 5: In Fig. 23, the intersection or the yellow area is between left edge of $f(\tau)$ and the right edge of $g(t + \tau)$, which is split into *seven* integrals. Hence, we have

$$\begin{aligned}
r(t) &= \left| \int_{t_0-2T_s}^{t_0-1.5T_s-t} d\tau - \int_{t_0-1.5T_s-t}^{t_0-0.5T_s} d\tau + \int_{t_0-0.5T_s}^{t_0-T_s-t} d\tau \right. \\
&\quad \left. - \int_{t_0-T_s-t}^{t_0-0.5T_s} d\tau + \int_{t_0-0.5T_s}^{t_0-T_s} d\tau \right. \\
&\quad \left. - \int_{t_0-T_s}^{t_0-0.5T_s-t} d\tau + \int_{t_0-0.5T_s-t}^{t_0-t} d\tau \right.
\end{aligned}$$

$$= |-7t + 2T_s \quad 0 \leq t < 0.5T_s \tag{64}$$

If we substitute our numerical values we get, $r(t) = 2T_s$, $t = 0$.

Case 6: In Fig. 24, the intersection or the yellow area is between left edge of the $g(t + \tau)$ and the right edge of the $f(\tau)$, which is split into *seven* integrals. Hence, we have

$$\begin{aligned}
r(t) &= \left| \int_{t_0-2T_s-t}^{t_0-1.5T_s} d\tau - \int_{t_0-1.5T_s-t}^{t_0-0.5T_s} d\tau + \int_{t_0-0.5T_s-t}^{t_0-T_s} d\tau \right. \\
&\quad \left. - \int_{t_0-T_s-t}^{t_0-0.5T_s} d\tau + \int_{t_0-0.5T_s-t}^{t_0-T_s} d\tau \right. \\
&\quad \left. - \int_{t_0-T_s}^{t_0-0.5T_s-t} d\tau + \int_{t_0-0.5T_s-t}^{t_0} d\tau \right.
\end{aligned}$$

$$= |7t + 2T_s \quad -0.5T_s \leq t < 0 \tag{65}$$

If we substitute our numerical values we get, $r(t) = 2T_s$, $t = 0$.

Case 7: In Fig. 25, the intersection or the yellow area is between left edge of the $g(t + \tau)$ and the right edge of the $f(\tau)$, which is split into *five* integrals. Hence, we have

$$\begin{aligned}
r(t) &= \left| -\int_{t_0-2T_s-t}^{t_0-T_s} d\tau + \int_{t_0-T_s-t}^{t_0-1.5T_s} d\tau - \int_{t_0-1.5T_s-t}^{t_0-0.5T_s} d\tau \right. \\
&\quad \left. + \int_{t_0-0.5T_s-t}^{t_0-T_s} d\tau - \int_{t_0-T_s-t}^{t_0} d\tau \right. \\
&= |-5t - 4T_s \quad -T_s \leq t < -0.5T_s
\end{aligned} \tag{66}$$

If we substitute our numerical values we get, $r(t) = -T_s$, $t = T_s$.

Case 8: In Fig. 26, the intersection or the yellow area is between left edge of the $g(t + \tau)$ and the right edge of the $f(\tau)$, which is split into *three* integrals. Hence, we have

$$\begin{aligned}
r(t) &= \int_{t_0-2T_s-t}^{t_0-0.5T_s} d\tau - \int_{t_0-0.5T_s-t}^{t_0-1.5T_s} d\tau + \int_{t_0-1.5T_s-t}^{t_0} d\tau \\
&= |3t + 4T_s \quad -1.5T_s \leq t < -T_s
\end{aligned} \tag{67}$$

If we substitute our numerical values we get, $r(t) = -T_s$, $t = T_s$.

Case 9: In Fig. 27, the intersection or the yellow area is between left edge of the $g(t + \tau)$ and the right edge of the $f(\tau)$, which is equal to *one* integral. Hence, we have

$$\begin{aligned}
r(t) &= -\int_{t_0-2T_s-t}^{t_0} d\tau \\
&= |-t - 2T_s \quad -2T_s \leq t < -1.5T_s
\end{aligned} \tag{68}$$

If we substitute our numerical values we get, $r(t) = -T_s$, $t = T_s$.

Case 10: In Fig. 28 it is shown the case with the function $f(t)$ is left behind the function $g(t + \tau)$. This occurs when

$$\begin{aligned}
r(t) &= f(t) * g(-t) \\
&= |0 \quad t < -2T_s
\end{aligned}$$

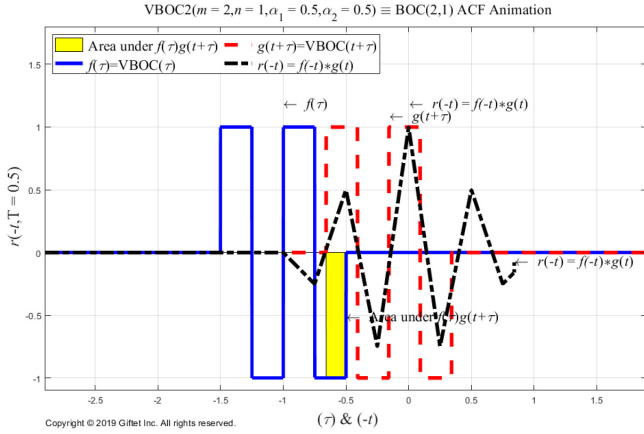


Fig. 27. BOC(2,1) ACF Case 9: the eight intersection between $f(\tau) = \text{VBOC2}_{(2,1,0.5,0.5)}(\tau)$ and $g(t + \tau) = \text{VBOC2}_{(2,1,0.5,0.5)}(t + \tau)$; $t_0 = -0.5$, $T = 0.5$.

