

Research Article

Efficient Computation of the Special Cases of the Generalized Bessel Function Distributions

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This paper discusses efficient and original computations of the special cases of the generalized Bessel function distributions of the second kind. The journal paper is a masterpiece in the field of numerical methods, numerical linear algebra, and computational mathematics. The entire work in this journal paper is original and never presented before in the literature. The main focus of this journal paper is to fill in the gap or to deal with the special cases that were not covered in the two premier publications of Progri 2016 and 2018. In this journal paper the closed form expression and the efficient computation of the probability density function (pdf) and the cumulative distribution function (cdf) for five special cases is provided. For each generalized Bessel distribution model the closed form expression of the cdfs is given by means of Legendre continued fraction or Taylor series expansion of the modified Bessel functions of the second kind, for both integer and real values of a parameter, which leads to incomplete gamma functions, hypergeometric series, the Kampé de Fériet functions (or double hypergeometric series), and in two special cases of the Srivastava's triple hypergeometric series. Efficient computations of the algorithms (which are implemented in MATLAB functions) are also presented in the paper. Numerical results are derived for each case to validate the theoretical models presented in the paper.

Index Terms—Bessel functions, modified Bessel functions, cumulative distribution function, Kampé de Fériet function, Srivastava's triple hypergeometric series, closed form expression, signal analysis, generalized functions, time-domain analysis, incomplete gamma functions, hypergeometric series.

1 Introduction

This journal paper discusses several special cases in the original closed form expressions of the generalized Bessel function cumulative distribution functions (cdf) of the second kind.

Why are these special cases so important? These special cases are important because they cover cases that were not discussed in either Progrid 2016 [1] and Progrid 2018 [2]. There was a provision made in Progrid 2016 [1] and an attempt to derive the main special case which is the one corresponding to the integer values of p . However, the derivations in Progrid 2016 [1] were not complete. Hence, the main focus on this journal paper is to derive this closed form expression of the generalized modified Bessel function distribution of the second kind for integer values of p .

Why was I not successful in deriving the entire, closed form expression of the generalized modified Bessel function distribution of the second kind for integer values of p in 2016?

There are two main reasons why I was not successful to derive the entire, closed form expression of the generalized modified Bessel function distribution of the second kind for integer values of p in 2016: I had a limited capability of the computation library and I was not following the right computation (or numerical) methodology.

First, what do I mean by not having an in depth understanding of the computation capability? The computation of the closed form expression of the generalized modified Bessel function distribution of the second kind for integer values of p requires the computation of the Kampé de Fériet function (or double hypergeometric series [1]-[16]) and in some cases the of Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[22] in MATLAB. Back in 2016 I was barely understanding the Kampé de Fériet function (or double hypergeometric series [1]-[16]) and I made an unsuccessful attempt to derive an inefficient implementation of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[22] which luckily I did not publish. This limitation in the computational capability was filled in 2018 with the arrival of the Progrid 2018 [2] where I developed a profoundly improved implementation in MATLAB of the Kampé de Fériet function (or double hypergeometric series [1]-[16]). Moreover, this limitation in the computational capability was entirely eliminated in 2019 as discussed in Appendices A through F in this paper.

Second, what is the right computational methodology? The computational methodology has to do with a sequence of steps, methods, algorithms, processes, etc. that are required to follow in order to produce the desired outcome in the computation of the closed form expression the generalized modified Bessel function distribution of the second kind for integer values of p . The main issue here is that this is not a trivial problem. Therefore, because this is not a trivial problem it requires a more profoundly sophisticated computational methodology. The organization of this journal paper is such that it enables the reader to understand the steps that are required to derive the closed form expression the generalized modified Bessel function distribution of the second kind for integer values of p .

In addition to the in depth original, novel, and exceptional organization of this journal paper I am also giving an in depth explanation of the insights that directed me to come up with such a wonderful in depth organization.

In 2016 I published an exceptional journal paper called "Exponential generalized Beta distribution" see (Progrid 2016, [3]) which would not have been possible without me working out a series of details in another journal paper (Progrid 2016, "Hypergeometric function partial derivatives" [4]). In (Progrid 2016, "Hypergeometric function partial derivatives" [4]) I had to develop a series of techniques for me to succeed in the computation of the "Exponential generalized Beta distribution" see (Progrid 2016, [3]). This is exactly what computation methodology is all about.

Therefore, by way of comparison and contrast, in order for me to develop this absolutely outstanding journal paper I had to work out a series of details in at least three other journal papers, Progrid 2016, [1], [4] and Progrid 2018, [2].

In Progrid 2016, [1] I was able to develop the insight that there are things that I can do in 2016 and there are things that I need to do later on. In Progrid 2018, [2] I was able to validate exactly that same computational mentality (for example the series expansion of incomplete gamma function in odd and even components (see Progrid, 2018 [2], (19)) that I initially architected in Progrid 2016, [1]. In Progrid 2016, [4] I was able to produce a series of techniques that I can use successfully in the implementation of the series of psi $\psi(k+1)$ Progrid (2016, [4] (127)) function not known by a lot of scholars. In this journal paper I was able to integrate the computation of $\psi(k+1)$ and $\psi(k+p+1)$ with the computation of the so called modified generalized Kampé de Fériet function because this was the ONLY way to achieve the highest possible performance in

computational speed and accuracy.

The issue, however, is that the efficient computation of these special cases cdfs require the computation of the several Kampé de Fériet functions [2], [10]-[13] and in some cases of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[22] in MATLAB. The main issue is the following: Is there an algorithm that we can apply to successfully produce an efficient MATLAB function of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[22]? Is this algorithm also computationally stable?

Since, the generalized Bessel functions cdfs require the computation of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[22] and the latter is not yet available in MATLAB 2018b [30], Giftet Inc. is also taking on the task to compute the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[22] as part of the Indoor Geolocation Systems MATLAB Library and publish the results of this work in Dr. Proгри pioneer publication [17].

The journal paper is also an extension of the discussion of "Efficient computation of generalized Bessel function distributions," in Proгри 2018, [2]. Therefore, things that already mentioned in Proгри 2018, [2] (the efficient computation of the Kampé de Fériet function (or double hypergeometric series [1]-[16]), I am not going to reiterate here. In this journal paper I developed another option for the computation of the Kampé de Fériet function (or double hypergeometric series [1]-[16]) called option 2. Option 2 is the recursive implementation of Option 1. The main purpose of Option 2 is to eliminate division or multiplication of very large numbers. Since, in Option 1 the number of terms included in the computation of the Kampé de Fériet function is limited to about fifty; in Option 2 there is no limit on the number of terms included in the computation.

In this journal paper Dr. Proгри masterfully, discusses the advantages and disadvantages of the direct closed form expressions vs. recursive closed form expressions.

Because of the level of detail, sophistication, brilliance in treating all these special cases, this paper presents an original work that falls into the category of *masterpiece marvels in analytical derivations and computational mathematics series* never published anywhere else.

This paper is organized as follows: in Sect. 2 special cases of the generalized Bessel function distribution of the second kind pdfs are discussed. The efficient computation of special cases of the generalized Bessel distribution of the second kind cdfs is

presented in Sect. 3. Section 4 contains numerical, theoretical results; Conclusion is provided in Sect. 5 along with a list of references. An inefficient implementation of $\mathcal{F}_{21}(x; a, 0, 0)$ is presented in Appendix A. An alternative implementation of $\mathcal{F}_{\text{GBessel2}}(x; a, 0, p)$ is discussed in Appendix B. Inefficient Implementation of $\mathcal{F}_{23}(x; a, d, 0)$ is presented in Appendix C. Recursive implementation of $\mathcal{F}_{22}(x; a, d, p)$ is given in Appendix D. In Appendix E, the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ is provided. In Appendix F, a super-efficient implementation of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ is presented.

2 Special Cases of the Generalized Bessel Function Distribution of the Second Kind PDFs

From Proгри 2016, (47) the generalized Bessel distribution of the second kind has its pdf given by

$$f_{\text{GBessel2}}(x; a, d, p) = \frac{|x|^p e^{-\frac{d}{a}|x|} K_p\left(\frac{|x|}{a}\right)}{C_2(p, d) a^{p2}} \quad (1)$$

for $-\infty < x < \infty$, $a > 0$, $d \geq 0$, and $p > -1/2$ where $K_p(x)$ (see Gradshteyn and Ryzhik [24] pg. 917 ex. 8.432 3.) is given by Proгри 2016, [1] (47) and $C_2(p, d)$ is given by Proгри 2016 [1], (56)

$$C_2(p, d) = \frac{\sqrt{\pi} 2^{p2} \Gamma(2p_1) F\left(2p_1, p_1; p_3; \frac{d-1}{d+1}\right)}{(d+1)^{2p_1} \Gamma(p_3)} \quad (2)$$

When $p > 0$ then the Proгри 2016 [1], (58)

$$f_{\text{GBessel2}}(0; a, d, p > 0) = \frac{2^{-2(d+1)} 2^{p1} \Gamma(p_3) \Gamma(p)}{\sqrt{\pi} a \Gamma(2p_1) F\left(2p_1, p_1; p_3; \frac{d-1}{d+1}\right)} \quad (3)$$

However from Proгри 2016 [1] we cannot know what happens to $f_{\text{GBessel2}}(0; a, d, p = 0)$ and $f_{\text{GBessel2}}(0; a, d, p < 0)$ when $-1/2 < p \leq 0$. The answer to these cases in given further down in the section.

Other important special cases consider when p is an integer and is not an integer when $d = 0$.

The special case when $d = 0$ and $p = 0$ is considered as a baseline special case because it serves as a test case for the other two special cases when: (1) $d = 0$ and $p \neq 0$, p a positive integer, which also serves as a test case for the special case when, (2) $d \neq 0$ and $p \neq 0$, p a positive integer.

Moreover, the special case when $d = 0$ and $p = 0$ also serves as test case when $d \neq 0$ and $p = 0$.

The special case when $d = 0$ and $p \neq 0$ not an integer serves as a test case for the general case when $d \neq 0$ and $p \neq$

0 not an integer which extensively discussed in both Proгри 2016 [1] and Proгри 2018 [2].

For other values of $d < 0$ and $p \leq -1/2$ for now we are going to assume that

$$f_{\text{GBessel2}}(x; a, d < 0, p \leq -1/2) = 0^{\text{iii}} \quad (4)$$

With that being said, in this section we consider five special cases of the generalized Bessel function distribution of the second kind.

2.1 Special Case: $d = 0$, $p \neq 0$ not an Integer

If we set $d = 0$ then from (1) we have

$$f_{\text{GBessel2}}(x; a, 0, p) = \frac{|x|^p K_p\left(\frac{|x|}{a}\right)}{C_2(p, 0) a^{p^2}} \quad (5)$$

Where $C_2(p, 0)$ (see (2) and Proгри 2016 [1], (67)-(70))

$$C_2(p, 0) = \frac{\sqrt{\pi} 2^{p^2} \Gamma(2p_1) F(2p_1, p_1; p_3; -1)}{\Gamma(p_3)} \equiv \sqrt{\pi} 2^p \Gamma(p_1) \quad (6)$$

First, we need to develop the series expansion of the modified Bessel function of the second kind, $K_p(t)$, based on the series expansion of the modified Bessel function of the first kind $I_p(t)$ (see Arfken, G.B., and Weber, H.J., 1995, [25],

$$\lim_{x \rightarrow 0} x^p K_p(x) = \lim_{x \rightarrow 0} \frac{\pi}{2} \frac{\sum_{k=0}^{\infty} \left[\frac{1}{k! \Gamma(1-p+k)} \frac{x^{2k}}{2^{2k} k^{2-p}} \frac{1}{k! \Gamma(p_2+k)} \frac{x^{2p} x^{2k}}{2^{2k}} \right]}{\sin \pi p} = \frac{\pi}{\Gamma(1-p) 2^{1-p} \sin \pi p} = 2^{p-1} \Gamma(p) \quad (12)$$

Substituting (6) and (12) into (5) and assuming that $p > 0$ yields

$$f_{\text{GBessel2}}(0; a, d, p) = \frac{\pi^{-\frac{1}{2}} (d+1)^{2p_1} \Gamma(p_3) 2^{p-1} \Gamma(p)}{a^{p^2} 2^{p^2} \Gamma(2p_1) F(2p_1, p_1; p_3; \frac{d-1}{d+1})} \quad (13)$$

So if we set $d = 0$ and $p > 0$ yields

$$f_{\text{GBessel2}}(0; a, 0, p > 0) = \frac{\Gamma(p)}{\sqrt{\pi} 2 \Gamma(p_1) a^{p^2}} \quad (14)$$

$$\lim_{x \rightarrow 0} x^{-q} K_{-q}(x) = \lim_{x \rightarrow 0} x^{-q} K_{-q}(x) = \lim_{x \rightarrow 0} \frac{\pi}{2} \frac{\sum_{k=0}^{\infty} \left[\frac{1}{k! \Gamma(1-q+k)} \frac{x^{2k} x^{-2q}}{2^{2k} 2^{-q}} \frac{1}{k! \Gamma(q_2+k)} \frac{x^{2q} x^{2k}}{2^{2k}} \right]}{\sin \pi q} \quad (16)$$

Which is identical to

$$\lim_{x \rightarrow 0} x^{-q} K_{-q}(x) = \lim_{x \rightarrow 0} x^{-q} K_{-q}(x) = \lim_{x \rightarrow 0} \frac{\pi}{2} \frac{\frac{1}{\Gamma(1-q)} \frac{x^{-2q}}{2^{-q}} \frac{1}{\Gamma(q_2)} \frac{1}{2^q}}{\sin \pi q} = \infty \quad (17)$$

Substituting (6) and (17) into (5) and for $p < 0$ yields

$$f_{\text{GBessel2}}(x; a, 0, p = -q < 0 \equiv q = -p > 0) = \infty \quad (18)$$

So if we set $d = 0$ and $p < 0$ yields

$$f_{\text{GBessel2}}(x; a, 0, p < 0) = \infty \quad (19)$$

This concludes the derivations of the special case for $d = 0$ and $p \neq 0$. Next we consider the special case when both $d = 0$ and $p = 0$.

(11.118) pg. 667 or Gradshteyn, Ryzhik, 2007 [24] pg. 928 ex. 8.485)

$$K_p(x) = \frac{\pi I_{-p}(x) - I_p(x)}{2 \sin \pi p} \quad (7)$$

where the series expansion of the modified Bessel function of the first kind, $I_p(t)$, (see Gradshteyn, Ryzhik, 2007 [24] pg. 919 ex. 8.445)

$$I_p(x) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(p_2+k)} \frac{x^{p+2k}}{2^{p+2k}} \quad (8)$$

$$I_{-p}(x) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(1-p+k)} \frac{x^{-p+2k}}{2^{-p+2k}} \quad (9)$$

where p is not an integer.

Substituting (8) and (9) into (7) yields

$$K_p(x) = \frac{\pi}{2} \frac{\sum_{k=0}^{\infty} \frac{1}{k! \Gamma(1-p+k)} \frac{x^{-p+2k}}{2^{-p+2k}} - \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(p_2+k)} \frac{x^{p+2k}}{2^{p+2k}}}{\sin \pi p} \quad (10)$$

Which is equivalent with

$$K_p(x) = \frac{\pi}{2} \frac{\sum_{k=0}^{\infty} \left[\frac{1}{k! \Gamma(1-p+k)} \frac{x^{2k} x^{-p}}{2^{2k} 2^{-p}} \frac{1}{k! \Gamma(p_2+k)} \frac{x^p x^{2k}}{2^{2k}} \right]}{\sin \pi p} \quad (11)$$

If we take the limit of

What happens when for $-1/2 < p < 0$? Which is equivalent with $0 < q \equiv -p < 1/2$. We already know that

$$K_q(x) = \frac{\pi}{2} \frac{\sum_{k=0}^{\infty} \left[\frac{1}{k! \Gamma(1-q+k)} \frac{x^{2k} x^{-q}}{2^{2k} 2^{-q}} \frac{1}{k! \Gamma(1+q+k)} \frac{x^q x^{2k}}{2^{2k}} \right]}{\sin \pi q} = K_{-q}(x) \quad (15)$$

If we take the limit of

2.2 Special Case: $d = 0$, $p = 0$

If we set $d = 0$ and $p = 0$ then from (1) we have

$$f_{\text{GBessel2}}(x; a, 0, 0) = \frac{K_0\left(\frac{|x|}{a}\right)}{C_2(0, 0) a} \quad (20)$$

When $p = 0$ we have from (see Gradshteyn, Ryzhik, 2007

[24] pg. 919 ex. 8.445 1. and 3.)

$$K_0(x) = -\ln\left(\frac{x}{2}\right) \left[I_0(x) \equiv \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2} \right] + \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2} \psi(k+1) \quad (21)$$

Which is equivalent with

$$K_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k} [\psi(k+1) - \ln(\frac{x}{2})]}{2^{2k}(k!)^2} \quad (22)$$

From $C_2(0,0)$ (see Proгри 2016, (67)-(70))

$$C_2(0,0) = \sqrt{\pi} \Gamma\left(\frac{1}{2}\right) = \pi \quad (23)$$

If we take the limit

$$\lim_{x \rightarrow 0} K_0(x) = \lim_{x \rightarrow 0} \sum_{k=0}^{\infty} \frac{x^{2k} [\psi(k+1) - \ln(\frac{x}{2})]}{2^{2k}(k!)^2} = \infty \quad (24)$$

Which is equivalent with

$$\lim_{x \rightarrow 0} K_0(x) = \lim_{x \rightarrow 0} \sum_{k=0}^{\infty} \left[\psi(1) - \ln\left(\frac{x}{2}\right) \right] = \infty \quad (25)$$

Substituting (23) and (25) into (20) yields

$$f_{\text{GBessel2}}(0; a, 0, 0) = \infty \quad (26)$$

Hence, the pdf of $f_{\text{GBessel2}}(x; a, 0, 0)$ if given by

$$f_{\text{GBessel2}}(x; a, 0, 0) = \begin{cases} \frac{K_0\left(\frac{|x|}{a}\right)}{\pi a} & x \neq 0 \\ \infty & x = 0 \end{cases} \quad (27)$$

This concludes the derivations of the special case for $d = 0$

$$K_p(x) = \frac{1}{2} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-p} + (-1)^{p+1} \sum_{k=0}^{\infty} \frac{\ln \frac{x}{2} \frac{\psi(k+1) + \psi(p+k+1)}{2}}{k!(p+k)!} \left(\frac{x}{2}\right)^{p+2k} \quad (31)$$

If we take the limit

$$\lim_{x \rightarrow 0} \frac{x^p K_p\left(\frac{x}{a}\right)}{C_2(p,0)a^{p/2}} = \lim_{x \rightarrow 0} \frac{x^p \left[\frac{1}{2} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \left(\frac{x}{2a}\right)^{2k-p} + (-1)^{p+1} \sum_{k=0}^{\infty} \frac{\ln \frac{x}{2a} \frac{\psi(k+1) + \psi(p+k+1)}{2}}{k!(p+k)!} \left(\frac{x}{2a}\right)^{p+2k} \right]}{C_2(p,0)a^{p/2}} \quad (32)$$

Which is equivalent with

$$\lim_{x \rightarrow 0} \frac{x^p K_p\left(\frac{x}{a}\right)}{C_2(p,0)a^{p/2}} = \lim_{x \rightarrow 0} \frac{\frac{(2a)^p}{2} (p-1)! + (-1)^{p+1} \frac{x^p}{p!} \left(\frac{x}{2a}\right)^p \left[\ln \frac{x}{2a} \frac{\psi(1) + \psi(p+1)}{2} \right]}{C_2(p,0)a^{p/2}} = \frac{2^{2p-1} p! (p-1)!}{\pi (2p)! a} = \frac{(p-1)!}{2a\sqrt{\pi} \Gamma\left(p+\frac{1}{2}\right)} \quad (33)$$

Hence, the pdf of $f_{\text{GBessel2}}(x; a, 0, p)$ if given by

$$f_{\text{GBessel2}}(x; a, 0, p) = \begin{cases} \frac{|x|^p K_p\left(\frac{|x|}{a}\right)}{C_2(p,0)a^{p/2}} & x \neq 0 \\ \frac{(p-1)!}{2a\sqrt{\pi} \Gamma(p_1)} & x = 0 \end{cases} \quad (34)$$

This concludes the derivations of the special case for $d = 0$ and p is a positive integer.