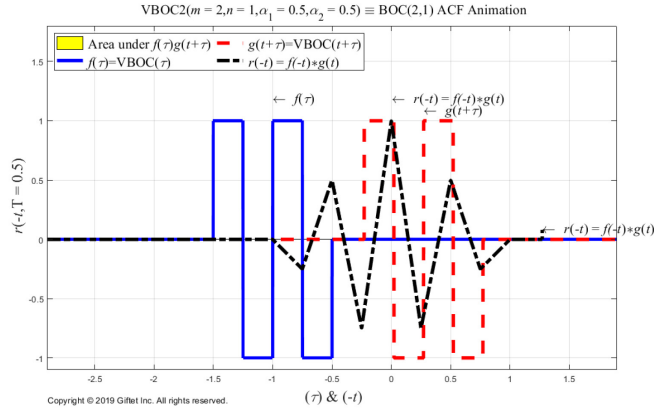


Fig. 28. BOC(2,1) ACF Case 10: no intersection after the integration between $f(\tau) = \text{VBOC2}_{(2,1,0.5,0.5)}(\tau)$ and $g(t + \tau) = \text{VBOC2}_{(2,1,0.5,0.5)}(t + \tau)$; $t_0 = -0.5$, $T = 0.5$.

$$= |0 \quad t < -1 \quad (69)$$

Finally, if we combine (60) through (69) we obtain

$$r(t) = \begin{cases} 0 & t > 2T_s \\ t - 2T_s & 1.5T_s \leq t < 2T_s \\ -3t + 4T_s & T_s \leq t < 1.5T_s \\ 5t - 4T_s & 0.5T_s \leq t < T_s \\ -7t + 2T_s & 0 \leq t < 0.5T_s \\ 7t + 2T_s & -0.5T_s \leq t < 0 \\ -5t - 4T_s & -T_s \leq t < -0.5T_s \\ 3t + 4T_s & -1.5T_s \leq t < -T_s \\ -t - 2T_s & -2T_s \leq t < -1.5T_s \\ 0 & t < -2T_s \end{cases} \quad (70)$$

The maximum value of the $r(t)$ occurs at $t = 0$ which is the same as $2T_s$. So, if we divide $r(t)$ by $2T_s$ we obtain the normalized autocorrelation function as follows

$$\hat{r}(t) = \begin{cases} 1 - \frac{7|t|}{2T_s} & 0 \leq |t| \leq 0.5T_s \\ -2 + \frac{5|t|}{2T_s} & 0.5T_s \leq |t| \leq T_s \\ 2 - \frac{3|t|}{2T_s} & T_s \leq |t| \leq 1.5T_s \\ -1 + \frac{|t|}{2T_s} & 1.5T_s \leq |t| \leq 2T_s \\ 0 & |t| > 2T_s \end{cases} \quad (71)$$

The normalized ACF $\hat{r}(t)$ has its maximum peak equal to one at the values of $t = 0$. Since the normalized ACF $\hat{r}(t)$ is invariant of t_0 ; hence, the ACF peak by itself does not give any distance information; it is only useful to decode the signal.

This concludes the discussion on the BOC(2,1) signal $f(t)$ and the computation and animation of its ACF waveform $r(t)$ and $\hat{r}(t)$.

3.2 Description of $\text{VBOC2}(2,1,\alpha,\beta_-)$ Signal and ACF Animation

The $\text{VBOC2}_{(2,1,\alpha,\beta_- \equiv 1-\alpha)}(t)$ signal satisfies the definition 1 of the BPSK signal whose first period of the $\text{VBOC2}_{(2,1,\alpha,\beta_-)}(t)$ signal, $f(t)$, is given by

$$f(t) = \begin{cases} 1 & t_0 - 2T_s \leq t < t_0 - \gamma_-T_s \\ -1 & t_0 - \gamma_-T_s \leq t < t_0 - T_s \\ 1 & t_0 - T_s \leq t < t_0 - \alpha T_s \\ -1 & t_0 - \alpha T_s \leq t < t_0 \end{cases} \quad (72)$$

where α , β_- , and γ_- are variable parameters that determine the asymmetrical offset of the $\text{VBOC2}_{(2,1,\alpha,\beta_-)}(t)$ waveform

$$0 \leq \alpha \leq 1 \quad (73)$$

$$\gamma_- \equiv 2 - \alpha; \Leftrightarrow \quad (74)$$

$$2 \leq \gamma_- \leq 3 \quad (75)$$

and t_0 as the time offset whose significance is explained later in the end of the section.

The ACF of the $\text{VBOC2}_{(2,1,\alpha,\beta_-)}(t)$ signal, $f(t)$, is defined exactly the same as in the case of the BPSK signal given by definition 2.

In contrast to the animation computation of the ACF of the BPSK, BOC(1,1), and $\text{VBOC1}_{(1,1,\alpha)}(t)$ signal, which contained *four*, *six*, and *eight* cases respectively, the animation computation of the $\text{VBOC2}_{(2,1,\alpha,\beta_-)}(t)$ signal, $f(t)$, contains *fourteen* cases.

Case 1: In Fig. 29 the function $f(\tau)$ is stationary shown with solid blue and $g(t + \tau)$ is depicted with dashed red that slides (or moves) towards $f(\tau)$, which is given by

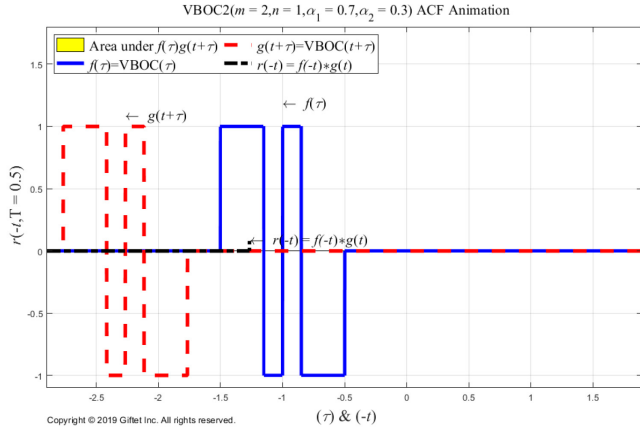


Fig. 29 VBOC2_(2,1,0.7,0.3)(t) ACF Case 1: no intersection between $f(\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

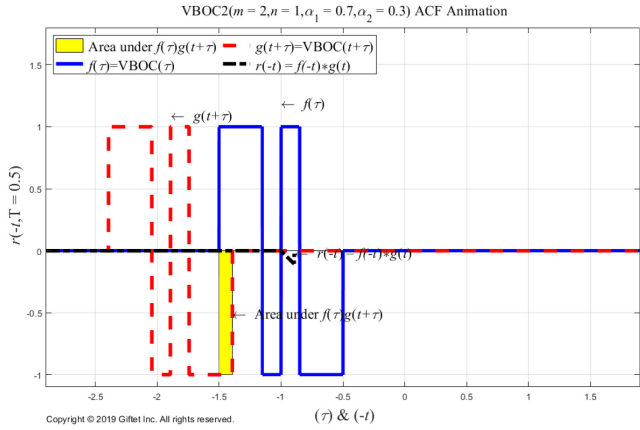


Fig. 30. VBOC2_(2,1,0.7,0.3)(t) ACF Case 2: The first intersection between $f(\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

$$g(t+\tau) = \begin{cases} 1 & t_0 - 2T_s \leq t+\tau < t_0 - \gamma T_s \\ -1 & t_0 - \gamma T_s \leq t+\tau < t_0 - T_s \\ 1 & t_0 - T_s \leq t+\tau < t_0 - \alpha T_s \\ -1 & t_0 - \alpha T_s \leq t+\tau < t_0 \end{cases}$$

$$= \begin{cases} 1 & t_0 - 2T_s - t \leq \tau < t_0 - \gamma T_s - t \\ -1 & t_0 - \gamma T_s - t \leq \tau < t_0 - T_s - t \\ 1 & t_0 - T_s - t \leq \tau < t_0 - \alpha T_s - t \\ -1 & t_0 - \alpha T_s - t \leq \tau < t_0 - t \end{cases} \quad (76)$$

In Case 1, ACF $r(t)$ is zero if and only if

$$r(t) = f(t) * g(-t)$$

$$= |0 \quad t_0 - 2T_s > t_0 - t$$

$$= |0 \quad t > 2T_s \quad (77)$$

Case 2: In Fig. 30 in addition to the function shown in Fig. 29, we show the intersection or the area of the non-zero product of $f(\tau)g(t+\tau)$ with yellow and with the solid dot line the integral or the ACF, $r(t)$. Since the product is equal to one then

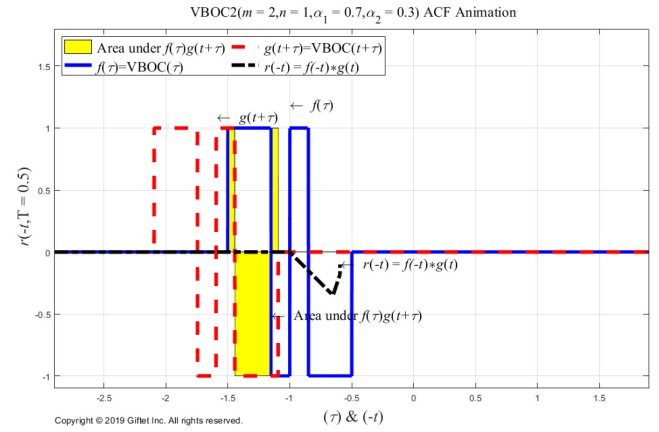


Fig. 31. VBOC2_(2,1,0.7,0.3)(t) ACF Case 3: The second intersection between $f(\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

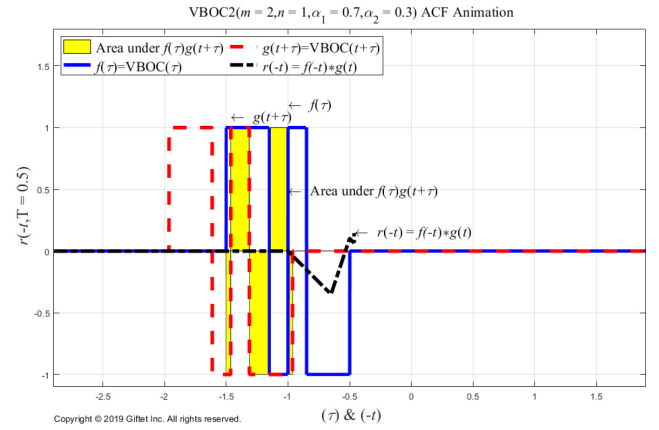


Fig. 32. VBOC2_(2,1,0.7,0.3)(t) ACF Case 4: The third intersection between $f(\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

$r(t)$ is the same as

$$r(t) = \left| - \int_{t_0-2T_s}^{t_0-t} d\tau \quad \gamma T_s \leq t < 2T_s \right.$$

$$= |t - 2T_s \quad \gamma T_s \leq t < 2T_s \quad (78)$$

If we substitute our numerical values we get, $r(t = 1) = 0$, and $r(t = \gamma T_s) = -\alpha T_s$.

Case 3: In Fig. 31, the intersection or the yellow area is between left edge of $f(\tau)$ and the right edge of $g(t+\tau)$, which is split into *three* integrals. Hence, we have

$$r(t) = \int_{t_0-2T_s}^{t_0-\alpha T_s-t} d\tau - \int_{t_0-\alpha T_s-t}^{t_0-\gamma T_s} d\tau + \int_{t_0-\gamma T_s}^{t_0-t} d\tau$$

$$= |-3t + 2(3 - 2\alpha)T_s \quad T_s \leq t < \gamma T_s \quad (79)$$

If we substitute our numerical values we get, $r(t) = (3 - 4\alpha)T_s = 0.1$, $t = T_s = 0.5$, $r(t = \gamma T_s) = -\alpha T_s$.