Next we consider the special case when $d \neq 0$ and $p = 0$.

2.4 Special Case: $d \neq 0$, $p = 0$

When $d \neq 0$ and $p = 0$ we have from (1)

and $p = 0$.

Next, we consider the special case when $d = 0$ and p is a positive integer.

2.3 Special Case: $d = 0$, p a Positive Integer

If we set $d = 0$ then from (1) we have

$$f_{\text{GBessel2}}(x; a, 0, p) = \frac{|x|^p K_p\left(\frac{|x|}{a}\right)}{C_2(p,0)a^{p/2}} \quad (28)$$

where $p \neq 0$ a positive integer or $\forall p \in \mathbb{Z}^+$ and $C_2(p, 0)$ (see Proгри 2016, (67)-(70)) and from a particular value of a gamma function we have

$$C_2(p, 0) = \sqrt{\pi} 2^p \Gamma(p_1) = \sqrt{\pi} 2^p \frac{(2p)!}{2^{2p} p!} \sqrt{\pi} = \pi \frac{(2p)!}{2^p p!} \quad (29)$$

Is this the right answer? If we also set $p = 0$ from (29) we get

$$C_2(0,0) = \pi \frac{0!}{2^0 0!} = \pi \quad (30)$$

Which is identical to (23) where we discussed the case for $p = 0$ and $d = 0$.

When $\forall p \in \mathbb{Z}^+$ we have from (see Gradshteyn, Ryzhik, 2007 [24] pg. 919 ex. 8.446)

$$f_{\text{GBessel2}}(x; a, d, 0) = \frac{e^{-\frac{d}{a}|x|} K_0\left(\frac{|x|}{a}\right)}{a C_2(0,d)} \quad (35)$$

for $-\infty < x < \infty$, $a > 0$, and $d \geq 0$, where $K_0(x)$ (see Gradshteyn, Ryzhik, 2007 [24] pg. 919 ex. 8.445 1. and 3.) is given by Proгри 2016 [1], (47).

From $C_2(0, d)$ (see Proгри 2016 [1], (56))

$$C_2(0, d) = \frac{4F\left(1, \frac{3}{2}, \frac{d-1}{2}\right)}{d+1} \quad (36)$$

Is this the right answer? If we also set $d = 0$ from (36) we

get (see Proгри 2016 (69))

$$C_2(0,0) = 4F\left(1, \frac{1}{2}; \frac{3}{2}; -1\right) = 4 \frac{\Gamma^2(\frac{3}{2})}{\Gamma(2)} = 4 \left(\frac{1}{2}\right)^2 \frac{(\sqrt{\pi})^2}{\Gamma(2)} = \pi \quad (37)$$

Which is identical to (23) where we discussed the case for $p = 0$ and $d = 0$. If we take the limit

$$\lim_{x \rightarrow 0} e^{-\frac{d}{a}|x|} K_0(x) = \lim_{x \rightarrow 0} \sum_{k=0}^{\infty} \frac{x^{2k} [\psi(k+1) - \ln(\frac{x}{2})]}{2^{2k} (k!)^2} = \infty \quad (38)$$

Which is equivalent with

$$\lim_{x \rightarrow 0} e^{-\frac{d}{a}|x|} K_0(x) = \lim_{x \rightarrow 0} \sum_{k=0}^{\infty} \left[\psi(1) - \ln\left(\frac{x}{2}\right) \right] = \infty \quad (39)$$

Substituting (36) and (39) into (35) yields

$$f_{\text{GBessel2}}(0; a, d, 0) = \infty \quad (40)$$

Hence, the pdf of $f_{\text{GBessel2}}(x; a, d, 0)$ if given by

$$f_{\text{GBessel2}}(x; a, d, 0) = \begin{cases} \frac{(d+1)e^{-\frac{d}{a}|x|} K_0\left(\frac{|x|}{a}\right)}{4F\left(1, \frac{3}{2}; \frac{d-1}{d+1}\right)a} & x \neq 0 \\ \infty & x = 0 \end{cases} \quad (41)$$

$$\lim_{x \rightarrow 0} \frac{x^p e^{-\frac{d}{a}|x|} K_p\left(\frac{|x|}{a}\right)}{C_2(p, d) a^{p2}} = \lim_{x \rightarrow 0} \frac{x^p \left[\frac{1}{2} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \left(\frac{x}{2a}\right)^{2k-p} + (-1)^{p+1} \sum_{k=0}^{\infty} \frac{\ln \frac{x}{2a} \frac{\psi(k+1) + \psi(p+k+1)}{2}}{k! (p+k)!} \left(\frac{x}{2a}\right)^{p+2k} \right]}{C_2(p, d) a^{p2}} \quad (44)$$

Which is equivalent with

$$\lim_{x \rightarrow 0} \frac{x^p e^{-\frac{d}{a}|x|} K_p\left(\frac{|x|}{a}\right)}{C_2(p, d) a^{p2}} = \lim_{x \rightarrow 0} \frac{\frac{(2a)^p}{2} (p-1)! + (-1)^{p+1} \frac{x^p}{p!} \left(\frac{x}{2a}\right)^p \left[\ln \frac{x}{2a} \frac{\psi(1) + \psi(p+1)}{2} \right]}{C_2(p, d) a^{p2}} = \frac{(p-1)! (d+1)^{2p1} \Gamma(p_3)}{\sqrt{\pi} 4 \Gamma(2p_1) F\left(2p_1, p_1; p_3; \frac{d-1}{d+1}\right) a} \quad (45)$$

Hence, the pdf of $f_{\text{GBessel2}}(x; a, 0, p)$ if given by

$$f_{\text{GBessel2}}(x; a, d, p) = \begin{cases} \frac{|x|^p e^{-\frac{d}{a}|x|} K_p\left(\frac{|x|}{a}\right)}{C_2(p, d) a^{p2}} & x \neq 0 \\ \frac{(p-1)! (d+1)^{2p1} \Gamma(p_3)}{\sqrt{\pi} 4 \Gamma(2p_1) F\left(2p_1, p_1; p_3; \frac{d-1}{d+1}\right) a} & x = 0 \end{cases} \quad (46)$$

This concludes the derivations of the special case for $d \neq 0$ and p is a positive integer.

Next we consider the special cases the generalized Bessel distribution of the second kind CDFs.

3 Special Cases of the Generalized Bessel Distribution of the Second Kind CDFs

The special cases of the generalized Bessel distribution of the second kind CDFs follow from the discussion given in Sec. 2.

3.1 Special Case: $d = 0$, $p \neq 0$

From the definition of the pdf of the generalized Bessel distribution of the second kind, we define its corresponding cdf as

This concludes the derivations of the special case for $d \neq 0$ and $p = 0$.

Next, we consider the final (or last) special case when $d \neq 0$ and p is a positive integer.

2.5 Special Case: $d \neq 0$, p a Positive Integer

If we set $d \neq 0$ then from (1) we have

$$f_{\text{GBessel2}}(x; a, d, p) = \frac{|x|^p e^{-\frac{d}{a}|x|} K_p\left(\frac{|x|}{a}\right)}{C_2(p, d) a^{p2}} \quad (42)$$

where $p \neq 0$ a positive integer or $\forall p \in \mathbb{Z}^+$ and $C_2(p, d)$ (see Proгри 2016 [1], (67)-(70)) and from a particular value of a gamma function we have

$$C_2(p, d) = \frac{\sqrt{\pi} 2^{p2} \Gamma(2p_1) F\left(2p_1, p_1; p_3; \frac{d-1}{d+1}\right)}{(d+1)^{2p1} \Gamma(p_3)} \quad (43)$$

When $\forall p \in \mathbb{Z}^+$ If we take the limit (see Gradshteyn, Ryzhik, 2007, [24], pg. 919 ex. 8.446)

$$F_{\text{GBessel2}}(x; a, 0, p) = \int_{-\infty}^x f_{\text{GBessel2}}(t; a, 0, p) dt \quad (47)$$

for $a > 0$, $d = 0$, and $p > -1/2$, $p \neq 0$, and p not an integer, which can also be written as

$$F_{\text{GBessel2}}(x; a, 0, p) = \frac{\int_{-\infty}^x |t|^p K_p(|t|) dt}{C_2(p, 0)} \quad (48)$$

where $C_2(p, 0)$ is given by (3).

Let us define

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = \frac{\int_0^x t^p K_p(t) dt}{C_2(p, 0)} \quad (49)$$

then

$$F_{\text{GBessel2}}(x; a, 0, p) = \frac{\begin{cases} 1 - 2\mathcal{F}_{\text{GBessel2}}(-x; a, 0, p) & x < 0 \\ 1 + 2\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) & x \geq 0 \end{cases}}{2} \quad (50)$$

Hence, if we compute the series expansion of (49); then, we have computed the series expansion of (50) via (49).

Substituting (7) into (48) yields

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = \frac{\pi \int_0^x t^p [I_{-p}(t) - I_p(t)] dt}{2 C_2(p, 0) \sin \pi p} \quad (51)$$

Or equivalently,

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = \frac{\pi}{2} \frac{\int_0^x t^p I_{-p}(t) dt - \int_0^x t^p I_p(t) dt}{c_2(p, 0) \sin \pi p} \quad (52)$$

Which is equivalent with

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = \frac{\int_0^x t^p I_{-p}(t) dt - \int_0^x t^p I_p(t) dt}{\tilde{c}_2(p, 0)} \quad (53)$$

Where

$$\tilde{c}_2(p, 0) = \frac{2c_2(p, 0) \sin \pi p}{\pi} = 2^{p_2} \pi^{-\frac{1}{2}} \Gamma(p_1) \sin \pi p \quad (54)$$

The second integral can be computed from using the series expansion of $I_p(t)$ (see Gradshteyn, Ryzhik, 2007 [24] pg. 919 ex. 8.445) we have

$$\int_0^x t^p I_p(t) dt = \int_0^x t^p \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(p_2 + k)} \frac{t^{p+2k}}{2^{p+2k}} dt \quad (55)$$

Or by changing the order of summation and integration produces

$$\int_0^x t^p I_p(t) dt = \sum_{k=0}^{\infty} \frac{\int_0^x t^{2p+2k} dt}{k! \Gamma(p_2 + k) 2^{p+2k}} \quad (56)$$

Which is identical to

$$\int_0^x t^p I_{-p}(t) dt = \sum_{k=0}^{\infty} \frac{\int_0^x t^{2k} dt}{k! \Gamma(-p+k+1) 2^{-p+2k}} \equiv \frac{x^{2p}}{\Gamma(1-p)} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k (1-p)_k} \frac{(x_1)^k}{k!} = \frac{x^{2p}}{\Gamma(1-p)} {}_1F_2(1/2; 3/2, 1-p; x_1) \quad (61)$$

Substituting (59) and (61) into (53) yields

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = \frac{\frac{x^{2p}}{\Gamma(1-p)} {}_1F_2(1/2; 3/2, 1-p; x_1) - \frac{x^{2p_1}}{p_1 2^{p_2} \Gamma(p_2)} {}_1F_2(p_1; p_3, p_2; x_1)}{\tilde{c}_2(p, 0)} \text{iv}; \quad x_1 = \frac{x^2}{4} \quad (62)$$

The question is as follows: “Can we obtain exactly the same answer from Progrid (2016 [1], (142)) by setting $d = 0$?”

Let us try it if we can. By setting $d = 0$ from Progrid (2016 [1], (142)) we have

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = x \left[\frac{{}_2F_{2;1;0} \left[\begin{matrix} -1, 1/2; 1 \\ 3/2, 1; 1-p; - \end{matrix}; x_1, 0 \right]}{c_{21}^{-1}(p, 0)} + \frac{x {}_2F_{2;1;0} \left[\begin{matrix} -1, 1/2; 1 \\ 2, 3/2; 1-p; - \end{matrix}; x_1, 0 \right]}{c_{22}^{-1}(p, 0)} + \frac{x^{2p} {}_2F_{2;2;0} \left[\begin{matrix} -; p_1; 1 \\ p_3, p_2; -; - \end{matrix}; x_1, 0 \right]}{c_{23}^{-1}(p, 0) \equiv -p_1 2^{p_2} \Gamma(p_2) \tilde{c}_2(p, 0)} + \frac{x^{2p_1} {}_2F_{2;2;0} \left[\begin{matrix} -p_1; 1 \\ p_4, p_3; -; - \end{matrix}; x_1, 0 \right]}{c_{24}^{-1}(p, 0)} \right] \quad (63)$$

Where

$$x_1 = \frac{x^2}{4}; \quad x_2 = \frac{d^2 x^2}{4} = 0 \quad (64)$$

where $C_{2i}(p, 0)$ for $i = \{1, 2, 3, 4\}$ are given by

$$C_{21}(p, 0) = \frac{2^p}{\Gamma(1-p) \tilde{c}_2(p, 0)}; \quad C_{22}(p, 0) = \frac{2^{p-1} 0}{\Gamma(1-p) \tilde{c}_2(p, 0)} = 0; \quad C_{23}(p, 0) = \frac{-C_{11}(p, 0) C_1}{\tilde{c}_2(p, 0)}; \quad C_{24}(p, 0) = \frac{-C_{12}(p, 0) C_1}{\tilde{c}_2(p, 0)} = 0; \quad (65)$$

Where $C_{11}(p, 0)$ and $C_{12}(p, 0)$ are given by

$$C_{11}(p, 0) = \frac{\Gamma(p_2)}{\Gamma(2p_2)}; \quad C_{12}(p, 0) = \frac{C_{11}(p, d) 0}{2p_2} = 0 \quad (66)$$

Substituting (65) and (66) into (63) produces

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = \frac{x {}_2F_{2;1;0} \left[\begin{matrix} -1, 1/2; 1 \\ 3/2, 1; 1-p; - \end{matrix}; x_1, 0 \right]}{c_{21}^{-1}(p, 0)} + \frac{x^{2p+1} {}_2F_{2;2;0} \left[\begin{matrix} -; p_1; 1 \\ p_3, p_2; -; - \end{matrix}; x_1, 0 \right]}{c_{23}^{-1}(p, 0) \equiv -p_1 2^{p_2} \Gamma(p_2) \tilde{c}_2(p, 0)} \quad (67)$$

By comparing and contrasting (67) with (62) we arrive at the conclusion that (67) is identical to (62) if and only if

$${}_2F_{2;1;0} \left[\begin{matrix} -; 1, 1/2; 1 \\ 3/2, 1; 1-p; - \end{matrix}; x_1, 0 \right] \equiv {}_1F_2[1/2; 3/2, 1-p; x_1] \quad (68)$$

And

$$F_{2:2;0}^{0:1;1} \left[\begin{matrix} -; p_1; 1; \\ p_3, p_2; -; -; x_1, 0 \end{matrix} \right] \equiv {}_1F_2(p_1; p_3, p_2; x_1) \quad (69)$$

From the definition of the Kampé de Fériet function [11]-[13] we have

$$F_{2:2;0}^{0:2;1} \left[\begin{matrix} -; 1, 1/2; 1; \\ 3/2, 1; 1-p; -; x_1, 0 \end{matrix} \right] = \sum_{k,m=0}^{\infty} \frac{(1)_k (\frac{1}{2})_k}{(\frac{3}{2})_{k+m} (1)_{k+m} (1-p)_k} \cdot \frac{x_1^k}{k!} \equiv \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{(\frac{3}{2})_k (1-p)_k} \cdot \frac{x_1^k}{k!} \equiv {}_1F_2 \left(\frac{1}{2}; \frac{3}{2}, 1-p; x_1 \right) \quad (70)$$

And

$$F_{2:2;0}^{0:1;1} \left[\begin{matrix} -; p_1; 1; \\ p_3, p_2; -; -; x_1, 0 \end{matrix} \right] = \sum_{k,m=0}^{\infty} \frac{(p_1)_k (1)_m}{(p_3)_{k+m} (p_2)_{k+m}} \cdot \frac{x_1^k}{k!} \equiv \sum_{k=0}^{\infty} \frac{(p_1)_k}{(p_3)_k (p_2)_k} \cdot \frac{x_1^k}{k!} \equiv {}_1F_2(p_1; p_3, p_2; x_1) \quad (71)$$

This concludes the derivations of the special case for $d = 0$ and $p \neq 0$. Next we consider the special case when both $d = 0$ and $p = 0$.

3.2 Special Case: $d = 0, p = 0$

From the definition of the pdf of the Generalized Bessel distribution of the second kind, we define its corresponding cdf as

$$F_{\text{GBessel2}}(x; a, 0, 0) = \int_{-\infty}^x f_{\text{GBessel2}}(t; a, 0, 0) dt \quad (72)$$

for $a > 0$, $d = 0$, and $p = 0$, which can also be written as

$$F_{\text{GBessel2}}(x; a, 0, 0) = \frac{\int_{-\infty}^x K_0(|t|) dt}{C_2(0,0)} \quad (73)$$

where $C_2(0,0)$ is given by (18).