Case 4: In Fig. 32, the intersection or the yellow area is

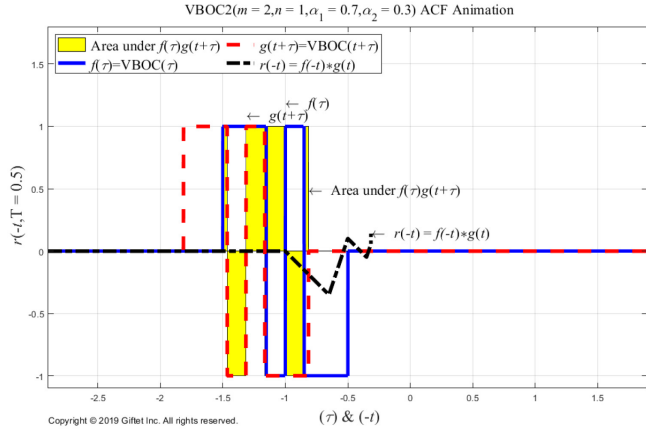


Fig. 33 VBOC2_(2,1,0.7,0.3)(t) ACF Case 5: The fourth intersection between $f(\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

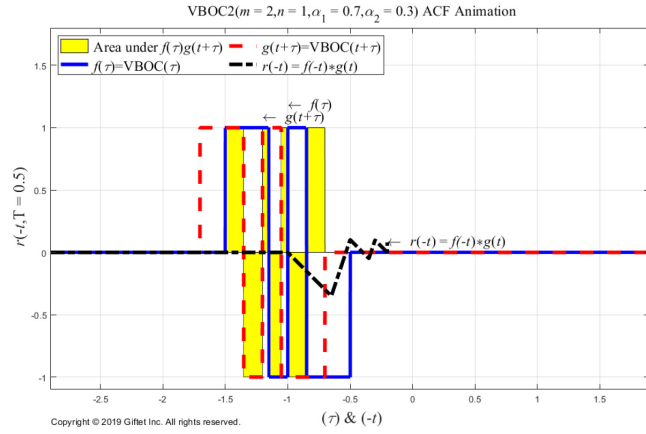


Fig. 34. VBOC2_(2,1,0.7,0.3)(t) ACF Case 6: The fifth intersection between $f(\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

between left edge of $f(\tau)$ and the right edge of $g(t+\tau)$, which is split into *five* integrals. Hence, we have

$$r(t) = \left| \begin{array}{l} -\int_{t_0-2T_s}^{t_0-T_s-t} d\tau + \int_{t_0-T_s-t}^{t_0-\alpha T_s-t} d\tau - \int_{t_0-\alpha T_s-t}^{t_0-\gamma-T_s} d\tau \\ \int_{t_0-\gamma-T_s}^{t_0-T_s} d\tau - \int_{t_0-T_s}^{t_0-t} d\tau \end{array} \right|$$

$$= |t + 2(1 - 2\alpha)T_s \quad \alpha T_s \leq t < T_s \quad (80)$$

If we substitute our numerical values we get, $r(t = T_s) = (3 - 4\alpha)T_s = 0.1$ and $r(t = \alpha T_s) = \alpha T_s + 2(1 - 2\alpha)T_s = (2 - 3\alpha)T_s = -0.05$.

Case 5: In Fig. 33, the intersection or the yellow area is between left edge of $f(\tau)$ and the right edge of $g(t+\tau)$, which is split into *seven* integrals. Hence, we have

$$r(t) = \left| \begin{array}{l} \int_{t_0-2T_s}^{t_0-\gamma-T_s-t} d\tau - \int_{t_0-\gamma-T_s-t}^{t_0-T_s-t} d\tau + \int_{t_0-T_s-t}^{t_0-\alpha T_s-t} d\tau \\ -\int_{t_0-\alpha T_s-t}^{t_0-T_s} d\tau + \int_{t_0-T_s}^{t_0-\gamma-T_s} d\tau - \int_{t_0-\gamma-T_s}^{t_0-t} d\tau \end{array} \right|$$

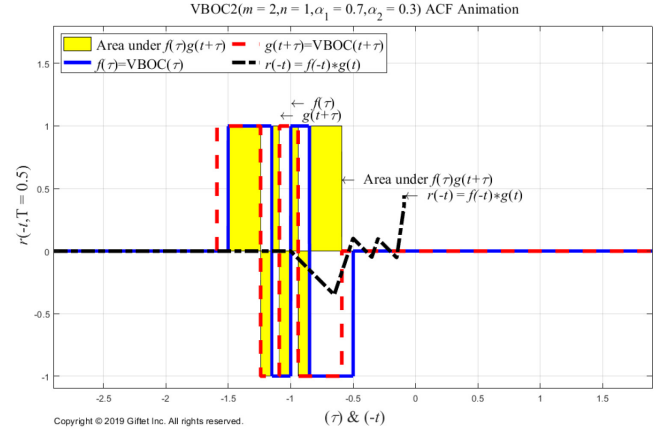


Fig. 35. VBOC2_(2,1,0.7,0.3)(t) ACF Case 7: The sixth intersection between $f(\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

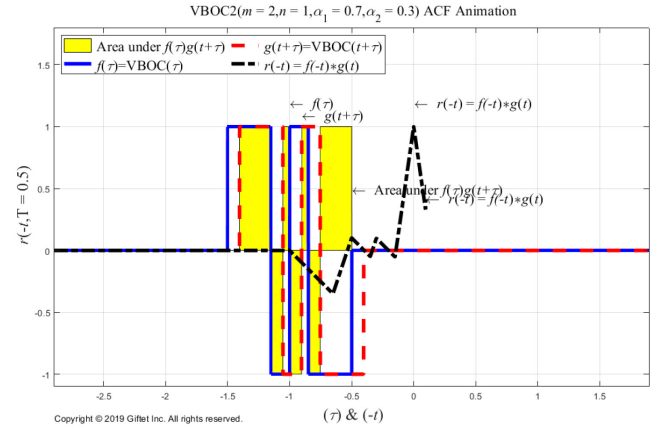


Fig. 36. VBOC2_(2,1,0.7,0.3)(t) ACF Case 8: The seventh intersection between $f(\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

$$= |2T_s - 3t \quad 2\beta_-T_s \leq t < \alpha T_s^{\text{xvii}} \quad (81)$$

Substituting our numerical values, we get, $r(t = \alpha T_s) = (2 - 3\alpha)T_s = -0.05$ and $r(t = 2\beta_-T_s) = (3\alpha - 2)2T_s = 0.1$.

Case 6: In Fig. 34, the intersection or the yellow area is between left edge of $f(\tau)$ and the right edge of $g(t+\tau)$, which is split into *seven* integrals. Hence, we have

$$r(t) = \left| \begin{array}{l} \int_{t_0-2T_s}^{t_0-\gamma-T_s-t} d\tau - \int_{t_0-\gamma-T_s-t}^{t_0-T_s-t} d\tau + \int_{t_0-T_s-t}^{t_0-\gamma-T_s} d\tau \\ -\int_{t_0-\gamma-T_s}^{t_0-\alpha T_s-t} d\tau + \int_{t_0-\alpha T_s-t}^{t_0-T_s} d\tau \\ -\int_{t_0-T_s}^{t_0-\alpha T_s} d\tau + \int_{t_0-\alpha T_s}^{t_0-t} d\tau \end{array} \right|$$

$$= |(4\alpha - 3)2T_s + t \quad \beta_-T_s \leq t < \beta_-2T_s \quad (82)$$

If we substitute our numerical values we get, $r(t = 2\beta_-T_s) = (3\alpha - 2)2T_s = 0.1$ and $r(t = \beta_-T_s) = (7\alpha - 5)T_s = -0.05$.

Case 7: In Fig. 35, the intersection or the yellow area is

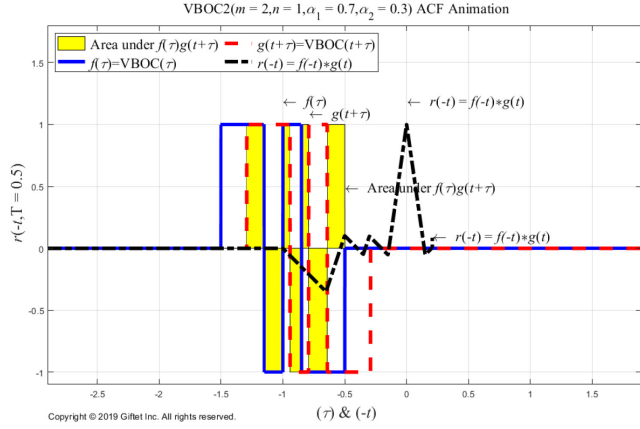


Fig. 37. $VBOC2_{(2,1,0.7,0.3)}(t)$ ACF Case 9: The eighth intersection between $f(\tau) = VBOC2_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = VBOC2_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

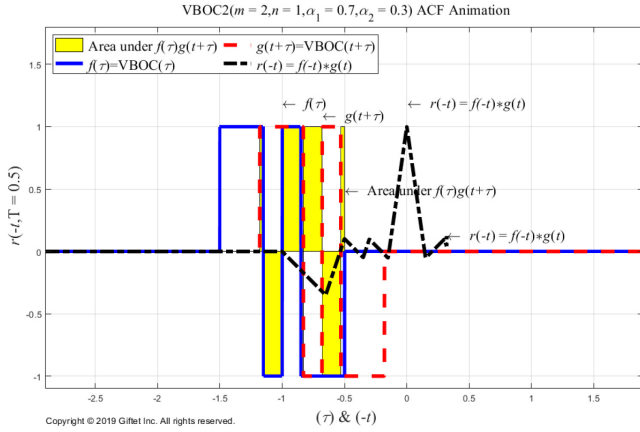


Fig. 38. $VBOC2_{(2,1,0.7,0.3)}(t)$ ACF Case 10: The ninth intersection between $f(\tau) = VBOC2_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = VBOC2_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

between left edge of $f(\tau)$ and the right edge of $g(t+\tau)$, which is split into seven integrals. Hence, we have

$$r(t) = \begin{cases} \int_{t_0-2T_s-t}^{t_0-\gamma-T_s} d\tau - \int_{t_0-\gamma-T_s}^{t_0-\gamma-T_s-t} d\tau + \int_{t_0-\gamma-T_s-t}^{t_0-\alpha T_s} d\tau \\ - \int_{t_0-\gamma-T_s}^{t_0-\alpha T_s-t} d\tau + \int_{t_0-\alpha T_s-t}^{t_0-T_s} d\tau \\ - \int_{t_0-T_s}^{t_0-\alpha T_s} d\tau + \int_{t_0-\alpha T_s}^{t_0-t} d\tau \end{cases}$$

$$= |2T_s - 7t \quad 0 \leq t < \beta_- T_s \quad (83)$$

If we substitute our numerical values we get, $r(t = \beta_- T_s) = (7\alpha - 5)T_s = -0.05$ and $r(t = 0) = 2T_s = 1$.

So far, we have completed half of the ACF. The other half is symmetrical to the half we have already computed. Nevertheless, we will continue doing the direct integration.

Case 8: In Fig. 36, the intersection or the yellow area is between left edge of $g(t+\tau)$ and the right edge of $f(\tau)$, which is split into *seven* integrals. Hence, we have

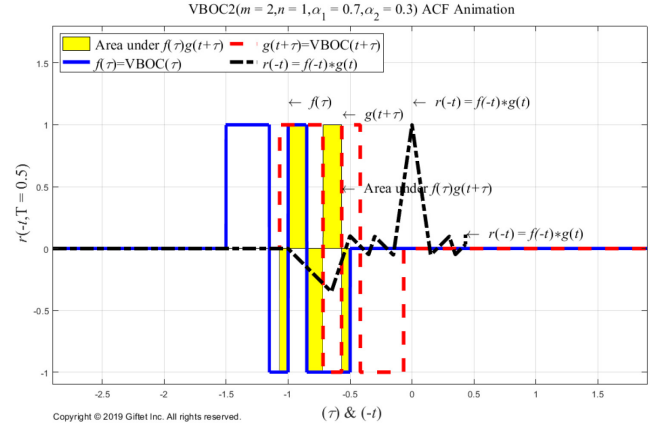


Fig. 39. $VBOC2_{(2,1,0.7,0.3)}(t)$ ACF Case 11: The tenth intersection between $f(\tau) = VBOC2_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = VBOC2_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