From (Gradshteyn, Ryzhik, 2007 [24], pg. 660, 6.5.11 12) we have

$$\int_0^{\infty} K_0(t) dt = \frac{\pi}{2} \quad (74)$$

Since,

$$K_0(|t|) = \begin{cases} K_0(t) & t > 0 \\ \infty & t = 0 \\ K_0(-t) & t < 0 \end{cases} \quad (75)$$

We have

$$\int_{-\infty}^{\infty} K_0(|t|) dt = \int_{-\infty}^0 K_0(-t) dt + \int_0^{\infty} K_0(t) dt \quad (76)$$

If we change the variables for the first integral we have

$$\int_{-\infty}^0 K_0(-t) dt = -\int_0^{\infty} K_0(t) d(-t) = \int_0^{\infty} K_0(t) dt = \frac{\pi}{2} \quad (77)$$

Substituting (74) and (77) into (76) yields

$$\int_{-\infty}^{\infty} K_0(|t|) dt = \frac{\pi}{2} + \frac{\pi}{2} = \pi \quad (78)$$

Substituting (18) and (78) into (73) and setting $x = \infty$ yields

$$F_{\text{GBessel2}}(\infty; a, 0, 0) = \frac{\int_{-\infty}^{\infty} K_0(|t|) dt}{C_2(0,0)} = \frac{\pi}{\pi} = 1 \quad (79)$$

Let us define

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \frac{\int_0^x K_0(t) dt}{C_2(0,0)} \quad (80)$$

then

$$F_{\text{GBessel2}}(x; a, 0, 0) = \begin{cases} 1 - 2\mathcal{F}_{\text{GBessel2}}(-x; a, 0, 0) & x < 0 \\ 1 + 2\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) & x \geq 0 \end{cases} \quad (81)$$

Hence, if we compute the series expansion of (80); then, we have computed the series expansion of (81) via (80).

Substituting (18) and (21) into (80) we obtain the following

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \frac{\int_0^x \sum_{k=0}^{\infty} \frac{t^{2k} [\psi(k+1) - \ln(\frac{t}{2})] dt}{2^{2k} (k!)^2}}{\pi} \quad (82)$$

which is equivalent with by changing the order of summation and integration

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \frac{\sum_{k=0}^{\infty} \frac{\int_0^x t^{2k} [\psi(k+1) - \ln(\frac{t}{2})] dt}{2^{2k} (k!)^2}}{\pi} \quad (83)$$

which is the same as

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \frac{\sum_{k=0}^{\infty} \frac{\psi(k+1) \int_0^x t^{2k} dt}{2^{2k} (k!)^2} - \sum_{k=0}^{\infty} \frac{\int_0^x t^{2k} \ln(\frac{t}{2}) dt}{2^{2k} (k!)^2}}{\pi} \quad (84)$$

The first integral is pretty straightforward to compute

$$\int_0^x t^{2k} dt = \frac{x^{2k+1}}{2k+1} \quad (85)$$

Substituting (85) into the first summation of (84) yields

$$\sum_{k=0}^{\infty} \frac{\psi(k+1) \int_0^x t^{2k} dt}{2^{2k} (k!)^2} = \sum_{k=0}^{\infty} \frac{\psi(k+1) x^{2k+1}}{2^{2k} (k!)^2} \quad (86)$$

The second integral is more difficult to compute directly, so we integrate by parts

$$\int_0^x \ln\left(\frac{t}{2}\right) t^{2k} dt = \ln\left(\frac{t}{2}\right) \frac{t^{2k+1}}{2k+1} \Big|_0^x - \int_0^x \frac{t^{2k}}{2k+1} dt \quad (87)$$

The first part of the integral can be computed from

$$\ln\left(\frac{t}{2}\right) \frac{t^{2k+1}}{2k+1} \Big|_0^x = \ln\left(\frac{x}{2}\right) \frac{x^{2k+1}}{2k+1} - \lim_{x \rightarrow 0} \ln\left(\frac{x}{2}\right) \frac{x^{2k+1}}{2k+1} \quad (88)$$

The lower limit of $\ln\left(\frac{t}{2}\right)\frac{t^{2k+1}}{2k+1}\Big|_0^x$ $k = 0, 1, \dots$ can be computed using L'Hôpital's rule [28]

$$\lim_{x \rightarrow 0} \ln\left(\frac{x}{2}\right)\frac{x^{2k+1}}{2k+1} = \lim_{x \rightarrow 0} \frac{1}{(-2k-1)} \frac{x^{2k+1}}{(2k+1)} = 0 \quad (89)$$

The second integral is equal to

$$\int_0^x \frac{t^{2k}}{2k+1} dt = \frac{x^{2k+1}}{(2k+1)^2} \quad (90)$$

Substituting (89) into (88) and then (88) and (90) into (87) yields

$$\int_0^x \ln\left(\frac{t}{2}\right) t^{2k} dt = \ln\left(\frac{x}{2}\right) \frac{x^{2k+1}}{2k+1} - \frac{x^{2k+1}}{(2k+1)^2} \quad (91)$$

Finally, substituting (86) and (91) into (84) we obtain the following

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \frac{\sum_{k=0}^{\infty} \frac{\psi(k+1) \frac{x^{2k+1}}{2^{2k}(k!)^2} - \sum_{k=0}^{\infty} \frac{\left[\ln\left(\frac{x}{2}\right) \frac{x^{2k+1}}{2k+1} - \frac{x^{2k+1}}{(2k+1)^2} \right]}{2^{2k}(k!)^2}}{\pi} = \frac{\sum_{k=0}^{\infty} \frac{\psi(k+1) \frac{x^{2k+1}}{2^{2k}(k!)^2} - \ln\left(\frac{x}{2}\right) \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^{2k}(k!)^2} + \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^{2k}(k!)^2 (2k+1)^2}}{\pi} \quad (92)$$

which is identical to

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \mathcal{F}_{21}(x; a, 0, 0) + \mathcal{F}_{22}(x; a, 0, 0) + \mathcal{F}_{23}(x; a, 0, 0) \quad (93)$$

Let us exploit these summations one at a time. We start with the second summation can be expressed in terms of the

generalized hypergeometric function [31]

$$\mathcal{F}_{22}(x; a, d, 0) = \frac{-\ln\left(\frac{x}{2}\right) \left[\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^{2k}(k!)^2} \equiv x \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2} \equiv x \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k} \left(\frac{1}{2}\right)_k k!} \equiv x \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(x_1 = \frac{x^2}{2}\right)^k}{\left(\frac{3}{2}\right)_k (1)_k k!} \equiv x {}_1F_2\left(\frac{1}{2}, \frac{3}{2}, 1; x_1\right) \right]}{\pi} \quad (94)$$

The third summation can be expressed in terms of the

generalized hypergeometric function [31] as

$$\mathcal{F}_{23}(x; a, 0, 0) = \frac{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^{2k}(k!)^2}}{\pi} = \frac{x \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k \frac{x^{2k}}{2^{2k}}}{\left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_k (1)_k k!}}{\pi} = \frac{x {}_2F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 1; x_1\right)}{\pi} \quad (95)$$

Now, we go back to compute the first summation

$$\mathcal{F}_{21}(x; a, 0, 0) = \frac{\sum_{k=0}^{\infty} \frac{\psi(k+1) \frac{x^{2k+1}}{2^{2k}(k!)^2}}{\pi} = \frac{x \sum_{k=0}^{\infty} \frac{\psi(k+1) \left(\frac{1}{2}\right)_k x^k}{\left(\frac{3}{2}\right)_k (1)_k k!}}{\pi} \quad (96)$$

In appendix A we consider an inefficient implementation of $\mathcal{F}_{21}(x; a, 0, 0)$.

The digamma function is computed recursively $\psi(k+1)$

$$\psi(k+1) = \frac{1}{k} + \psi(k) \quad (97)$$

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \begin{cases} x \left\{ \sum_{k=0}^{\infty} \frac{\psi(k+1) \left(\frac{1}{2}\right)_k x^k}{\left(\frac{3}{2}\right)_k (1)_k k!} \equiv \psi_1 F_2\left(\frac{1}{2}, \frac{3}{2}, 1; x_1\right) \right\} - \ln\left(\frac{x}{2}\right) {}_1F_2\left(\frac{1}{2}, \frac{3}{2}, 1; x_1\right) + {}_2F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 1; x_1\right) \end{cases} & x > 0^{\text{v,vi}} \\ 0 & x = 0 \end{cases} \quad (98)$$

Equation (98) is an original expression not published anywhere else; however, care must be shown in the computation of the $\mathcal{F}_{21}(x; a, 0, 0)$.

This concludes the derivations of the special case for $d = 0$ and $p = 0$.

Next we consider the special case when for $d = 0$ and p a positive integer.

3.3 Special Case: $d = 0$, $p \neq 0$ a Positive Integer

There are two approaches that are equivalent in both performance and speed to solve this special case: one is the recursive approach discussed here, the other is the direct approach discussed in Appendix B.

In order to compute the $\mathcal{F}_{\text{GBessel2}}(x; a, 0, p)$ recursively, we

will have to employ some properties of the modified Bessel function of the second kind (see Gradshteyn, Ryzhik, 2007, [24], pg. 929 ex. 8.486 10. and 11.)

$$K_{p+1}(t) = K_{p-1}(t) + \frac{2p}{t} K_p(t); \quad p = 1, 2, \dots, \infty \quad (99)$$

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p_2) = \frac{\int_0^x t^{p_2} K_{p_2}(t) dt}{C_2(p_2, 0)} = \frac{\int_0^x t^{p_2} K_{p-1}(t) dt + 2p \int_0^x t^{p-1} K_p(t) dt}{C_2(p_2, 0)} = \frac{\int_0^x t^{p_2} K_{p-1}(t) dt}{C_2(p_2, 0)} + \frac{p}{p_1} \mathcal{F}_{\text{GBessel2}}(x; a, d, p) \quad (101)$$

Next, let us evaluate the integral by parts and substituting (101) into the integration

$$\int_0^x t^{p_2} K_{p-1}(t) dt = -\int_0^x t^{p_2} K_{p_2}(t) dt - 2 \int_0^x t^{p_2} \frac{dK_p(t)}{dt} dt = -\frac{\mathcal{F}_{\text{GBessel2}}(x; a, d, p_2)}{C_2^{-1}(p_2, 0)} - 2t^{p_2} K_p(t) \Big|_0^x + 2p_2 \int_0^x t^{p-1} K_p(t) dt \quad (102)$$

Next, substituting (102) into (101) we obtain the recursive algorithm to compute fast, accurate, and very efficiently, the

$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p)$ $d = 0$, $p \neq 0$ a positive Integer

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p_2) = \mathcal{F}_{\text{GBessel2}}(x; a, d, p) - \frac{x^{p_2} K_p(x)}{C_2(p_2, 0)} \rightarrow \mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \sum_{i=0}^{p-1} \left[\mathcal{F}_{\text{GBessel2}}(x; a, d, i) - \frac{x^{i+1} K_i(x)}{C_2(i+1, 0)} \right] \quad (103)$$

To know whether (102) is correct, we test it for $p = 0$

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \begin{cases} x \left[\frac{\sum_{k=0}^{\infty} \frac{\psi(k+1) \left(\frac{1}{2}\right)_k x_1^k}{\left(\frac{3}{2}\right)_k (1)_k k!} = {}_{\psi,1}F_2 \left((1), \frac{1}{2}, \frac{3}{2}; 1; x_1 \right) \right] - \ln\left(\frac{x}{2}\right) {}_1F_2 \left(\frac{1}{2}, \frac{3}{2}; 1; x_1 \right) + {}_2F_3 \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{3}{2}, \frac{3}{2}, 1; x_1 \right) \Big] & x > 0 \\ \pi & x = 0 \end{cases} \quad (104)$$

Equation (104) is identical to (98) thus completing the proof of this test point.

This concludes the derivations of the special case for $d = 0$ and p is a positive integer.

Next we consider the special case when for $d \neq 0$ and $p = 0$.

3.4 Special Case: $d \neq 0$, $p = 0$

When $d \neq 0$ and $p = 0$, $\mathcal{F}_{\text{GBessel2}}(x; a, d, 0)$ is given by (41).

Following from Progi 2016 [1], (50)-(53), substituting (21) into (121) of Progi 2016 [1]

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, 0) = \frac{\int_0^x e^{-td} \sum_{k=0}^{\infty} \frac{t^{2k} [\psi(k+1) - \ln(\frac{t}{2})]}{2^{2k} (k!)^2} dt}{C_2(0, d)} \quad (105)$$

which is equivalent with by changing the order of summation and integration

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, 0) = \frac{\sum_{k=0}^{\infty} \frac{\int_0^x t^{2k} e^{-td} [\psi(k+1) - \ln(\frac{t}{2})] dt}{2^{2k} (k!)^2}}{C_2(0, d)} \quad (106)$$

Substituting (110) into the first summation of (107) yields,

$$\mathcal{F}_{21}(x; a, d, 0) = \frac{\sum_{k=0}^{\infty} \frac{\psi(k+1) x^{2k+1} e^{-dx} \left[\sum_{m=0}^{\infty} \frac{x_2^m}{\left(k+\frac{3}{2}\right)_m (k+1)_m} + \frac{dx}{2(k+1)} \sum_{m=0}^{\infty} \frac{x_2^m}{(k+2)_m \left(k+\frac{3}{2}\right)_m} \right]}{2^{2k} (k!)^2}}{C_2(0, d)} \quad (111)$$

Which is equivalent with

$$K_{p-1}(t) = -K_{p+1}(t) - 2 \frac{d}{dt} K_p(t); \quad p = 1, 2, \dots, \infty \quad (100)$$

Subsuming (99) into Progi 2016, (121) we obtain

which is identical to

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, 0) = \frac{\sum_{k=0}^{\infty} \frac{\psi(k+1) \int_0^x t^{2k} e^{-td} dt}{2^{2k} (k!)^2} - \sum_{k=0}^{\infty} \frac{\int_0^x \frac{t^{2k} \ln(\frac{t}{2})}{e^{td}} dt}{2^{2k} (k!)^2}}{C_2(0, d)} \quad (107)$$

The first integral is pretty straightforward to compute (see Progi 2016 [1], (132))

$$\int_0^x t^{2k} e^{-td} dt = \frac{\gamma(2k+1, dx)}{d^{2k+1}} \quad (108)$$

Substituting (108) into the first summation of (107) yields

$$\sum_{k=0}^{\infty} \frac{\psi(k+1) \int_0^x t^{2k} e^{-td} dt}{2^{2k} (k!)^2} = \sum_{k=0}^{\infty} \frac{\psi(k+1) \gamma(2k+1, dx)}{2^{2k} (k!)^2} \quad \text{vii} \quad (109)$$

Before computing the first summation of (109), first, from Progi, 2018 [2], (19) for $p = 0$ we have

$$\gamma(2k+1, dx) = \frac{\left[\sum_{m=0}^{\infty} \frac{x_2^m}{\left(k+\frac{3}{2}\right)_m (k+1)_m} + \frac{dx \sum_{m=0}^{\infty} \frac{x_2^m}{(k+2)_m \left(k+\frac{3}{2}\right)_m}}{2(k+1)} \right]}{(dx)^{-2k-1} e^{dx} (2k+1)} \quad (110)$$

$$\mathcal{F}_{21}(x; a, d, 0) = \frac{x e^{-dx} \left[\sum_{k,m=0}^{\infty} \frac{\psi(k+1) \left(\frac{1}{2}\right)_k (1)_m x_1^k x_2^m}{\left(\frac{3}{2}\right)_{k+m} (1)_{k+m}} + \frac{dx}{2} \sum_{k,m=0}^{\infty} \frac{\psi(k+1) \left(\frac{1}{2}\right)_k (1)_m x_1^k x_2^m}{(2)_{k+m} \left(\frac{3}{2}\right)_{k+m}} \right]}{C_2(0, d)} \quad (112)$$

Equation (112) can be written with the help of the modified generalized Kampé de Fériet function [11]-[13] as follows

$$\sum_{k=0}^{\infty} \frac{\psi(k+1) \frac{\gamma(2k+1, dx)}{d^{2k+1}}}{2^{2k} (k!)^2} = x e^{-dx} \left[F_{2;0;0}^{0;\psi,1;1} \left[-; (1), 1/2; 1; x_1, x_2 \right] + \frac{dx}{2} F_{2;0;0}^{0;\psi,1;1} \left[-; (1), 1/2; 1; 2, 3/2; -; -; x_1, x_2 \right] \right] \quad (113)$$

Substituting (113) into (109) produces the final expression of $\mathcal{F}_{21}(x; a, d, 0)$

$$\mathcal{F}_{21}(x; a, d, 0) = \frac{x e^{-dx} \left[F_{2;0;0}^{0;\psi,1;1} \left[-; (1), 1/2; 1; x_1, x_2 \right] + \frac{dx}{2} F_{2;0;0}^{0;\psi,1;1} \left[-; (1), 1/2; 1; 2, 3/2; -; -; x_1, x_2 \right] \right]}{C_2(0, d)} \quad (114)$$

Equation (114) eliminates a singularity for $d = 0$, which the recursive approach was not able to; hence, (114) is much better implementation than the recursive algorithm described in the endnote viii.

The second summation function can be computed from the second integral.

The second integral is more difficult to compute directly, so we integrate by parts

$$\int_0^x \ln\left(\frac{t}{2}\right) t^{2k} e^{-td} dt = \left[\ln\left(\frac{t}{2}\right) \frac{\gamma(2k+1, td)}{d^{2k+1}} \right]_0^x = \ln\left(\frac{x}{2}\right) \frac{\gamma(2k+1, dx)}{d^{2k+1}} - \lim_{x \rightarrow 0} \ln\left(\frac{x}{2}\right) \frac{\gamma(2k+1, dx)}{d^{2k+1}} - \int_0^x \frac{t^{-1} \gamma(2k+1, td)}{d^{2k+1}} dt \quad (115)$$

The lower limit of $\ln(x/2) \gamma(2k+1, dx)/d^{2k+1}$ $k = 0, 1, \dots$ can be computed using L'Hôpital's rule (see also (89))

$$\lim_{x \rightarrow 0} \ln\left(\frac{x}{2}\right) \frac{\gamma(2k+1, dx)}{d^{2k+1}} = \lim_{x \rightarrow 0} \ln\left(\frac{x}{2}\right) \frac{d^{2k+1} x^{2k+1} e^{-dx} \sum_{n=0}^{\infty} \frac{(1)_n (-dx)^n}{(2+2k)_n n!}}{(2k+1)} = \frac{d^{2k+1} \lim_{x \rightarrow 0} \ln\left(\frac{x}{2}\right) x^{2k+1}}{(2k+1)} = 0 \quad (116)$$

Substituting (115), (116) into the second summation of (107) yields

$$\mathcal{F}_{22}(x; a, d, 0) + \mathcal{F}_{23}(x; a, d, 0) = \frac{\sum_{k=0}^{\infty} \frac{\int_0^x \frac{t^{2k} \ln\left(\frac{t}{2}\right) dt}{e^{td}}}{2^{2k} (k!)^2}}{C_2(0, d)} = \frac{\ln\left(\frac{x}{2}\right) \sum_{k=0}^{\infty} \frac{\gamma(2k+1, dx)}{2^{2k} (k!)^2} - \sum_{k=0}^{\infty} \frac{\int_0^x \frac{t^{-1} \gamma(2k+1, td) dt}{d^{2k+1}}}{2^{2k} (k!)^2}}{C_2(0, d)} \quad (117)$$

We start with the second summation. First, substituting (110) into the first summation yields

$$\sum_{k=0}^{\infty} \frac{\gamma(2k+1, dx)}{d^{2k+1} 2^{2k} (k!)^2} = x e^{-dx} \left[\sum_{k,m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k (1)_m}{\left(\frac{3}{2}\right)_{k+m} (1)_{k+m}} \frac{x_1^k x_2^m}{k! m!} + \frac{dx}{4} \sum_{k,m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k (1)_m}{(2)_{k+m} \left(\frac{3}{2}\right)_{k+m}} \frac{x_1^k x_2^m}{k! m!} \right] \quad (118)$$

Equation (118) can be written with the help of the Kampé de Fériet function [11]-[13] as follows

$$\sum_{k=0}^{\infty} \frac{\gamma(2k+1, dx)}{d^{2k+1} 2^{2k} (k!)^2} = x e^{-dx} \left[F_{2;0;0}^{0;1;1} \left[-; 1/2; 1; x_1, x_2 \right] + \frac{dx}{2} F_{2;0;0}^{0;1;1} \left[-; 1/2; 1; 2, 3/2; -; -; x_1, x_2 \right] \right] \quad (119)$$

Substituting (119) into the second summation yields,

$$\mathcal{F}_{22}(x; a, d, 0) = \frac{-\ln\left(\frac{x}{2}\right) \sum_{k=0}^{\infty} \frac{\gamma(2k+1, dx)}{d^{2k+1} 2^{2k} (k!)^2}}{C_2(0, d)} = \frac{-\ln\left(\frac{x}{2}\right) x e^{-dx} \left[F_{2;0;0}^{0;1;1} \left[-; 1/2; 1; x_1, x_2 \right] + \frac{dx}{2} F_{2;0;0}^{0;1;1} \left[-; 1/2; 1; 2, 3/2; -; -; x_1, x_2 \right] \right]}{C_2(0, d)} \quad (120)$$

For the second summation of (119) we employ the series expansion of the incomplete gamma function and after performing the integration we obtain

$$\mathcal{F}_{23}(x; a, d, 0) = \frac{\sum_{k=0}^{\infty} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n d^n x^{2k+n+1}}{(2k+n+1)^2}}{2^{2k} (k!)^2}}{C_2(0, d)} = \frac{x \sum_{k,n=0}^{\infty} \frac{\frac{x^{2k}}{2^{2k}} \frac{(-dx)^n}{k! n!}}{(2k+n+1)^2 k!}}{C_2(0, d)} \quad (121)$$

Next, we perform the expansion into even and odd terms we have

$$\mathcal{F}_{23}(x; a, d, 0) = \frac{x \sum_{k,m=0}^{\infty} \frac{x_1^k (-dx)^{2m}}{(2k+2m+1)^2 k!} + x \sum_{k,m=0}^{\infty} \frac{x_1^k (-dx)^{2m+1}}{(2k+2m+1+1)^2 k!}}{C_2(0,d)} \quad (122)$$

Which is equivalent with

$$\mathcal{F}_{23}(x; a, d, 0) = \frac{x \sum_{k,m=0}^{\infty} \frac{x_1^k (-dx)^{2m}}{(2k+2m+1)^2 k!} + dx^2 \sum_{k,m=0}^{\infty} \frac{x_1^k (-dx)^{2m}}{(2k+2m+2)^2 k!}}{C_2(0,d)} \quad (123)$$

Which is identical to

$$\mathcal{F}_{23}(x; a, d, 0) = \frac{\frac{1}{4} x \sum_{k,m=0}^{\infty} \frac{x_1^k x_2^m}{(k+m+\frac{1}{2})^2 k!} - \frac{dx^2}{8} \sum_{k,m=0}^{\infty} \frac{x_1^k x_2^m}{(k+m+1)^2 k!}}{C_2(0,d)} \quad (124)$$

Equations (124) can be written as follows

$$\mathcal{F}_{23}(x; a, d, 0) = \frac{x \left[\sum_{k,m=0}^{\infty} \frac{(\frac{1}{2})_{k+m} (\frac{1}{2})_{k+m} x_1^k x_2^m}{(\frac{3}{2})_{k+m} (\frac{3}{2})_{k+m} (1)_k (\frac{1}{2})_m} - \frac{dx}{4} \sum_{k,m=0}^{\infty} \frac{(1)_{k+m} (1)_{k+m} x_1^k x_2^m}{(2)_{k+m} (2)_{k+m} (1)_k (\frac{3}{2})_m} \right]}{C_2(0,d)} \quad (125)$$

Finally, Dr. Proгри has successfully derived for the first time Kampé de Fériet function [11]-[13] as follows that this summation which can be written with the help of the

$$\mathcal{F}_{23}(x; a, d, 0) = \frac{x \left[F_{2:0;0}^{2:0;0} \left[\begin{matrix} 1/2, 1/2; -; -; \\ 3/2, 3/2; 1; 1/2; \end{matrix} ; x_1, x_2 \right] - \frac{dx}{4} F_{2:1;1}^{2:0;0} \left[\begin{matrix} 1, 1; -; -; \\ 2, 2; 1; 3/2; \end{matrix} ; x_1, x_2 \right] \right]}{C_2(0,d)} \quad (126)$$

Finally, substituting (114), (120), and (126) into (107) produces the desired closed form expression of

$$\mathcal{F}_{\text{GBessel2}}(x; a, d \neq 0, 0) = \begin{cases} \frac{x e^{-dx} \left[F_{2:0;0}^{0;\psi;1;1} \left[\begin{matrix} -(1), 1/2; 1; \\ 3/2, 1; -; -; \end{matrix} ; x_1, x_2 \right] + \frac{dx}{2} F_{2:0;0}^{0;\psi;1;1} \left[\begin{matrix} -(1), 1/2; 1; \\ 2, 3/2; -; -; \end{matrix} ; x_1, x_2 \right] \right.}{C_2(0,d)} & x > 0 \\ -\ln\left(\frac{x}{2}\right) x e^{-dx} \left[F_{2:0;1;1}^{0;1;1} \left[\begin{matrix} -1/2; 1; \\ 3/2, 1; -; -; \end{matrix} ; x_1, x_2 \right] + \frac{dx}{2} F_{2:0;1;1}^{0;1;1} \left[\begin{matrix} -1/2; 1; \\ 2, 3/2; -; -; \end{matrix} ; x_1, x_2 \right] \right. \\ \left. + x \left[F_{2:1;1}^{2:0;0} \left[\begin{matrix} 1/2, 1/2; -; -; \\ 3/2, 3/2; 1; 1/2; \end{matrix} ; x_1, x_2 \right] - \frac{dx}{4} F_{2:1;1}^{2:0;0} \left[\begin{matrix} 1, 1; -; -; \\ 2, 2; 1; 3/2; \end{matrix} ; x_1, x_2 \right] \right] \right] & x < 0 \end{cases} \quad (127)$$

Equation (127) is an original expression not published anywhere else; however, care must be shown in the computation of the Kampé de Fériet function [2].

equation? While numerically we will test this at the numerical, theoretical results section, we can still test analytically for at least one data point, $d = 0$.

The obvious question is as follows. Is (127) the right

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \frac{x F_{2:0;0}^{0;1;1} \left[\begin{matrix} -\psi(1), 1/2; 1; \\ 3/2, 1; -; -; \end{matrix} ; x_1, x_2 \right] - x \ln\left(\frac{x}{2}\right) F_{1:1;1}^{1;0;0} \left[\begin{matrix} 1/2; -; -; \\ 3/2, 1; 1/2; \end{matrix} ; x_1, x_2 \right] + x F_{2:1;1}^{2:0;0} \left[\begin{matrix} 1/2, 1/2; -; -; \\ 3/2, 3/2; 1; 1/2; \end{matrix} ; x_1, x_2 \right]}{C_2(0,0)} \Big|_{d=0} \quad (128)$$

Evaluating the two Kampé de Fériet function [11]-[13] at $d = 0$ yields

$$F_{2:0;0}^{0;1;1} \left[\begin{matrix} -\psi(1), 1/2; 1; \\ 3/2, 1; -; -; \end{matrix} ; x_1, x_2 \right] \Big|_{d=0} = \sum_{k,m=0}^{\infty} \frac{\psi(1+k) (\frac{1}{2})_k (1)_m x_1^k x_2^m}{(\frac{3}{2})_{k+m} (1)_{k+m} k! m!} = \sum_{k=0}^{\infty} \frac{\psi(1+k) (\frac{1}{2})_k x_1^k}{(\frac{3}{2})_k (1)_k} = {}_1F_2[(1) 1/2; 3/2, 1; x_1] \quad (129)$$

$$F_{2:0;0}^{0;1;1} \left[\begin{matrix} -1/2; 1; \\ 3/2, 1; -; -; \end{matrix} ; x_1, x_2 \right] \Big|_{d=0} = \sum_{k,m=0}^{\infty} \frac{(\frac{1}{2})_k (1)_m x_1^k x_2^m}{(\frac{3}{2})_{k+m} (1)_{k+m} k! m!} = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k x_1^k}{(\frac{3}{2})_k (1)_k} = {}_1F_2(1/2; 3/2, 1; x_1) \quad (130)$$

$$F_{2:0;1}^{2:0;0} \left[\begin{matrix} 1/2, 1/2 : -; -; \\ 3/2, 3/2 : 1; 1/2; \end{matrix} ; x_1, 0 \right] = \sum_{k,m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{k+m} \left(\frac{1}{2}\right)_{k+m} \frac{x_1^k}{k! m!}}{\left(\frac{3}{2}\right)_{k+m} \left(\frac{3}{2}\right)_{k+m} (1)_k \left(\frac{1}{2}\right)_m} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k \frac{x_1^k}{k!}}{\left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_k (1)_k} = {}_2F_3(1/2, 1/2; 3/2, 3/2, 1; x_1) \quad (131)$$

Substituting (129)-(131) into (128) yields

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = x \frac{\psi_1 F_2[(1)1/2; 3/2, 1; x_1] - \ln\left(\frac{x}{2}\right) {}_1F_2(1/2; 3/2, 1; x_1) + {}_2F_3(1/2, 1/2; 3/2, 3/2, 1; x_1)}{C_2(0, 0)\pi} \quad (132)$$

Equation (132) is identical to (98); thus, completing the proof for $d = 0$.

This concludes the derivations of the special case for $d \neq 0$ and $p = 0$.

Next we consider some remarks the most important special case when $d \neq 0$ and p is a positive integer.

3.5 Special Case: $d \neq 0$, p a Positive Integer

When $d \neq 0$ and p a positive, $f_{\text{GBessel2}}(x; a, d, p)$ is given by (46).

Following from Progi 2016 [1], (50)-(53), substituting (31) into (121) of Progi 2016 [1]

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\int_0^x t^p e^{-td} \left[\frac{1}{2} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \left(\frac{t}{2}\right)^{2k-p} + (-1)^p \sum_{k=0}^{\infty} \frac{\psi(k+1) + \psi(p+k+1)}{k!(p+k)!} \left(\frac{t}{2}\right)^{p+2k} \right] dt}{C_2(p, d)} \quad (133)$$

which is equivalent with by changing the order of summation and integration

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{2^{p-1} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \int_0^x e^{-td} t^{2k} dt + \frac{(-1)^p}{2^p} \sum_{k=0}^{\infty} \frac{[\psi(k+1) + \psi(p+k+1)] \int_0^x e^{-td} t^{2p+2k} dt}{k!(p+k)! 2^{2k}} + \frac{(-1)^{p/2}}{2^p} \sum_{k=0}^{\infty} \frac{\int_0^x \ln \frac{t}{2} e^{-td} t^{2p+2k} dt}{k!(p+k)! 2^{2k}}}{C_2(p, d)} \quad (134)$$

The first integral is pretty straightforward to compute (see 108).

first summation of (134) and performing a series of identities yields

We start with the first summation. Substituting, (110) into the

$$\mathcal{F}_{21}(x; a, d, p) = \frac{\sum_{k=0}^{p-1} \frac{(p-k-1)! \gamma(2k+1, dx)}{(-1)^{-k-2k} k!}}{2^{1-p} C_2(p, d)} = \begin{cases} \frac{\sum_{k,m=0}^{p-1, \infty} \frac{(-x_1)^k x_2^m}{\left(\frac{3}{2}\right)_{k+m} (1)_{k+m}} + \frac{dx}{2} \sum_{k,m=0}^{p-1, \infty} \frac{(-x_1)^k x_2^m}{(2)_{k+m} \left(\frac{3}{2}\right)_{k+m}}}{x^{-1} \Gamma^{-1}(p) e^{dx} 2^{1-p} C_2(p, d)} & p \neq 0 \\ 0 & p = 0 \end{cases} \quad (135)$$

Equation (135) can be written with the help of the generalized Kampé de Fériet function [11]-[13] as follows

$$\mathcal{F}_{21}(x; a, d, p) = \begin{cases} \frac{\Gamma(p) 2^{p-1} x e^{-dx} \left[F_{2:0;0}^{0:2;1} \left[\begin{matrix} (p-1) \\ 0 \end{matrix} ; \begin{matrix} -; -; 1 \\ \frac{3}{2} 1; -; -; \end{matrix} ; -x_1, x_2 \right] + \frac{dx}{2} F_{2:0;0}^{0:2;1} \left[\begin{matrix} (p-1) \\ 0 \end{matrix} ; \begin{matrix} -; -; 1 \\ 2, \frac{3}{2}; -; -; \end{matrix} ; -x_1, x_2 \right] \right]}{C_2(p, d)} & p \neq 0 \\ 0 & p = 0 \end{cases} \quad (136)$$

Equation (136) eliminates the singularity at $d = 0$ which was present in the left side of (135).

The second integral is also pretty straightforward to compute

$$\int_0^x e^{-td} t^{2p+2k} dt = \frac{\gamma(2p_1+2k, dx)}{d^{2p_1+2k}} \quad (137)$$

Substituting (137) into the second summation of (134) yields

$$\sum_{k=0}^{\infty} \frac{[\psi(k+1) + \psi(p_2+k)] \int_0^x e^{-td} t^{2p+2k} dt}{k!(p+k)! 2^{2k}} = \sum_{k=0}^{\infty} \frac{[\psi(k+1) + \psi(p_2+k)] \gamma(2p_1+2k, dx)}{2^{2k} k!(p+k)!} \quad (138)$$

The recursive implementation algorithm is considered in Appendix D. Since, the direct approach is numerically more accurate and computationally just as efficient as the recursive algorithm, for this particular case the direct approach is

recommended.

Substituting Progi, 2018 [2], (19) into the second summation of (134) yields,

$$\mathcal{F}_{22}(x; a, d, p) = \frac{\frac{(-1)^p}{2^p} \sum_{k=0}^{\infty} \frac{[\psi(k+1) + \psi(p_2+k)]}{k!(p+k)! 2^{2k}} \left[\frac{(dx)^{2p_1+2k} e^{-dx}}{2^{p_1+2k}} \left[\sum_{m=0}^{\infty} \frac{x_2^m}{(p_3+k)_m (p_2+k)_m} + \frac{dx}{2(p_2+k)} \sum_{m=0}^{\infty} \frac{x_2^m}{(p_4+k)_m (p_3+k)_m} \right] \right]}{C_2(p, d)} \quad (139)$$

Which is equivalent with

$$\mathcal{F}_{22}(x; a, d, p) = \frac{\frac{(-1)^p x^{2p_1} e^{-dx}}{2^{p_2} 2^{p_1} \Gamma(p_2)} \left[\sum_{m=0}^{\infty} \frac{[\psi(k+1) + \psi(p_2+k)] (p_1)_k \frac{x_1^k}{k!} x_2^m}{(p_3)_{k+m} (p_2)_{k+m}} + \frac{dx}{2^{p_2}} \sum_{m=0}^{\infty} \frac{[\psi(k+1) + \psi(p_2+k) (p_1)_k] \frac{x_1^k}{k!} x_2^m}{(p_4)_{k+m} (p_3)_{k+m}} \right]}{C_2(p, d)} \quad (140)$$

Equation (140) can be written with the help of the modified, generalized Kampé de Fériet function [11]-[13] as follows

$$\sum_{k=0}^{\infty} \frac{[\psi(k+1) + \psi(p_2+k)] \frac{\gamma(2k+1, dx)}{d^{2k+1}}}{2^{2k} (k!)^2} = \frac{x^{2p_1} e^{-dx} \left[F_{2:0;0}^{0:\psi,1;1} \left[\begin{matrix} -(1)+(p_2), p_1; 1 \\ p_3, p_2; -; - \end{matrix}; x_1, x_2 \right] + \frac{dx}{2^{p_2}} F_{2:0;0}^{0:\psi,1;1} \left[\begin{matrix} -(1)+(p_2), p_1; 1 \\ p_4, p_3; -; - \end{matrix}; x_1, x_2 \right] \right]}{2^{p_1} \Gamma(p_2)} \quad (141)$$

Substituting (141) into (140) produces the final expression of $\mathcal{F}_{22}(x; a, d, p)$

$$\mathcal{F}_{22}(x; a, d, p) = \frac{\frac{(-1)^p x^{2p_1} e^{-dx}}{2^{p_4} 2^{p_1} \Gamma(p_2)} \left[F_{2:0;0}^{0:\psi,1;1} \left[\begin{matrix} -(1)+(p_2), p_1; 1 \\ p_3, p_2; -; - \end{matrix}; x_1, x_2 \right] + \frac{dx}{2^{p_2}} F_{2:0;0}^{0:\psi,1;1} \left[\begin{matrix} -(1)+(p_2), p_1; 1 \\ p_4, p_3; -; - \end{matrix}; x_1, x_2 \right] \right]}{C_2(p, d)} \quad (142)$$

Equation (142) eliminates a singularity for $d = 0$, which the recursive approach was not able to; hence, (142) is much better implementation than the recursive algorithm described in Appendix D.

The third summation function of (134) can be computed from the third integral.