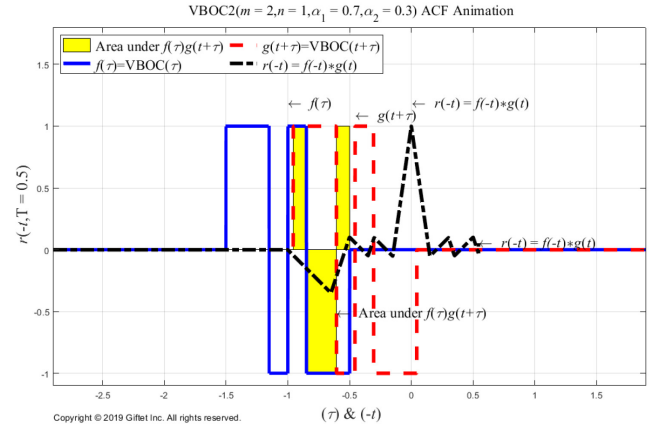


Fig. 40. $VBOC2_{(2,1,0.7,0.3)}(t)$ ACF Case 12: The eleventh intersection between $f(\tau) = VBOC2_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = VBOC2_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

$$r(t) = \begin{cases} \int_{t_0-2T_s-t}^{t_0-\gamma-T_s} d\tau - \int_{t_0-\gamma-T_s}^{t_0-\gamma-T_s-t} d\tau + \int_{t_0-\gamma-T_s-t}^{t_0-\alpha T_s} d\tau \\ - \int_{t_0-\gamma-T_s}^{t_0-\alpha T_s-t} d\tau + \int_{t_0-\alpha T_s-t}^{t_0-T_s} d\tau \\ - \int_{t_0-T_s}^{t_0-\alpha T_s} d\tau + \int_{t_0-\alpha T_s}^{t_0-t} d\tau \end{cases}$$

$$= |2T_s + 7t \quad -\beta_- T_s \leq t < 0 \quad (84)$$

If we substitute our numerical values we get, $r(t = -\beta_- T_s) = (7\alpha - 5)T_s = -0.05$ and $r(t = 0) = 2T_s = 1$.

Case 9: In Fig. 37, the intersection or the yellow area is between left edge of $g(t+\tau)$ and the right edge of $f(\tau)$, which is split into *seven* integrals. Hence, we have

$$r(t) = \begin{cases} \int_{t_0-2T_s-t}^{t_0-\gamma-T_s} d\tau - \int_{t_0-\gamma-T_s}^{t_0-\gamma-T_s-t} d\tau + \int_{t_0-\gamma-T_s-t}^{t_0-\alpha T_s} d\tau \\ - \int_{t_0-\gamma-T_s}^{t_0-\alpha T_s-t} d\tau + \int_{t_0-\alpha T_s-t}^{t_0-T_s} d\tau \\ - \int_{t_0-T_s}^{t_0-\alpha T_s} d\tau + \int_{t_0-\alpha T_s}^{t_0-t} d\tau \end{cases}$$

$$= |(4\alpha - 3)2T_s - t \quad -2\beta_- T_s \leq t < -\beta_- T_s \quad (85)$$

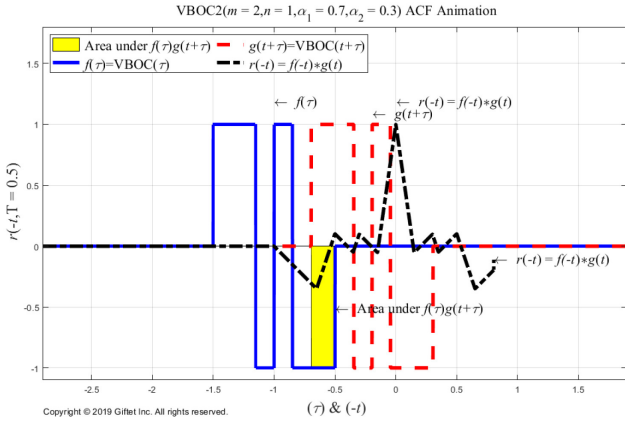


Fig. 41 VBOC2_(2,1,0.7,0.3)(t) ACF Case 13: The twelfth intersection between $f(\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

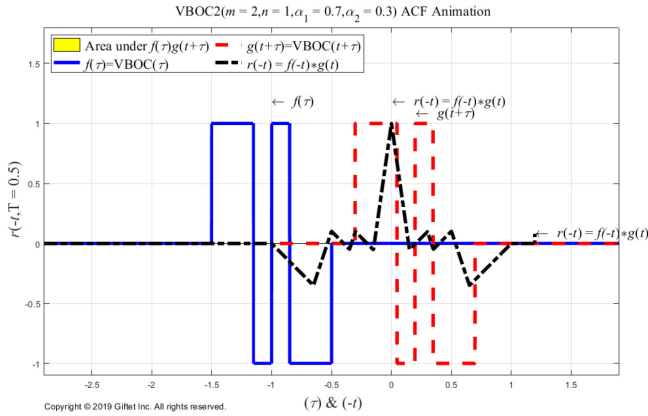


Fig. 42. VBOC2_(2,1,0.7,0.3)(t) ACF Case 14: No intersection after the integration between $f(\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(\tau)$ and $g(t+\tau) = \text{VBOC2}_{(2,1,0.7,0.3)}(t+\tau)$; $t_0 = -0.5$, $T = 0.5$.

If we substitute our numerical values we get, $r(t = -2\beta_-T_s) = (3\alpha - 2)2T_s = 0.1$ and $r(t = -\beta_-T_s) = (7\alpha - 5)T_s = -0.05$.

Case 10: In Fig. 38, the intersection or the yellow area is between left edge of $g(t+\tau)$ and the right edge of $f(\tau)$, which is split into *seven* integrals. Hence, we have

$$r(t) = \begin{cases} \int_{t_0-2T_s-t}^{t_0-\gamma_-T_s} d\tau - \int_{t_0-\gamma_-T_s}^{t_0-T_s} d\tau + \int_{t_0-T_s}^{t_0-\alpha T_s} d\tau \\ - \int_{t_0-\alpha T_s}^{t_0-\gamma_-T_s-t} d\tau + \int_{t_0-\gamma_-T_s-t}^{t_0-T_s-t} d\tau \\ - \int_{t_0-T_s-t}^{t_0} d\tau + \int_{t_0-\alpha T_s-t}^{t_0} d\tau \end{cases}$$

$$= |2T_s + 3t \quad -\alpha T_s \leq t < -2\beta_-T_s^{\text{xviii}} \quad (86)$$

Substituting our numerical values, we get, $r(t = \alpha T_s) = (2 - 3\alpha)T_s = -0.05$ and $r(t = 2\beta_-T_s) = (3\alpha - 2)2T_s = 0.1$.