The second integral is more difficult to compute directly, so we integrate by parts

$$\int_0^x \ln \frac{t}{2} e^{-td} t^{2p+2k} dt = \left[\ln \left(\frac{t}{2} \right) \frac{\gamma(\alpha, td)}{d^\alpha} \right]_0^x = \ln \left(\frac{x}{2} \right) \frac{\gamma(\alpha, dx)}{d^\alpha} - \lim_{x \rightarrow 0} \ln \left(\frac{x}{2} \right) \frac{\gamma(\alpha, dx)}{d^\alpha} - \int_0^x \frac{t^{-1} \gamma(\alpha, td)}{d^\alpha} dt \quad (143)$$

The lower limit of $\ln(x/2) \gamma(\alpha, dx)/d^\alpha$ $k = 0, 1, \dots$ can be computed using L'Hôpital's rule (see also (89)) [28]

$$\lim_{x \rightarrow 0} \ln \left(\frac{x}{2} \right) \frac{\gamma(\alpha, dx)}{d^\alpha} = \lim_{x \rightarrow 0} \ln \left(\frac{x}{2} \right) \frac{d^\alpha x^\alpha e^{-dx} \sum_{n=0}^{\infty} \frac{(1)_n (dx)^n}{(\alpha+1)_n n!}}{\alpha} = \frac{d^\alpha \lim_{x \rightarrow 0} \ln \left(\frac{x}{2} \right) x^\alpha}{\alpha} = 0 \quad (144)$$

Substituting (143), (144) into the second summation of (134) yields

$$\mathcal{F}_{23}(x; a, d, p) + \mathcal{F}_{24}(x; a, d, p) = \frac{\sum_{k=0}^{\infty} \frac{\int_0^x \ln \frac{t}{2} e^{-td} t^{2p+2k} dt}{2^{2k} k! (p+k)!}}{C_2(p, d)} = \frac{\ln \left(\frac{x}{2} \right) \sum_{k=0}^{\infty} \frac{\frac{\gamma(\alpha, dx)}{d^\alpha}}{2^{2k} k! (p+k)!} - \sum_{k=0}^{\infty} \frac{\int_0^x \frac{t^{-1} \gamma(\alpha, td)}{d^\alpha} dt}{2^{2k} k! (p+k)!}}{C_2(p, d)} \quad (145)$$

Next, we work on the first summation of (145). Substituting Progr 2018 [2], (19) into the second summation yields

$$\sum_{k=0}^{\infty} \frac{\frac{\gamma(2p_1+2k, dx)}{d^{2p_1+2k}}}{2^{2k} k! (p+k)!} = \frac{e^{-dx}}{2^{p_1} \Gamma(p_2)} \left[\sum_{m=0}^{\infty} \frac{(p_1)_k \frac{x_1^k}{k!} x_2^m}{(p_3)_{k+m} (p_2)_{k+m}} + \frac{dx}{2^{p_2}} \sum_{m=0}^{\infty} \frac{\frac{x_1^k}{k!} x_2^m}{(p_4)_{k+m} (p_3)_{k+m}} \right] \quad (146)$$

Equation (146) can be written with the help of the Kampé de Fériet function [11]-[13] as follows

$$\sum_{k=0}^{\infty} \frac{\frac{\gamma(2p_1+2k, dx)}{d^{2p_1+2k}}}{2^{2k} k! (p+k)!} = \frac{x^{2p_1} e^{-dx}}{2^{p_1} \Gamma(p_2)} \left[F_{2:0;0}^{0:1;1} \left[\begin{matrix} -; p_1; 1 \\ p_3, p_2; -; - \end{matrix}; x_1, x_2 \right] + \frac{dx}{2^{p_2}} F_{2:0;0}^{0:1;1} \left[\begin{matrix} -; p_1; 1 \\ p_4, p_3; -; - \end{matrix}; x_1, x_2 \right] \right] \quad (147)$$

Substituting (147) into the first summation of (145) yields,

$$\mathcal{F}_{23}(x; a, d, p) = \frac{\frac{(-1)^{p_2} \ln \left(\frac{x}{2} \right) \sum_{k=0}^{\infty} \frac{\frac{\gamma(2p_1+2k, dx)}{d^{2p_1+2k}}}{2^{2k} k! (p+k)!}}{2^p}}{C_2(p, d)} = \frac{\ln \left(\frac{x}{2} \right) \frac{(-1)^{p_2} x^{2p_1} e^{-dx}}{2^{p_2} 2^{p_1} \Gamma(p_2)} \left[F_{2:0;0}^{0:1;1} \left[\begin{matrix} -; p_1; 1 \\ p_3, p_2; -; - \end{matrix}; x_1, x_2 \right] + \frac{dx}{2^{p_2}} F_{2:0;0}^{0:1;1} \left[\begin{matrix} -; p_1; 1 \\ p_4, p_3; -; - \end{matrix}; x_1, x_2 \right] \right]}{C_2(p, d)} \quad (148)$$

For the computation of the second summation of (150) we employ the series expansion of the incomplete gamma function and after performing the integration we obtain

$$\mathcal{F}_{24}(x; a, d, p) = \frac{\frac{(-1)^p \sum_{k=0}^{\infty} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n d^n x^{2k+2p_1+n}}{(2k+2p_1+n)^2}}{2^{2k} k! (p+k)!}}{2^p}}{C_2(p, d)} = \frac{\frac{(-1)^{p_2} x^{2p_1}}{2^p} \sum_{k,n=0}^{\infty} \frac{\frac{x^{2k} (-dx)^n}{k! n!}}{(2k+2p_1+n)^2 (p+k)!}}{C_2(p, d)} \quad (149)$$

Next, we perform the expansion into even and odd terms we have

$$\mathcal{F}_{24}(x; a, d, p) = \frac{\frac{(-1)^p}{2^p} x^{2p_1} \left[\sum_{k,m=0}^{\infty} \frac{x_1^k (-dx)^{2m}}{k! \frac{(2m)!}{m!}} + \sum_{k,m=0}^{\infty} \frac{x_1^k (-dx)^{2m+1}}{k! \frac{(2m+1)!}{m!}} \right]}{C_2(p, d)} \quad (150)$$

Which is equivalent with

$$\mathcal{F}_{24}(x; a, d, p) = \frac{\frac{(-1)^p}{2^p} x^{2p_1} \left[\sum_{k,m=0}^{\infty} \frac{x_1^k (-dx)^{2m}}{k! \left(\frac{1}{2}\right)^m \frac{2^{2m} m!}{m!}} + dx \sum_{k,m=0}^{\infty} \frac{x_1^k (-dx)^{2m}}{k! (2m+1) \left(\frac{1}{2}\right)^m \frac{2^{2m} m!}{m!}} \right]}{C_2(p, d)} \quad (151)$$

Which is identical to

$$\mathcal{F}_{24}(x; a, d, p) = \frac{\frac{(-1)^p}{2^p} x^{2p_1} \left[\frac{1}{4} \sum_{k,m=0}^{\infty} \frac{x_1^k x_2^m}{k! \left(\frac{1}{2}\right)^m m!} - \frac{dx}{8} \sum_{k,m=0}^{\infty} \frac{x_1^k x_2^m}{k! \left(m+\frac{1}{2}\right) \left(\frac{1}{2}\right)^m m!} \right]}{C_2(p, d)} \quad (152)$$

Equation (152) can be written as follows

$$\mathcal{F}_{24}(x; a, d, p) = \frac{\frac{(-1)^p}{2^p} x^{2p_1} \left[\frac{1}{4p_1 p_1} \sum_{k,m=0}^{\infty} \frac{(p_1)_{k+m} (p_1)_{k+m} x_1^k x_2^m}{(p_3)_{k+m} (p_3)_{k+m} k! \frac{(1)}{2} m!} - \frac{dx}{4p_2 p_2} \sum_{k,m=0}^{\infty} \frac{(p_2)_{k+m} (p_2)_{k+m} x_1^k x_2^m}{(p_4)_{k+m} (p_4)_{k+m} k! \frac{(3)}{2} m!} \right]}{C_2(p, d)} \quad (153)$$

Finally, Dr. Progridi has successfully derived for the first time Kampé de Fériet function [11]-[13] as follows that this summation which can be written with the help of the

$$\mathcal{F}_{24}(x; a, d, p) = \frac{\frac{(-1)^p}{2^{p_4}} x^{2p_1} \left[\frac{1}{p_1^2 F_{2:1;1}^{2:0;0}} \left[p_1, p_1; -; -; \right] - \frac{dx}{p_2^2 F_{2:1;1}^{2:0;0}} \left[p_2, p_2; -; -; \right] \right]}{C_2(p, d)} \quad (154)$$

Finally, substituting (136), (142), (148) and (154) into (136) $\mathcal{F}_{\text{GBessel2}}(x; a, d, p)$ produces the desired closed form expression of

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\left[\Gamma(p) 2^{p-1} x e^{-dx} \left[F_{2:0;0}^{0:2;1} \left[\begin{matrix} (p-1) \\ 0 \end{matrix} \right] \left[\begin{matrix} -p-\frac{1}{2}; 1; \\ \frac{3}{2}; 1; -; -; \end{matrix} \right] - x_1, x_2 \right] + \frac{dx}{2} F_{2:0;0}^{0:2;1} \left[\begin{matrix} (p-1) \\ 0 \end{matrix} \right] \left[\begin{matrix} -p-\frac{1}{2}; 1; \\ 2, \frac{3}{2}; -; -; \end{matrix} \right] - x_1, x_2 \right] \right. \\ \left. + \frac{(-1)^p x^{2p_1} e^{-dx}}{2^{p_4} p_1 \Gamma(p_2)} \left[F_{2:0;0}^{0:\psi;1;1} \left[\begin{matrix} -(1)+(p_2), p_1; 1; \\ p_3, p_2; -; -; \end{matrix} \right] x_1, x_2 \right] + \frac{dx}{2p_2} F_{2:0;0}^{0:\psi;1;1} \left[\begin{matrix} -(1)+(p_2), p_1; 1; \\ p_4, p_3; -; -; \end{matrix} \right] x_1, x_2 \right] \right. \\ \left. + \ln\left(\frac{x}{2}\right) \frac{(-1)^p x^{2p_1} e^{-dx}}{2^{p_2} p_1 \Gamma(p_2)} \left[F_{2:0;0}^{0:1;1;1} \left[\begin{matrix} -p; 1; 1; \\ p_3, p_2; -; -; \end{matrix} \right] x_1, x_2 \right] + \frac{dx}{2p_2} F_{2:0;0}^{0:1;1;1} \left[\begin{matrix} -p; 1; 1; \\ p_4, p_3; -; -; \end{matrix} \right] x_1, x_2 \right] \right. \\ \left. + \frac{(-1)^p x^{2p_1}}{2^{p_4} \Gamma(p_2)} \left[\frac{1}{p_1^2 F_{2:1;1}^{2:0;0}} \left[p_1, p_1; -; -; \right] - \frac{dx}{p_2^2 F_{2:1;1}^{2:0;0}} \left[p_2, p_2; -; -; \right] \right] \right] \\ C_2(p, d) \quad x > 0 \\ 0, \quad x < 0 \quad (155)$$

Equation (155) is an original expression not published anywhere else; however, care must be shown in the computation of the Kampé de Fériet function [2].

The obvious question is as follows. Is (155) the right equation? While numerically we will test this at the numerical,

theoretical results section, we can still test analytically for at least three data point: (1) $d = 0, p \neq 0$; (2) $d \neq 0, p = 0$; (3) $d = 0, p = 0$.

First, let us set $d = 0, p \neq 0$ and evaluate (160), we obtain

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = \frac{\left[\Gamma(p) 2^{p-1} x F_{2:0;0}^{0:2;1} \left[\begin{matrix} (p-1) \\ 0 \end{matrix} \right] \left[\begin{matrix} -p-\frac{1}{2}; 1; \\ \frac{3}{2}; 1; -; -; \end{matrix} \right] - x_1, 0 \right] + \frac{(-1)^p x^{2p_1}}{2^{p_4} p_1 \Gamma(p_2)} F_{2:0;0}^{0:\psi;1;1} \left[\begin{matrix} -(1)+(p_2), p_1; 1; \\ p_3, p_2; -; -; \end{matrix} \right] x_1, x_2 \right] \\ + \ln\left(\frac{x}{2}\right) \frac{(-1)^p x^{2p_1}}{2^{p_2} p_1 \Gamma(p_2)} F_{2:0;0}^{0:1;1;1} \left[\begin{matrix} -p; 1; 1; \\ p_3, p_2; -; -; \end{matrix} \right] x_1, 0 \right] + \frac{(-1)^p x^{2p_1}}{2^{p_4} \Gamma(p_2)} \frac{1}{p_1^2 F_{2:1;1}^{2:0;0}} \left[p_1, p_1; -; -; \right] x_1, 0 \right] \\ C_2(p, 0) \quad x > 0 \\ 0, \quad x < 0 \quad (156)$$

Evaluating the four Kampé de Fériet functions [11]-[13] at $d = 0$ yields

$$F_{2:0;0}^{0:2;1} \left[\begin{matrix} (p-1) \\ 0 \end{matrix} \middle| \begin{matrix} -; p-, \frac{1}{2}; 1; \\ \frac{3}{2}, 1; -; -; \end{matrix} -x_1, 0 \right] = \sum_{k,m=0}^{p-1, \infty} \frac{(p-k)(1)_k (\frac{1}{2})_m (-x_1)^{k+m}}{(\frac{3}{2})_{k+m} (1)_{k+m}} = \sum_{k=0}^{p-1} \frac{(p-k)(\frac{1}{2})_k (-x_1)^k}{(\frac{3}{2})_k} \quad (157)$$

$$F_{2:0;0}^{0:1;1} \left[\begin{matrix} -; (1) + (p_2), p_1; 1; \\ p_3, p_2; -; -; \end{matrix} x_1, x_2 \right] = \sum_{k=0}^{\infty} \frac{[\psi(1+k) + \psi(p_2+k)](p_1)_k \frac{x_1^k}{k!}}{(p_3)_k (p_2)_k} = {}_1F_2[(1) + (p_2), p_1; p_3, p_2; x_1] \quad (158)$$

$$F_{2:0;0}^{0:1;1} \left[\begin{matrix} -; p_1; 1; \\ p_3, p_2; -; -; \end{matrix} x_1, 0 \right] = \sum_{k,m=0}^{\infty} \frac{(p_1)_k (1)_m \frac{x_1^{k+m}}{k! m!}}{(p_3)_{k+m} (p_2)_{k+m}} = \sum_{k=0}^{\infty} \frac{(p_1)_k \frac{x_1^k}{k!}}{(p_3)_k (p_2)_k} = {}_1F_2(p_1; p_3, p_2; x_1) \quad (159)$$

$$F_{2:1;1}^{2:0;0} \left[\begin{matrix} p_1, p_1; -; -; \\ p_3, p_3; p_2; \frac{1}{2}; \end{matrix} x_1, 0 \right] = \sum_{k,m=0}^{\infty} \frac{(p_1)_{k+m} (p_1)_{k+m} \frac{x_1^{k+m}}{k! m!}}{(p_3)_{k+m} (p_3)_{k+m} (p_2)_k (\frac{1}{2})_m} = \sum_{k=0}^{\infty} \frac{(p_1)_k (p_1)_k \frac{x_1^k}{k!}}{(p_3)_k (p_3)_k (p_2)_k} = {}_2F_3(p_1, p_1; p_3, p_3, p_2; x_1) \quad (160)$$

Substituting (157)-(160) into (158) yields

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = \frac{\Gamma(p) 2^{p-1} x \sum_{k=0}^{p-1} \frac{(p-k)(\frac{1}{2})_k (-x_1)^k}{(\frac{3}{2})_k} + \frac{(-1)^p x^{2p_1}}{2^{p_4} p_1 \Gamma(p_2)} {}_1F_2[(1) + (p_2), p_1; p_3, p_2; x_1] + \ln\left(\frac{x}{2}\right) \frac{(-1)^{p_2} x^{2p_1}}{2^{p_2} p_1 \Gamma(p_2)} {}_1F_2(p_1; p_3, p_2; x_1) + \frac{(-1)^p x^{2p_1}}{2^{p_4} \Gamma(p_2) p_1^2} {}_2F_3(p_1, p_1; p_3, p_3, p_2; x_1)}{C_2(p, 0)} \quad (161)$$

Equation (161) produces exactly the same answer as (103); thus, completing the proof for $d = 0, p \neq 0$.

Second, let us set $d \neq 0, p = 0$ and evaluate (155), we obtain

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, 0) = \left\{ \begin{array}{l} x e^{-dx} \left[F_{2:0;0}^{0:1;1} \left[\begin{matrix} -; (1), \frac{1}{2}; 1; \\ \frac{3}{2}, 1; -; -; \end{matrix} x_1, x_2 \right] + \frac{dx}{2} F_{2:0;0}^{0:1;1} \left[\begin{matrix} -; (1), \frac{1}{2}; 1; \\ 2, \frac{3}{2}; -; -; \end{matrix} x_1, x_2 \right] \right] \\ - \ln\left(\frac{x}{2}\right) x e^{-dx} \left[F_{2:0;0}^{0:1;1} \left[\begin{matrix} -; \frac{1}{2}; 1; \\ \frac{3}{2}, 1; -; -; \end{matrix} x_1, x_2 \right] + \frac{dx}{2} F_{2:0;0}^{0:1;1} \left[\begin{matrix} -; \frac{1}{2}; 1; \\ 2, \frac{3}{2}; -; -; \end{matrix} x_1, x_2 \right] \right] \\ + x \left[F_{2:1;1}^{2:0;0} \left[\begin{matrix} p_1, p_1; -; -; \\ \frac{3}{2}, \frac{3}{2}; 1; \frac{1}{2}; \end{matrix} x_1, x_2 \right] - \frac{dx}{4} F_{2:1;1}^{2:0;0} \left[\begin{matrix} 1, 1; -; -; \\ 2, 2; 1; \frac{3}{2}; \end{matrix} x_1, x_2 \right] \right] \end{array} \right\} \quad (162)$$

$$\left\{ \begin{array}{ll} C_2(0, d) & x > 0 \\ 0, & x < 0 \end{array} \right.$$

Equation (162) is identical to (127); thus, completing the proof for $d \neq 0, p = 0$.

since, we have considered the proof $d = 0, p = 0$ from (162) which is identical to (104) or (98).

Finally, we consider the case for $d = 0, p = 0$ from (161),

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \left\{ \begin{array}{ll} x \left[{}_1F_2\left((1), \frac{3}{2}; 1; x_1\right) - \ln\left(\frac{x}{2}\right) {}_1F_2\left(\frac{1}{2}, \frac{3}{2}; 1; x_1\right) + {}_2F_3\left(\frac{1}{2}, \frac{3}{2}, \frac{3}{2}; 1; x_1\right) \right] & x > 0 \\ C_2(0, 0) \equiv \pi & \\ 0 & x \leq 0 \end{array} \right. \quad (163)$$

Equation (163) is identical to (98); thus, completing the proof for $d = 0, p = 0$.

This concludes the derivations of the special case for $d \neq 0$ and $p \neq 0, p$ a positive integer.

Next we consider the numerical, theoretical results.

4 Numerical, Theoretical Results

Proгри 2016 [1] numerical, theoretical results section contains the first set of simulation results of pdfs and approximated numerical cdfs of the generalized Bessel function distributions of the first and second kinds based analytical derivations

presented in Proгри 2016 [1]; i.e., in Proгри 2016, we did not provide any simulation results of the exact closed form expression of the cdfs of the generalized Bessel function distributions of the first and second kinds.

Hence, in this section we present for the first time the MATLAB simulation results of the of the exact closed form expression of the cdfs of the generalized Bessel function distributions of the first and second kinds based on closed form expressions (117) and (142) in Proгри 2016 and (25) in the current journal paper.

Moreover, we want to ensure the reader that when we produced the simulation results of this section every single equation of Proгри 2016 and of the current journal paper has

been tested; hence, the simulation results go far beyond the simulation results of just closed form expressions (117) and (142) in Proгри 2016 and (25) in the current journal paper.

4.1 *Special Case: $d = 0$, $p \neq 0$ not an Integer*

In this subsection we present, numerical results of the generalized Bessel function of the first and second kinds using the exact closed form expression found in this paper and in Proгри 2016 [1].

Figure 1(a) shows the plots of (top) the generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 0.5$ for the integral and linear numerical approximation; (bottom) cdf using (62) in the current journal paper and (142) from Proгри 2016 [1].

Figure 1(b) shows absolute errors of the generalized Bessel of the first kind cdf for $a = 2$, $d = 0$, $p = 0.5$.

Absolute errors of the generalized Bessel function distribution of the second kinds are shown in Figs. 1(b) and 2(b).

Figure 2 (a) shows the plots of (top) the generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = -0.25$ for the integral and linear numerical approximation; (bottom) cdf using (62) in the current journal paper and (142) from Proгри 2016 [1].

Figure 2 (b) shows absolute errors of the generalized Bessel of the second kind cdf for $a = 2$, $d = 0$, $p = -0.25$; i.e., p not an integer.

The important results here are that even for non-integer values of p the generalized Bessel of the first and second kinds pdfs and cdfs are validated. For this special case the MATLAB integral function provides an erroneous solution due to the presence of infinity or singularity.

4.2 *Special Case: $d = 0$, $p = 0$*

The pdf and cdfs and the absolute errors of the generalized Bessel function distribution of the second kinds corresponding to the special case of $d = 0$, and $p = 0$ are shown in Fig. 3.

Figure 3 (a) shows the plots of (top) the generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 0$ for the integral and linear numerical approximation; (bottom) cdfs using (98) and (103) in the current journal paper.

Figure 3 (b) shows absolute errors of the generalized Bessel of the second kind cdf for $a = 2$, $d = 0$, $p = 0$; i.e., p an integer.

For this special case the cdfs computed from using (98) and (103) are fourteen to sixteen orders of magnitude as accurate as

the MATLAB integral function.

4.3 *Special Case: $d = 0$, p a Positive Integer*

The pdf and cdfs and the absolute errors of the generalized Bessel function distribution of the second kinds corresponding to the special case of $d = 0$, and p a positive integer are shown in Fig. 4.

Figure 4 (a) shows the plots of (top) the generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 2$ for the integral and linear numerical approximation; (bottom) cdfs using (103) Option 1 and (187) Option 2 in the current journal paper.

Figure 4 (b) shows absolute errors of the generalized Bessel of the second kind cdf for $a = 2$, $d = 0$, $p = 2$; i.e., p an integer.

For this special case the cdfs computed from using (103) and (187) are fourteen to sixteen orders of magnitude as accurate as the MATLAB integral function.

4.4 *Special Case: $d \neq 0$, $p = 0$*

The pdf and cdfs and the absolute errors of the generalized Bessel function distribution of the second kinds corresponding to the special case of $d = 0.5$, and $p = 0$ shown in Fig. 5.

Figure 5 (a) shows the plots of (top) the generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0.5$, and $p = 0$ for the integral and linear numerical approximation; (bottom) cdfs using (127) Option 2 and (155) Option 2 in the current journal paper.

Figure 5 (b) shows absolute errors of the generalized Bessel of the second kind cdf for $a = 2$, $d = 0.5$, $p = 0$; i.e., p an integer.

For this special case the cdfs computed from using (127) and (155) are fourteen to sixteen orders of magnitude as accurate as the MATLAB integral function.

4.5 *Special Case: $d \neq 0$, p a Positive Integer*

The pdf and cdfs and the absolute errors of the generalized Bessel function distribution of the second kinds corresponding to the special case of $d \neq 0$, and $p = 0$ a positive integer are shown in Fig. 6.

Figure 6 (a) shows the plots of (top) the generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0.5$, and $p = 2$ for the integral and linear numerical approximation; (bottom) cdfs using (155) Option 2 and (155) Option 3 in the current journal paper.

TABLE I: COMPUTATION TIME OF CDFS

No	τ_{int} (s)	τ_{lap} (ms)	τ_{cf1} (s)	τ_{cf2} (s)	τ_{Σ} (s)
1a	4.15	1.2	0.005	0.013	4.17
1b	4.16	3.1	0.014	0.029	4.20
2	0.13	0.1	0.004	0.01	0.14
3	0.16	3.5	0.008	0.012	0.18
4	4.14	0.1	0.22	0.13	4.54
5	4.29	0.01	0.31	0.17	4.62

TABLE II: COMPUTATION TIME OF CDF (155) VIA OPTIONS 1-4

N	τ_{int} (s)	τ_{lap} (ms)	τ_{cf1} (s)	τ_{cf2} (s)	τ_{Σ} (s)
15	4.2	3.1	0.35 (1)	1.56 (2)	6.12
25	4.29	0.6	0.50 (1)	1.66 (2)	6.45
50	4.19	3.2	1.21 ^{ix} (1)	2.00 (2)	7.41
15	4.18	0.1	38.26 (3)	38.14 (4)	80.58
25	4.21	0.1	60.08 (3)	60.42 (4)	124.71
50	4.23	0.032	116.12 (3)	115.89 (4)	236.24

TABLE III: SAME AS II USING PROGRI'S IMPLEMENTATION

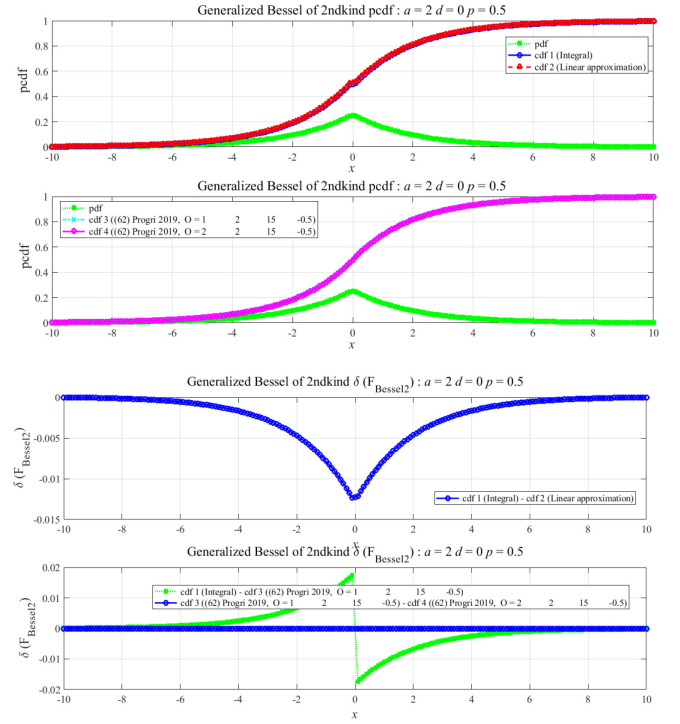
N	τ_{int} (s)	τ_{lap} (ms)	τ_{cf1} (s)	τ_{cf2} (s)	τ_{Σ} (s)
15	4.2	3.1	0.27 (1)	0.24 (2)	4.69
25	4.29	0.6	0.45 (1)	0.32 (2)	4.92
50	4.14	3.2	1.21 ^x (1)	0.74 (2)	6.08
15	4.18	0.1	0.18 (3)	0.16 (4)	4.50
25	4.21	0.1	0.20 (3)	0.17 (4)	4.52
50	4.23	0.032	0.25 (3)	0.22 (4)	4.61

Figure 6 (b) shows absolute errors of the generalized Bessel of the second kind cdf for $a = 2$, $d = 0.5$, $p = 2$; i.e., p an integer.

For this special case the cdfs computed from using (127) and (155) are fourteen to sixteen orders of magnitude as accurate as the MATLAB integral function.

Figures 1-6 are necessary because they show that the cdf accuracy performance is outstanding; hence, it remains to test the computational speed performance.

The computational times, performed in an AMD A6-3420 APU with Radeon™ HD Graphics 1.5 GHz processor 64-bit operating system, of all the cdfs is considered in Tab. I. In the first row of Tab. I the computation times of special case discussed in Fig. 1 are reported. And so forth.

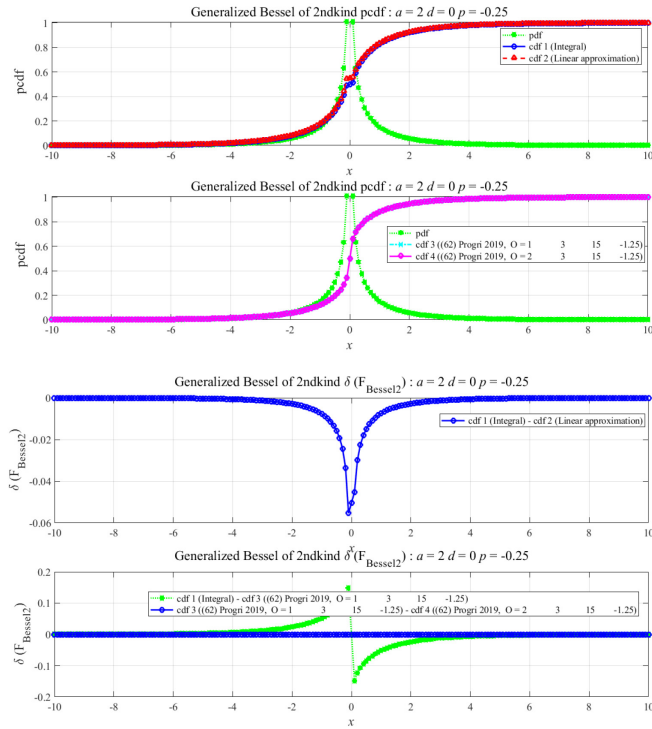


(a) (top) original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 0.5$, using the integral and linear numerical approximation; (bottom) original generalized Bessel of the second kind pdf and cdf for $a = 12$, $d = 0$, and $p = 0.5$ for (62) and (142) from Progrid 2016 [1] Option 3 of the Kampé de Fériet function.

(b) (top) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 0.5$, integral – linear numerical approximation; (bottom) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 0.5$, for (62) – (142) from Progrid 2016 [1] Option 3 of the Kampé de Fériet function; p not an integer.

FIGURE 1: Numerical calculations of the generalized Bessel of the (a) second and (b) second kind absolute error pdf and cdf for (62) and (142) from Progrid 2016 [1] Option 3 of the Kampé de Fériet function for $a = 2$, $d = 0$, and $p = 0.5$.

There are six columns in Tab. I: The first column corresponds to the number of rows; the second column corresponds to the MATLAB integral function computations, the third column corresponds to the linear approximation, the fourth to the first closed form expression discussed in Sect. 3, the fifth to the total computation time. Based on the computation times reported in Tab. I, 6/6 times, the computation times of the cdfs based on the closed form expression provided in Sect. 3 is significantly (9-26 $\times$) faster than the MATLAB integral function.

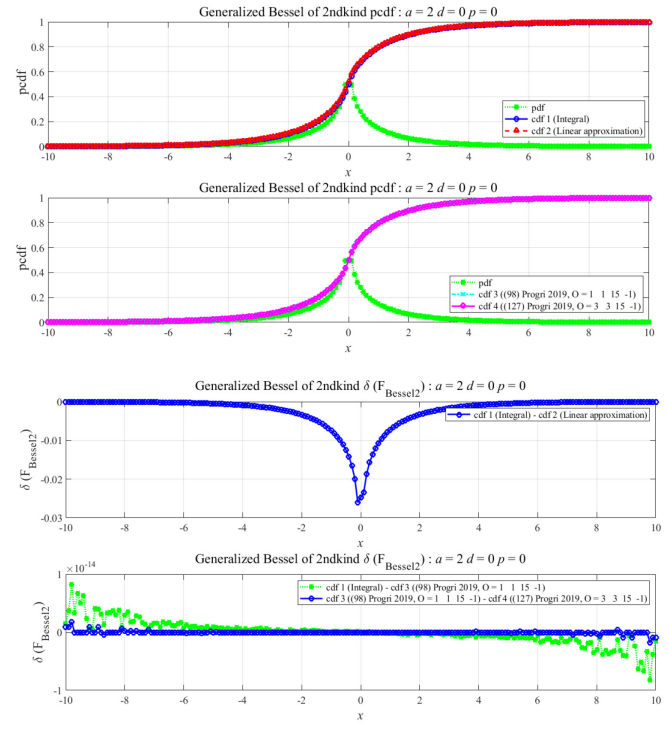


(a) (top) original generalized Bessel of the second kind pdf^{xii} and cdf for $a = 2$, $d = 0$, and $p = -0.25$, for the integral^{xii} and linear numerical approximation; (bottom) original generalized Bessel of the second kind pdf and cdf for $a = 12$, $d = 0$, and $p = -0.25$ for (62) and (142) from Progrid 2016 [1] Option 3 of the Kampé de Fériet function.

(b) (top) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = -0.25$, integral – linear numerical approximation; (bottom) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = -0.25$, for (62) – (142) from Progrid 2016 [1] Option 3 of the Kampé de Fériet function; p not an integer.

FIGURE 2: Numerical calculations of the generalized Bessel of the (a) second and (b) second kind absolute error pdf and cdf for (62) Option 2 of the Kampé de Fériet function and (142) from Progrid 2016 [1] Option 3 of the Kampé de Fériet function for $a = 2$, $d = 0$, and $p = -0.25$; p not an integer.

For example. If we consider 15 terms during the computation of the cdf via (155) (see Tab. II) then: (1) the integral approach is 42 thousand times less efficient than the linear approximation; (2) The implementation of the the Kampé de Fériet function via options 1 and 2 is 12/2.5 times more efficient than the direct integral via MATLAB; (3) The implementation of the Kampé de Fériet function using the MATLAB hypergeom function via options 3 and 4 is 9 times less efficient than the direct integral via MATLAB.

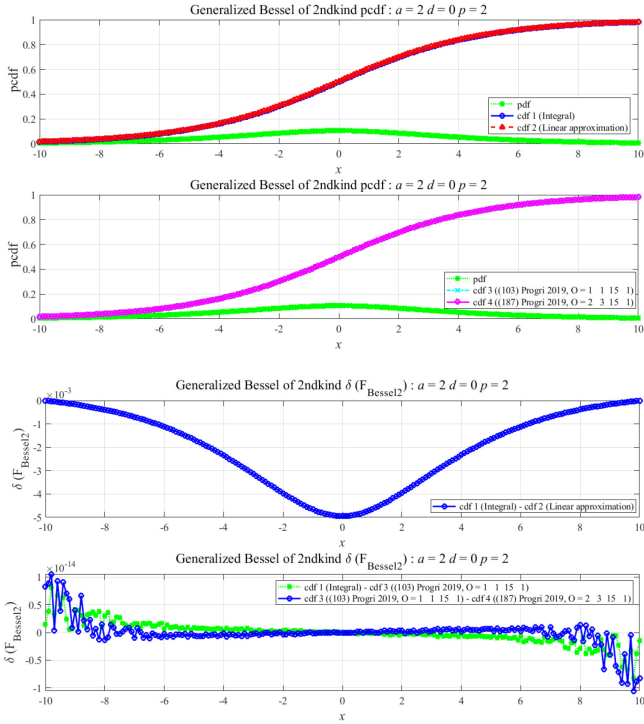


(a) (top) original generalized Bessel of the second kind pdf^{xiii} and cdf for $a = 2$, $d = 0$, and $p = 0$, for the integral^{xiv} and linear numerical approximation; (bottom) original generalized Bessel of the second kind pdf and cdf for $a = 12$, $d = 0$, and $p = 0$ for (98) Option 1 of the Kampé de Fériet function and (127)^{xv} Option 3 of the Kampé de Fériet function; p an integer.

(b) (top) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 0$, integral – linear numerical approximation; (bottom) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 0$, for (98) Option 1 of the Kampé de Fériet function – (127) Option 3 of the Kampé de Fériet function; p an integer.

FIGURE 3: Numerical calculations of the generalized Bessel of the (a) second and (b) second kind absolute error pdf and cdf for (98) Option 1 of the Kampé de Fériet function and (127) Option 3 of the Kampé de Fériet function for $a = 2$, $d = 0$, and $p = 0$; p an integer.

The implementation of the Kampé de Fériet function using the Progrid's phypergeom function via options 3 and 4 in Tab. III is 527 times more efficient than the save via MATLAB hypergeom function. In summary, the Progrid implementation of the Kampé de Fériet function (see Progrid 2018 [2], Appendix A) is the optimum in performance and speed (option 4); but it requires Progrid implementation of the generalized hypergeometric function.