Case 11: In Fig. 39, the intersection or the yellow area is between left edge of $g(t+\tau)$ and the right edge of $f(\tau)$, which is split into *five* integrals. Hence, we have

$$r(t) = \begin{cases} - \int_{t_0-2T_s-t}^{t_0-\gamma_-T_s} d\tau + \int_{t_0-\gamma_-T_s}^{t_0-\alpha T_s} d\tau - \int_{t_0-\alpha T_s}^{t_0-\gamma_-T_s-t} d\tau \\ + \int_{t_0-\gamma_-T_s-t}^{t_0-T_s-t} d\tau - \int_{t_0-T_s-t}^{t_0} d\tau \end{cases}$$

$$= |-t + 2(1 - 2\alpha)T_s \quad -T_s \leq t < -\alpha T_s \quad (87)$$

If we substitute our numerical values we get, $r(t = -T_s) = (3 - 4\alpha)T_s = 0.1$ and $r(t = -\alpha T_s) = \alpha T_s + 2(1 - 2\alpha)T_s = (2 - 3\alpha)T_s = -0.05$.

Case 12: In Fig. 40, the intersection or the yellow area is between left edge of $g(t+\tau)$ and the right edge of $f(\tau)$, which is split into *three* integrals. Hence, we have

$$r(t) = \int_{t_0-2T_s-t}^{t_0-\alpha T_s} d\tau - \int_{t_0-\alpha T_s}^{t_0-\gamma_-T_s-t} d\tau + \int_{t_0-\gamma_-T_s-t}^{t_0} d\tau$$

$$= |3t + 2(3 - 2\alpha)T_s \quad -\gamma_-T_s \leq t < -T_s \quad (88)$$

If we substitute our numerical values we get, $r(t = -T_s) = (3 - 4\alpha)T_s = 0.1$, and $r(t = -\gamma_-T_s) = -\alpha T_s$.

Case 13: In Fig. 41, the intersection or the yellow area is between left edge of $g(t+\tau)$ and the right edge of $f(\tau)$, which results into *one* integral. Hence, we have

$$r(t) = \left| - \int_{t_0-2T_s-t}^{t_0} d\tau \quad -2T_s \leq t < -\gamma_-T_s \right.$$

$$= |-t - 2T_s \quad -2T_s \leq t < -\gamma_-T_s \quad (89)$$

If we substitute our numerical values we get, $r(t = -1) = 0$, and $r(t = -\gamma_-T_s) = -\alpha T_s$.

Case 14: In Fig. 42 it is shown the case with the function $f(t)$ is left behind the function $g(t)$. This occurs when

$$r(t) = f(t) * g(-t)$$

$$= |0 \quad t < -2T_s \quad (90)$$

Finally, if we combine (77) through (90) we obtain for $\alpha > 2/3$ the following

$$r(t) = \begin{cases} 0 & t > 2T_s \\ t - 2T_s & \gamma_-T_s \leq t < 2T_s \\ 2(3 - 2\alpha)T_s - 3t & T_s \leq t < \gamma_-T_s \\ t + 2(1 - 2\alpha)T_s & \alpha T_s \leq t < T_s \\ 2T_s - 3t & 2\beta_-T_s \leq t < \alpha T_s \\ (4\alpha - 3)2T_s + t & \beta_-T_s \leq t < 2\beta_-T_s \\ 2T_s - 7t & 0 \leq t < \beta_-T_s \\ 2T_s + 7t & -\beta_-T_s \leq t < 0 \\ (4\alpha - 3)2T_s - t & -2\beta_-T_s \leq t < -\beta_-T_s \\ 2T_s + 3t & -\alpha T_s \leq t < -2\beta_-T_s \\ 2(1 - 2\alpha)T_s - t & -T_s \leq t < -\alpha T_s \\ 3t + 2(3 - 2\alpha)T_s & -\gamma_-T_s \leq t < -T_s \\ -t - 2T_s & -2T_s \leq t < -\gamma_-T_s \\ 0 & t < -2T_s \end{cases} \quad (91)$$

The maximum value of the $r(t)$ occurs at $t = 0$ which is the same as $2T_s$. So, if we divide $r(t)$ by $2T_s$ we obtain the

normalized ACF as follows for $\alpha > 2/3$

$$\hat{r}(t) = \begin{cases} 1 - \frac{7|t|}{2T_s} & 0 \leq |t| \leq \beta_- T_s \\ 4\alpha - 3 + \frac{|t|}{2T_s} & \beta_- T_s \leq |t| < 2\beta_- T_s \\ 1 - \frac{3|t|}{2T_s} & 2\beta_- T_s \leq |t| \leq \alpha T_s \\ \frac{|t|}{2T_s} + 1 - 2\alpha & \alpha T_s \leq |t| \leq T_s \\ 3 - 2\alpha - \frac{3|t|}{2T_s} & T_s \leq |t| \leq \gamma_- T_s \\ -1 + \frac{|t|}{2T_s} & \gamma_- T_s \leq |t| \leq 2T_s \\ 0 & |t| > 2T_s \end{cases} \quad (92)$$

The normalized ACF $\hat{r}(t)$ has its maximum peak equal to one at the values of $t = 0$. Since the normalized ACF $\hat{r}(t)$ is invariant of t_0 ; hence, the ACF peak by itself does not give any distance information; it is only useful to decode the signal.

This concludes the discussion on the $VBOC2_{(2,1,\alpha,\beta_-)}(t)$ signal $f(t)$ and the computation and animation of its ACF waveform $r(t)$ and $\hat{r}(t)$.