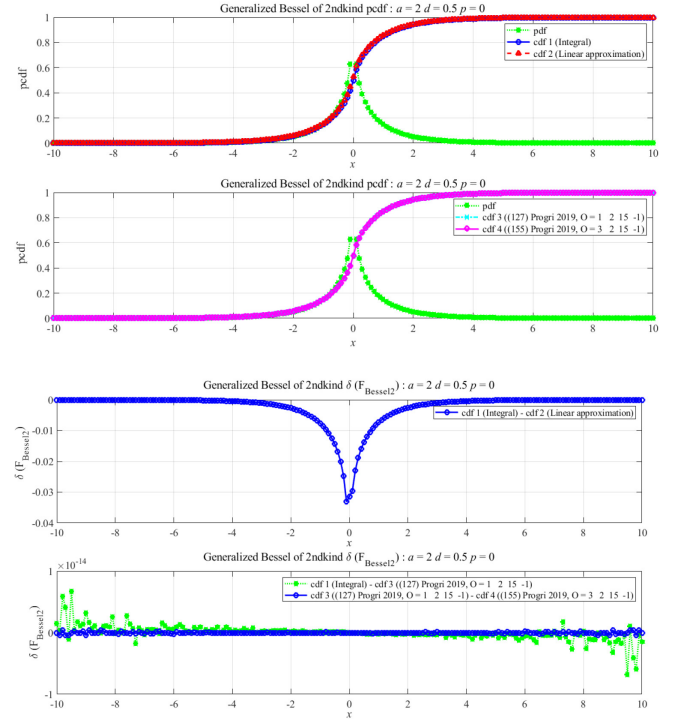
(a) (top) original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 2$, for the integral and linear numerical approximation; (bottom) original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 2$ for (103) Option 1 or the recursive algorithm and (187) Option 3 of the Kampé de Fériet function; p an integer.

(b) (top) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 3$, integral – linear numerical approximation; (bottom) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0$, and $p = 2$, for (103) Option 1 or the recursive algorithm – (187) Option 3 of the Kampé de Fériet function; p an integer.

FIGURE 4: Numerical calculations of the generalized Bessel of the (a) second and (b) second kind absolute error pdf and cdf for (103) Option 1 or the recursive algorithm and (187) Option 3 of the Kampé de Fériet function for $a = 2$, $d = 0$, and $p = 2$; p an integer.

5 Conclusions

The main focus on this masterpiece journal paper is to validate the entire derivations of the closed form expressions provided in Sects. 2 and 3 namely: (1) the cdf of the generalized Bessel of the second kind pdfs/cdfs in the current journal paper corresponding to five special cases: one for non-integer values of p and four for integer values of p .



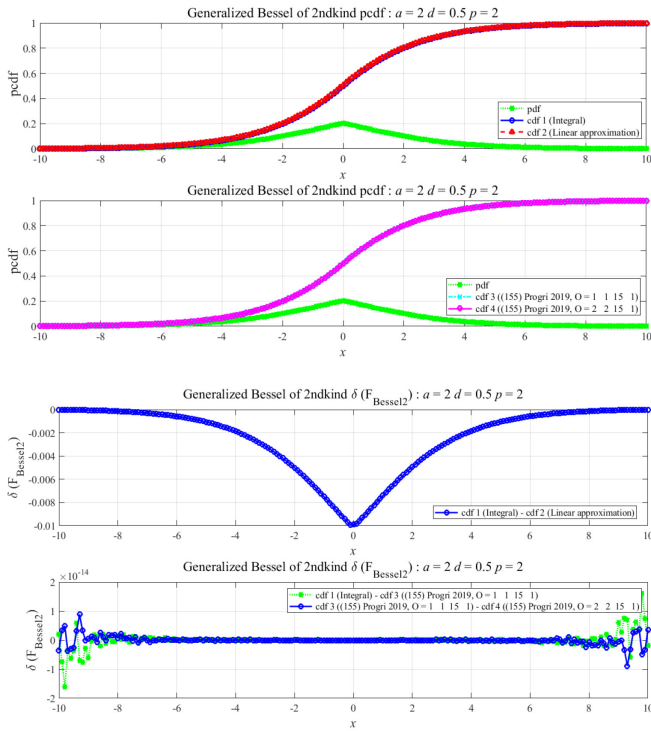
(a) (top) original generalized Bessel of the second kind pdf^{xvi} and cdf for $a = 2$, $d = 0.5$, and $p = 0$, for the integral^{xvii} and linear numerical approximation; (bottom) original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0.5$, and $p = 0$ for (127) Option 2 of the Kampé de Fériet function and (155) Option 2 of the Kampé de Fériet function; $p = 0$ an integer.

(b) (top) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0.5$, and $p = 0$, integral – linear numerical approximation; (bottom) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0.5$, and $p = 0$, for (127) Option 2 of the Kampé de Fériet function – (155) Option 2 of the Kampé de Fériet function; p an integer.

FIGURE 5: Numerical calculations of the generalized Bessel of the (a) second and (b) second kind absolute error pdf and cdf for (127) Option 2 of the Kampé de Fériet function and (155) Option 2 of the Kampé de Fériet function for $a = 2$, $d = 0.5$, and $p = 0$; p an integer.

Option 1 does not produce any output when the number of terms exceeds fifty. Option 2 is the most accurate and the most computationally stable option for the implementation of the Kampé de Fériet function. Option 2 was added for the first time to perform the computations of this journal paper.

For cases discussed in Figs. 1 and 2, the direct integration is not accurate due to the presence of infinity or singularity.



(a) (top) original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0.5$, and $p = 2$, for the integral and linear numerical approximation; (bottom) original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0.5$, and $p = 2$ for (155) Option 1 of the Kampé de Fériet function and (155) Option 2 of the Kampé de Fériet function; p an integer.

(b) (top) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0.5$, and $p = 2$, integral – linear numerical approximation; (bottom) absolute error of the original generalized Bessel of the second kind pdf and cdf for $a = 2$, $d = 0.5$, and $p = 2$, for (155) Option 1 of the Kampé de Fériet function – (155) Option 2 of the Kampé de Fériet function; p an integer.

FIGURE 6: Numerical calculations of the generalized Bessel of the (a) second and (b) second kind absolute error pdf and cdf for (155) Option 1 of the Kampé de Fériet function and (155) Option 2 of the Kampé de Fériet function for $a = 2$, $d = 0.5$, and $p = 2$; p an integer.

We can say with absolute certainty that all the closed form expressions provided in Progrid 2016 [1], 2018 [2] have been simulated and tested. The numerical results obtained from the closed form expressions in Fig. 1 to 6 are several orders of magnitude more accurate than those obtained from the linear approximation simulation results presented initially in Progrid 2016 [1].

Numerical results are derived for each case to validate the theoretical models presented in the paper.

For each generalized Bessel distribution model, the closed form expression of cdf is given by means of series expansion of the modified Bessel functions which leads to incomplete gamma functions, hypergeometric series, and the Kampé de Fériet functions.

It is also shown that for most practical applications the generalized Bessel function distribution cdf of the first and second kinds can be represented in closed form by means of the family of well-known the Kampé de Fériet functions to within fourteen to fifteen orders of magnitude accuracy.

In summary, the Progrid implementation of the generalized hypergeometric function and the Kampé de Fériet function is the optimum in performance and speed.

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This journal paper is dedicated to four special men in my life: my grandfather, Xhevdet Progrid, my dear father, Fiqiri Progrid, my father's first cousin Dr. Peter Demir, and the 41st President of the United States of America, George H.W. Bush.

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8 Appendix A: Inefficient Implementation of $\mathcal{F}_{21}(x; a, 0, 0)$

There is another inefficient algorithm for computing

$$\mathcal{F}_{21}(x; a, 0, 0) = \frac{\sum_{k=0}^{\infty} \frac{\psi(k+1) x^{2k+1}}{2^{2k} (k!)^2}}{\pi} = \frac{x \sum_{k=0}^{\infty} \frac{\left\{ \frac{k}{k+1} F \left[\begin{matrix} k+1, 1, 1; 1 \end{matrix} \right] + \psi(1) \right\} \left(\frac{1}{2} \right)_k \frac{x^{2k}}{k 2^{2k}}}}{\pi} = \frac{x \sum_{k=0}^{\infty} \frac{k F \left[\begin{matrix} k+1, 1, 1; 1 \end{matrix} \right] \left(\frac{1}{2} \right)_k \frac{x^{2k}}{k 2^{2k}} + x \psi(1) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} \right)_k \frac{x^{2k}}{k 2^{2k}}}}{\pi} \quad (165)$$

The second summation of (165) can be computed from generalized hypergeometric function [31].

$$x \psi(1) \left[\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} \right)_k \frac{x^{2k}}{k 2^{2k}}}{\left(\frac{3}{2} \right)_k (1)_k k!} \equiv x {}_1F_2 \left(\frac{1}{2}, \frac{3}{2}, 1; x_1 \right) \right] \quad (166)$$

$$\sum_{k=0}^{\infty} \frac{k F \left[\begin{matrix} k+1, 1, 1; 1 \end{matrix} \right] \left(\frac{1}{2} \right)_k \frac{x^{2k}}{k 2^{2k}}}{(k+1) \left(\frac{3}{2} \right)_k (1)_k k!} = \sum_{k=0}^{\infty} \frac{k \sum_{m=0}^{\infty} \frac{(k+1)m(1)m(1)m^1 m! \left(\frac{1}{2} \right)_k \frac{x^{2k}}{k 2^{2k}}}{(k+2)m(2)m m! \left(\frac{3}{2} \right)_k (1)_k k!}}{(k+1) \left(\frac{3}{2} \right)_k (1)_k k!} = \sum_{k=1}^{\infty} \frac{k \sum_{m=0}^{\infty} \frac{(k+1)m(1)m(1)m^1 m! \left(\frac{1}{2} \right)_k \frac{x^{2k}}{k 2^{2k}}}{(k+2)m(2)m m! \left(\frac{3}{2} \right)_k (1)_k k!}}{(k+1) \left(\frac{3}{2} \right)_k (1)_k k!} \quad (167)$$

Which is identical to

$$\sum_{k=0}^{\infty} \frac{k F \left[\begin{matrix} k+1, 1, 1; 1 \end{matrix} \right] \left(\frac{1}{2} \right)_k \frac{x^{2k}}{k 2^{2k}}}{(k+1) \left(\frac{3}{2} \right)_k (1)_k k!} = \sum_{k=0}^{\infty} \frac{(k+1) \sum_{m=0}^{\infty} \frac{(k+2)m(1)m(1)m^1 m! \left(\frac{1}{2} \right)_k \frac{x^{2(k+1)}}{k+1 2^{2(k+1)}}}{(k+2) \left(\frac{3}{2} \right)_{k+1} (1)_{k+1} (k+1)!}}{(k+1) \left(\frac{3}{2} \right)_k (1)_k k!} = \frac{x^2}{2^2 6} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2)_{k+m} \left(\frac{3}{2} \right)_k (1)_m (1)_m \left(\frac{x^2}{2^2} \right)^k \frac{1^m}{k! m!}}{(3)_{k+m} \left(\frac{5}{2} \right)_k (2)_k (2)_m \frac{1^m}{k! m!}} \quad (168)$$

Finally, Dr. Proгри has successfully derived for the first time that this summation which can be written with the help of the

Kampé de Fériet function [11]-[13] as follows

$$\sum_{k=0}^{\infty} \frac{k F \left[\begin{matrix} k+1, 1, 1; 1 \end{matrix} \right] \left(\frac{1}{2} \right)_k \frac{x^{2k}}{k 2^{2k}}}{(k+1) \left(\frac{3}{2} \right)_k (1)_k k!} = \frac{x^2}{2^2 6} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2)_{k+m} \left(\frac{3}{2} \right)_k (1)_m (1)_m \left(\frac{x^2}{2^2} \right)^k \frac{1^m}{k! m!}}{(3)_{k+m} \left(\frac{5}{2} \right)_k (2)_k (2)_m \frac{1^m}{k! m!}} = \frac{x^2}{2^2 6} F_{1;2;1}^{1;1;2} \left[\begin{matrix} 2; \frac{3}{2}; 1, 1; \\ 3; \frac{5}{2}; 2, 2; \end{matrix} x_1, 1 \right] \quad (169)$$

Substituting (166) and (169) into (165)

$$\mathcal{F}_{21}(x; a, 0, 0) = \frac{\sum_{k=0}^{\infty} \frac{\psi(k+1) x^{2k+1}}{2^{2k} (k!)^2}}{\pi} = \frac{\frac{x^3}{2^3 3} F_{1;2;1}^{1;1;2} \left[\begin{matrix} 2; \frac{3}{2}; 1, 1; \\ 3; \frac{5}{2}; 2, 2; \end{matrix} x_1, 1 \right] + x \psi(1) {}_1F_2 \left(\frac{1}{2}, \frac{3}{2}, 1; x_1 \right)}{\pi} \quad (170)$$

Equation (170) is just as accurate as (96) but is computationally much more expensive; hence, it is not recommended in implementation.

9 Appendix B: Alternative Implementation of $\mathcal{F}_{\text{GBessel2}}(x; a, 0, p)$

I initially developed the alternative implementation of $\mathcal{F}_{\text{GBessel2}}(x; a, 0, p)$ then later on I discovered the recursive

$\mathcal{F}_{21}(x; a, 0, 0)$. The main reason for showing this is to make sure that scholars understand the difference of both algorithms.

From Proгри (2016, [4] (127)), we have $\psi(k+1)$

$$\psi(k+1) = \frac{k}{k+1} F \left[\begin{matrix} k+1, 1, 1; 1 \end{matrix} \right] + \psi(1) \quad (164)$$

Substituting (164) into (96) yields

It remains to compute the first summation as follows

algorithm which is discussed in Sect. 3.2. It is also important to note that both the recursive approach and the direct computation algorithm, discussed here, produce similar performance in terms of accuracy and computational load. The only drawback to the direct approach is that it is more complex than the recursive algorithm.

When $d = 0$ and p a positive, $f_{\text{GBessel2}}(x; a, 0, p)$ is given by (34).

Following from Proгри 2016 [1], (50)-(53), substituting (31) into (121) of Proгри 2016 [1]

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = \frac{\int_0^x t^p \left[\frac{1}{2} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \left(\frac{t}{2}\right)^{2k-p} + (-1)^p \sum_{k=0}^{\infty} \frac{\frac{\psi(k+1)+\psi(p+k+1)}{2} - \ln \frac{t}{2}}{k!(p+k)!} \left(\frac{t}{2}\right)^{p+2k} \right] dt}{C_2(p, 0)} \quad (171)$$

Changing the order of summation and integration we obtain

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{2^{p-1} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k! 2^{2k}} \int_0^x t^{2k} dt + \frac{(-1)^p}{2^{p/2}} \sum_{k=0}^{\infty} \frac{[\psi(k+1)+\psi(p+k+1)] \int_0^x t^{2p+2k} dt}{k!(p+k)! 2^{2k}} + \frac{(-1)^{p/2}}{2^{p/2}} \sum_{k=0}^{\infty} \frac{\int_0^x \ln \frac{t}{2} t^{2p+2k} dt}{k!(p+k)! 2^{2k}}}{C_2(p, d)} \quad (172)$$

The first integral is pretty straightforward to compute (see 90). first summation of (172) and performing a series of identities yields

We start with the first summation. Substituting, (90) into the

$$\mathcal{F}_{21}(x; a, 0, p) = \frac{\sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{2^{1-p} C_2(p, 0)} \frac{x^{2k+1}}{2^{2k+1}}}{\frac{C_2(p, 0)}{0}} = \begin{cases} \frac{x 2^{p-1} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \left(\frac{1}{2}\right)^k (-x_1)^k}{C_2(p, 0)} & p \neq 0 \\ 0 & p = 0 \end{cases} \quad (173)$$

Equation (173) can be written with the help of the modified generalized hypergeometric function as follows

$$\mathcal{F}_{21}(x; a, 0, p) = \begin{cases} \frac{2^{p-1} x F \left[\begin{matrix} (p-1) \\ 0 \end{matrix} \middle| \begin{matrix} p-\frac{1}{2} \\ \frac{3}{2} \end{matrix} \middle| -x_1 \right]}{C_2(p, d)} & p \neq 0 \\ 0 & p = 0 \end{cases} \quad (174)$$

The second integral is also pretty straightforward to compute

Substituting (175) into the second summation of (172) yields

$$\int_0^x t^{2p+2k} dt = \frac{x^{2p+2k+1}}{2p+2k+1} \quad (175)$$

$$\sum_{k=0}^{\infty} \frac{[\psi(k+1)+\psi(p+k+1)] \int_0^x t^{2p+2k} dt}{k!(p+k)! 2^{2k}} = \sum_{k=0}^{\infty} \frac{[\psi(k+1)+\psi(p+k+1)] \frac{x^{2p+2k+1}}{2p+2k+1}}{2^{2k} k!(p+k)!} \quad (176)$$

Substituting (176) into the second summation of (172) yields,

$$\mathcal{F}_{22}(x; a, d, p) = \frac{(-1)^p \sum_{k=0}^{\infty} \frac{[\psi(k+1)+\psi(p+k+1)] \frac{x^{2p+2k+1}}{2p+2k+1}}{2^{2k} k!(p+k)!}}{C_2(p, d)} = \frac{(-1)^p [\psi(1)+\psi(p+1)] \frac{x^{2p+1}}{2p+1} + x^{2p+1} \sum_{k=1}^{\infty} \frac{[\psi(k+1)+\psi(p+k+1)] x_1^k}{(2p+2k+1) k!(p+k)!}}{C_2(p, d)} \quad (177)$$

Where, here we use the continued fraction expansion of the rational function; hence, we have

$$\mathcal{F}_{22}(x; a, d, p) = \frac{(-1)^p x^{2p+1}}{2^{p+1} \Gamma(p_2)} \frac{\psi(1)+\psi(p+1)+\sum_{k=1}^{\infty} \frac{[\psi(k+1)+\psi(p+k+1)] (p_1)_k x_1^k}{(p_3)_k (p_2)_k}}{C_2(p, d)} \quad (178)$$

Equation (178) can be computed recursively.