4 Conclusions

This paper is the first complete discussion on the animation of eight special cases of the ACF of $VBOC2(\alpha_1, \alpha_2)$ GMGM, namely, two corresponding to BPSK, two corresponding to BOC(1,1), two corresponding to $VBOC1(1,1,\alpha)$, one corresponding to BOC(2,1), and one $VBOC2(2,1,\alpha,\beta_-)$.

It is clearly demonstrated both analytically and graphically (or by means of the animation) that the ACF of all these waveforms does not depend on the initial time offset, t_0 ; *the ACF peak by itself does not give any distance information; it is only useful to decode the signal.*

This is the very first publication that shows that the definitions of the $VBOC1(1,1,\alpha)$ and $VBOC2(2,1,\alpha,\beta_-)$ are not unique and this insight came from the description of the corresponding ACFs by means of animation.

This paper does not contradict the finds of any of my previous publications [32]-[41], it only adds extensive clarity and removes any ambiguity in understanding the content of any of my previous publications [32]-[41] and all of this came from description of the ACF by means of animation.

Hence, the case can be made that $VBOC2_{(2,1,\alpha_1,\alpha_2)}$ is a much better signal design than either or both $VBOC1_{(1,1,\alpha)}$ or/and $VBOC2_{(1,1,\alpha,1-\alpha)}$; i.e., it cannot be worse than any of the previously published signal designs in Proгри 2014, [34]-2018, [40] because anything that was previously published is included as special case. The detailed discussion of

$VBOC2_{(2,1,\alpha_1,\alpha_2)}$ is subject to a very special journal paper. This journal paper serves as a building block or setting the stage for the complete and detailed discussion of $VBOC2_{(2,1,\alpha_1,\alpha_2)}$.

Further details of the animation can be obtained from Proгри 2019, [43]-[54].

5 Acknowledgement

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This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Proгри, my dear father, Fiqiri Proгри, my father's first cousin Dr. Peter Demir, and the forty-first President of the United States of America, George H.W. Bush.

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ⁱ This journal paper is a *masterpiece marvel in signal analysis, modeling, simulation, and animation series* because it is the first journal paper where all these tools are discussed together as a whole and in complete harmony with each other.

ⁱⁱ Henry Nyquist (/ˈnaɪkwɪst/, Swedish: [ˈnyːkvɪst]; February 7, 1889 – April 4, 1976) was a Swedish-born American electronic engineer who made important contributions to communication theory [3].

He worked at AT&T's Department of Development and Research from 1917 to 1934, and continued when it became Bell Telephone Laboratories that year, until his retirement in 1954 [3].

Nyquist received the IRE Medal of Honor in 1960 for “fundamental contributions to a quantitative understanding of thermal noise, data transmission and negative feedback.” In October 1960 he was awarded the Stuart Ballantine Medal of the Franklin Institute “for his theoretical analyses and practical inventions in the field of communications systems during the past forty years including, particularly, his original work in the theories of telegraph transmission, thermal noise in electric conductors, and in the history of feedback systems.” In 1969 he was awarded the National Academy of Engineering's fourth Founder's Medal “in recognition of his many fundamental contributions to engineering.” In 1975 Nyquist received together with Hendrik Bode the Rufus Oldenburger Medal from the American Society of Mechanical Engineers [3].

ⁱⁱⁱ The Bell Telephone Company, a common law joint stock company, was organized in Boston, Massachusetts on July 9, 1877, by Alexander Graham Bell's father-in-law Gardiner Greene Hubbard, who also helped organize a sister company — the New England Telephone and Telegraph Company. The Bell Telephone Company was started on the basis of holding “potentially valuable patents,” principally Bell's master telephone patent #174465 [5].

^{iv} Sir Maurice George Kendall, FBA (6 September 1907 – 29 March 1983) was a British statistician, widely known for his contribution to statistics. The Kendall tau rank correlation is named after him [6].

In 1938 and 1939 he began work, along with Bernard Babington-Smith, on the question of random number generation, developing both one of the first early mechanical devices to produce random digits, and formulated a series of tests for statistical randomness in a given set of digits which, with some small

modifications, became fairly widely used [6].

^v Maurice Stevenson Bartlett FRS (18 June 1910 – 8 January 2002) was an English statistician who made particular contributions to the analysis of data with spatial and temporal patterns. He is also known for his work in the theory of statistical inference and in multivariate analysis [8].

^{vi} Norman Levinson (August 11, 1912 in Lynn, Massachusetts – October 10, 1975 in Boston) was an American mathematician. Some of his major contributions were in the study of Fourier transforms, complex analysis, non-linear differential equations, number theory, and signal processing. He worked closely with Norbert Wiener in his early career [6]. Levinson was a doctoral student of Norbert Wiener.

^{vii} Norbert Wiener (November 26, 1894 – March 18, 1964) was an American mathematician and philosopher. He was a professor of mathematics at the Massachusetts Institute of Technology (MIT). A child prodigy, Wiener later became an early researcher in stochastic and mathematical noise processes, contributing work relevant to electronic engineering, electronic communication, and control systems [14].

^{viii} Andrey Nikolaevich Kolmogorov (Russian: Андрей Никола́евич Колмогоров, IPA: [ɐnˈdrʲej nʲɪkɐˈlajzɨvʲɪtɕ kəlmeˈɡorəf] (About this soundlisten), 25 April 1903 – 20 October 1987) was a Soviet mathematician who made significant contributions to the mathematics of probability theory, topology, intuitionistic logic, turbulence, classical mechanics, algorithmic information theory and computational complexity [15].

^{ix} The animation file can be obtained from [43].

^x The animation file can be obtained from [44].

^{xi} The animation file can be obtained from [45].

^{xii} The animation files can be obtained from [46], [47].

^{xiii} The animation files can be obtained from [48], [49].

^{xiv} The animation files can be obtained from [50], [51].

^{xv} The animation file can be obtained from [52].

^{xvi} The animation file can be obtained from [53].

^{xvii} This condition is true if and only if (or iff) $\alpha > \frac{2}{3} = 6.6667$.

^{xviii} Ditto xvii.