The second integral is more difficult to compute directly, so

The third summation function of (172) can be computed from the third integral. we integrate by parts

$$\int_0^x \ln \frac{t}{2} t^{2p+2k} dt = \left[\ln \left(\frac{t}{2} \right) \frac{t^{2p+2k+1}}{2p+2k+1} \right]_0^x = \ln \left(\frac{x}{2} \right) \frac{x^{2p+2k+1}}{2p+2k+1} - \lim_{x \rightarrow 0} \ln \left(\frac{x}{2} \right) \frac{x^{2p+2k+1}}{2p+2k+1} - \frac{1}{2p+2k+1} \int_0^x t^{2p+2k} dt \quad (179)$$

The lower limit of $\ln \left(\frac{x}{2} \right) \frac{x^{2p+2k+1}}{2p+2k+1}$ $k = 0, 1, \dots$ can be computed using L'Hôpital's rule (see also (89)) [28]

$$\lim_{x \rightarrow 0} \ln \left(\frac{x}{2} \right) \frac{x^{2p+2k+1}}{2p+2k+1} = -\frac{2p+2k+2}{2p+2k+1} \lim_{x \rightarrow 0} \frac{x^{-1}}{x^{-2p-2k-2}} = -\frac{2p+2k+2}{2p+2k+1} \lim_{x \rightarrow 0} x^{2p+2k+2} = 0 \quad (180)$$

Substituting (179), (180) into the second summation of (172) yields

$$\mathcal{F}_{23}(x; a, d, p) + \mathcal{F}_{24}(x; a, d, p) = \frac{\sum_{k=0}^{\infty} \frac{\int_0^x \ln \frac{t}{2} t^{2p+2k} dt}{2^{2k} k! (p+k)!}}{C_2(p, d)} = \frac{\ln\left(\frac{x}{2}\right) \sum_{k=0}^{\infty} \frac{x^{2p+2k+1}}{2^{2k} k! (p+k)!} - \sum_{k=0}^{\infty} \frac{1}{2^{2k} k! (p+k)!} \int_0^x t^{2p+2k} dt}{C_2(p, d)} \quad (181)$$

Next, we work on the first summation of (181). Substituting (175) into the second summation yields

$$\sum_{k=0}^{\infty} \frac{x^{2p+2k+1}}{2^{2k} k! (p+k)!} = \frac{x^{2p_1}}{2\Gamma(p_2)} \frac{1}{p_1} \sum_{k=0}^{\infty} \frac{(p_1)_k x_1^k}{(p_3)_k (p_2)_k} \quad (182)$$

Equation (182) can be written with the help of the generalized hypergeometric function [31] as follows

$$\sum_{k=0}^{\infty} \frac{x^{2p+2k+1}}{2^{2k} k! (p+k)!} = \frac{x^{2p_1}}{2p_1 \Gamma(p_2)} F[p_1; p_3, p_2; x_1] \quad (183)$$

Substituting (183) into the first summation of (181) yields,

$$\mathcal{F}_{23}(x; a, d, p) = \frac{\frac{(-1)^{p_2} \ln\left(\frac{x}{2}\right) \sum_{k=0}^{\infty} \frac{x^{2p+2k+1}}{2^{2k} k! (p+k)!}}{C_2(p, d)}}{\frac{\ln\left(\frac{x}{2}\right) \frac{(-1)^{p_2} x^{2p_1}}{2^{p_2} p_1 \Gamma(p_2)} F[p_1; p_3, p_2; x_1]}{C_2(p, d)}} \quad (184)$$

For the computation of the second summation of (181) we and after performing the integration we obtain
employ the series expansion of the incomplete gamma function

$$\mathcal{F}_{24}(x; a, d, p) = \frac{\frac{(-1)^p \sum_{k=0}^{\infty} \frac{x^{2k+2p_1}}{2^{2k} k! (p+k)!}}{C_2(p, d)}}{\frac{(-1)^{p_2} x^{2p_1} \sum_{k=0}^{\infty} \frac{(p_1)_k (p_1)_k x_1^k}{(p_3)_k (p_3)_k (p_2)_k}}{C_2(p, d)}} \quad (185)$$

Equation (185) can be written with the help of the generalized hypergeometric function, ${}_2F_3$, [31] as follows

$$\mathcal{F}_{24}(x; a, d, p) = \frac{(-1)^p}{2^{p_4} p_1^2 \Gamma(p_2) C_2(p, d)} {}_2F_3[p_1, p_1; p_3, p_3, p_2; x_1] \quad (186)$$

Finally, substituting (174), (178), (184) and (186) into (172) $\mathcal{F}_{\text{GBessel2}}(x; a, 0, p)$
produces the desired closed form expression of

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, p) = \begin{cases} \frac{\left[2^{p-1} x F \left[\begin{matrix} (p-1) \\ 0 \end{matrix} \middle| \begin{matrix} p-\frac{1}{2} \\ \frac{3}{2} \end{matrix} \right] - x_1 \right] + \frac{(-1)^p x^{2p_1}}{2^{p_4} p_1 \Gamma(p_2)} \left[\psi(1) + \psi(p+1) + \sum_{k=1}^{\infty} \frac{[\psi(k+1) + \psi(p+k+1)] (p_1)_k x_1^k}{(p_3)_k (p_2)_k k!} \right] \\ + \ln\left(\frac{x}{2}\right) \frac{(-1)^{p_2} x^{2p_1}}{2^{p_2} p_1 \Gamma(p_2)} F[p_1; p_3, p_2; x_1] + \frac{(-1)^p x^{2p_1}}{2^{p_4} p_1^2 \Gamma(p_2)} {}_2F_3[p_1, p_1; p_3, p_3, p_2; x_1] \right]}{C_2(p, d)} & x > 0 \\ 0, & x < 0 \end{cases} \quad (187)$$

Equation (187) is an original expression not published anywhere else; however, care must be shown in the computation of generalized hypergeometric functions [31].

The obvious question is as follows. Is (187) the right

equation? While numerically we will test this at the numerical, theoretical results section, we can still test analytically for at least one data point: (1) $d = 0, p = 0$.

Hence, for $d = 0, p = 0$ from (187), we obtain.

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = \begin{cases} \frac{x \left[\psi(1) + \sum_{k=1}^{\infty} \left[\frac{1}{k} + \psi(k) \right] \frac{x_1^k}{(1)_k} - \ln\left(\frac{x}{2}\right) {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, 1; x_1\right) + {}_2F_3\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}, 1; x_1\right) \right]}{C_2(0, 0) \equiv \pi} & x > 0 \\ 0 & x \leq 0 \end{cases} \quad (188)$$

Equation (188) is identical to (98); thus, completing the proof for $d = 0, p = 0$.

This concludes the derivations of the special case for $d = 0$ and $p \neq 0, p$ a positive integer.

10 Appendix C: Inefficient Implementation of $\mathcal{F}_{23}(x; a, d, 0)$

If we were to employ the continued fraction expansion of the incomplete gamma function in $\mathcal{F}_{23}(x; a, d, 0)$ we have

The second integral is equal to

$$\int_0^x \frac{t^{-1} \gamma(2k+1, td)}{d^{2k+1}} dt = \int_0^x t^{-1} \frac{t^{2k+1} e^{-dt}}{(2k+1)} \left[\sum_{m=0}^{\infty} \frac{\left(\frac{d^2 t^2}{2^2}\right)^m}{\left(k+\frac{3}{2}\right)_m (k+1)_m} + \frac{dt}{2(k+1)} \sum_{m=0}^{\infty} \frac{\left(\frac{d^2 t^2}{2^2}\right)^m}{(k+2)_m \left(k+\frac{3}{2}\right)_m} \right] dt \quad (189)$$

Which is equivalent with

$$\int_0^x \frac{t^{-1} \gamma(2k+1, td)}{d^{2k+1}} dt = \frac{\left[\sum_{m=0}^{\infty} \frac{\int_0^x t^{2k+2m} e^{-dt} dt \left(\frac{d^2}{2^2}\right)^m}{\left(k+\frac{3}{2}\right)_m (k+1)_m} + \frac{d}{2(k+1)} \sum_{m=0}^{\infty} \frac{\int_0^x t^{2k+2m+1} e^{-dt} dt \left(\frac{d^2}{2^2}\right)^m}{(k+2)_m \left(k+\frac{3}{2}\right)_m} \right]}{(2k+1)} \quad (190)$$

Utilizing the result from Progi 2016 [1], (81) we obtain

$$\int_0^x \frac{t^{-1} \gamma(2k+1, td)}{d^{2k+1}} dt = \frac{\left[\sum_{m=0}^{\infty} \frac{\gamma(2k+2m+1, dx) \left(\frac{d^2}{2^2}\right)^m}{d^{2k+2m+1} \left(k+\frac{3}{2}\right)_m (k+1)_m} + \frac{d}{2(k+1)} \sum_{m=0}^{\infty} \frac{\gamma(2k+2m+2, dx) \left(\frac{d^2}{2^2}\right)^m}{d^{2k+2m+2} (k+2)_m \left(k+\frac{3}{2}\right)_m} \right]}{(2k+1)} \quad (191)$$

Finally, substituting (191) into (117) yields,

$$\mathcal{F}_{23}(x; a, d, 0) = \frac{\sum_{k=0}^{\infty} \frac{\left[\sum_{m=0}^{\infty} \frac{\gamma(2k+2m+1, dx) \left(\frac{1}{2^2}\right)^m}{\left(k+\frac{3}{2}\right)_m (k+1)_m} + \frac{1}{2(k+1)} \sum_{m=0}^{\infty} \frac{\gamma(2k+2m+2, dx) \left(\frac{1}{2^2}\right)^m}{(k+2)_m \left(k+\frac{3}{2}\right)_m} \right]}{d^{2k+1} (2k+1) 2^{2k} (k!)^2}}{C_2(0, d)} \quad (192)$$

Which is identical to

$$\mathcal{F}_{23}(x; a, d, 0) = \frac{\sum_{k,m=0}^{\infty} \frac{\gamma(2k+2m+1, dx)}{d^{2k+1} 2^{2k+2m} (k!)^2} + \frac{1}{2} \sum_{k,m=0}^{\infty} \frac{\gamma(2k+2m+2, dx)}{d^{2k+1} 2^{2k+2m} (k!)^2}}{(k+\frac{3}{2})_m (k+1)_m (2k+1)} \quad (193)$$

Next, we work on the first summation of (193). Substituting (19) into the third summation yields, an expansion of $\gamma(2k+2m+1, dx)$ from Progi 2018, [2]

$$\sum_{k,m=0}^{\infty} \frac{\gamma(2k+2m+1, dx)}{\left(k+\frac{3}{2}\right)_m (k+1)_m (2k+1)} = \sum_{k,m=0}^{\infty} \frac{\frac{(dx)^{2k+2m+1} e^{-dx}}{(2k+2m+1)} \left[\sum_{n=0}^{\infty} \frac{x_2^n}{\left(k+m+\frac{3}{2}\right)_n (k+m+1)_n} + \frac{dx}{2(k+m+1)} \sum_{n=0}^{\infty} \frac{x_2^n}{(k+m+2)_n \left(k+m+\frac{3}{2}\right)_n} \right]}{d^{2k+1} 2^{2k+2m} (k!)^2}}{\left(k+\frac{3}{2}\right)_m (k+1)_m (2k+1)} \quad (194)$$

Which is equivalent with

$$\sum_{k,m=0}^{\infty} \frac{\gamma(2k+2m+1, dx)}{\left(k+\frac{3}{2}\right)_m (k+1)_m (2k+1)} = x e^{-dx} \left[\sum_{k,m,n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{k+m} \left(\frac{1}{2}\right)_k (1)_m (1)_n \frac{x_1^k x_2^m x_2^n}{k! m! n!}}{\left(\frac{3}{2}\right)_{k+m+n} (1)_{k+m+n} \left(\frac{3}{2}\right)_{k+m}} + \frac{dx}{2} \sum_{k,m,n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{k+m} \left(\frac{1}{2}\right)_k (1)_m (1)_n \frac{x_1^k x_2^m x_2^n}{k! m! n!}}{(2)_{k+m+n} \left(\frac{3}{2}\right)_{k+m+n} \left(\frac{3}{2}\right)_{k+m}} \right] \quad (195)$$

Equation (195) can be written efficiently by the help of the Appendix E) as follows
Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see

$$\mathcal{F}_{231}(x; a, d, 0) = \frac{\sum_{k,m=0}^{\infty} \frac{\gamma(2k+2m+1, dx)}{\left(k+\frac{3}{2}\right)_m (k+1)_m (2k+1)}}{C_2(0, d)} = \frac{x e^{-dx} \left[F^{(3)} \left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; 1; 1; \\ \frac{3}{2}, 1, \frac{3}{2}; -; -; -; \end{matrix} ; x_1, x_2, x_2 \right] + \frac{dx}{2} F^{(3)} \left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; 1; 1; \\ 2, \frac{3}{2}, \frac{3}{2}; -; -; -; \end{matrix} ; x_1, x_2, x_2 \right] \right]}{C_2(0, d)} \quad (196)$$

The last summation can also be written by means of the of the Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix E).

$$\frac{1}{2} \sum_{k,m=0}^{\infty} \frac{\gamma(2k+2m+2, dx)}{\left(k+\frac{3}{2}\right)_m \left(k+\frac{3}{2}\right)_m (k+1) (2k+1)} = \frac{1}{2} \sum_{k,m=0}^{\infty} \frac{\frac{(dx)^{2k+2m+2} e^{-dx}}{(2k+2m+2)} \left[\sum_{n=0}^{\infty} \frac{x_2^n}{(k+m+2)_n \left(k+m+\frac{3}{2}\right)_n} + \frac{dx}{2(k+m+\frac{3}{2})} \sum_{n=0}^{\infty} \frac{x_2^n}{\left(k+m+\frac{5}{2}\right)_n (k+m+2)_n} \right]}{d^{2k+1} 2^{2k+2m} (k!)^2}}{\left(k+\frac{3}{2}\right)_m \left(k+\frac{3}{2}\right)_m (k+1) (2k+1)} \quad (197)$$

Which is equivalent with

$$\frac{1}{2} \sum_{k,m=0}^{\infty} \frac{\frac{\gamma(2k+2m+2, dx)}{d^{2k+1} 2^{2k+2m} (k!)^2}}{(k+2)_m \left(k+\frac{3}{2}\right)_m (k+1)(2k+1)} = \frac{dx^2 e^{-dx}}{4} \left[\sum_{k,m,n=0}^{\infty} \frac{(1)_{k+m} \left(\frac{1}{2}\right)_k (1)_m (1)_n \frac{x_1^k x_2^m x_3^n}{k! m! n!}}{(2)_{k+m+n} \left(\frac{3}{2}\right)_{k+m+n} (2)_{k+m}} + \frac{dx}{3} \sum_{k,m,n=0}^{\infty} \frac{(1)_{k+m} \left(\frac{1}{2}\right)_k (1)_m (1)_n \frac{x_1^k x_2^m x_3^n}{k! m! n!}}{\left(\frac{5}{2}\right)_{k+m+n} (2)_{k+m+n} (2)_{k+m}} \right] \quad (198)$$

Equation (198) can be written efficiently by the help of the Appendix E) as follows
Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see

$$\mathcal{F}_{232}(x; a, d, 0) = \frac{\sum_{k,m=0}^{\infty} \frac{\frac{\gamma(2k+2m+1, dx)}{d^{2k+1} 2^{2k+2m} (k!)^2}}{\left(k+\frac{3}{2}\right)_m (k+1)m(2k+1)}}{C_2(0, d)} = \frac{dx^2 e^{-dx} \left[F^{(3)} \left[\begin{matrix} -::1; -; -; \frac{1}{2}; 1; 1; \\ 2, \frac{3}{2}; 2; -; -; -; -; \end{matrix} ; x_1, x_2, x_2 \right] + \frac{dx}{3} F^{(3)} \left[\begin{matrix} -::1; -; -; \frac{1}{2}; 1; 1; \\ 5, 2; 2; -; -; -; -; \end{matrix} ; x_1, x_2, x_2 \right] \right]}{4C_2(0, d)} \quad (199)$$

Finally, substituting (196) and (199) into (193) produces a computationally inefficient implementation of $\mathcal{F}_{23}(x; a, d, 0)$

$$\mathcal{F}_{23}(x; a, d, 0) = \begin{cases} e^{-dx} x \left[\frac{F^{(3)} \left[\begin{matrix} -::\frac{1}{2}; -; -; \frac{1}{2}; 1; 1; \\ 3, 1; \frac{3}{2}; -; -; -; -; \end{matrix} ; x_1, x_2, x_2 \right] + \frac{dx}{2} F^{(3)} \left[\begin{matrix} -::\frac{1}{2}; -; -; \frac{1}{2}; 1; 1; \\ 2, \frac{3}{2}; \frac{3}{2}; -; -; -; -; \end{matrix} ; x_1, x_2, x_2 \right]}{\frac{dx}{4} \left[F^{(3)} \left[\begin{matrix} -::1; -; -; \frac{1}{2}; 1; 1; \\ 2, \frac{3}{2}; 2; -; -; -; -; \end{matrix} ; x_1, x_2, x_2 \right] + \frac{dx}{3} F^{(3)} \left[\begin{matrix} -::1; -; -; \frac{1}{2}; 1; 1; \\ 5, 2; 2; -; -; -; -; \end{matrix} ; x_1, x_2, x_2 \right] \right]} & x > 0 \\ 0, & x < 0 \end{cases} \quad (200)$$

Equation (200) is an original expression not published anywhere else; however, care must be shown in the computation of the Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ (see Appendix F).

The obvious question is as follows. Is (200) the right equation? While numerically we will test this at the numerical, theoretical results section, we can still test analytically for at least one data point, $d = 0$.

$$\mathcal{F}_{23}(x; a, d = 0, 0) = x \frac{F^{(3)} \left[\begin{matrix} -::\frac{1}{2}; -; -; \frac{1}{2}; 1; 1; \\ 3, 1; \frac{3}{2}; -; -; -; -; \end{matrix} ; x_1, 0, 0 \right]}{C_2(0, 0)} \quad (201)$$

Evaluating the Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ [18]-[22] (see Appendix E) at $d = 0$ yields

$$F^{(3)} \left[\begin{matrix} -::\frac{1}{2}; -; -; \frac{1}{2}; 1; 1; \\ 3, 1; \frac{3}{2}; -; -; -; -; \end{matrix} ; x_1, x_2, x_2 \right] \bigg|_{d=0} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k \frac{x_1^k}{k!}}{\left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_k (1)_k} = {}_2F_3 \left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}, 1; x_1 \right) \quad (202)$$

Substituting (202) into (200) yields

$$\mathcal{F}_{\text{GBessel2}}(x; a, 0, 0) = x \frac{{}_2F_3 \left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}, 1; x_1 \right)}{C_2(0, 0) \equiv \pi} \quad (203)$$

Equation (203) is identical to (95); thus, completing the proof for $d = 0$.

This concludes the derivations of the special case for $d \neq 0$ and $p = 0$ via the inefficient implementation of $\mathcal{F}_{23}(x; a, d, 0)$.

recursively based on the recursive expansions of the incomplete gamma function; however, this approach is not as accurate as the expansion via the modified generalized the Kampé de Fériet double hypergeometric series developed by Progrri 2019.

From (138) we have,

11 Appendix D: Recursive Implementation of $\mathcal{F}_{22}(x; a, d, p)$

The function $\mathcal{F}_{22}(x; a, d, p)$ can also be implemented

$$\mathcal{F}_{22}(x; a, d, p) = \frac{(-1)^p \sum_{k=0}^{\infty} \frac{[\psi(k+1) + \psi(p+k+1)] \gamma(2p+2k+1, dx)}{2^{2k} (k!)^2}}{2^{2p} C_2(p, d)} = \frac{(-1)^p [\psi(1) + \psi(p+1)] \gamma(2p+1, dx)}{2^{2p} d^{2p+1}} + \sum_{k=1}^{\infty} \frac{[\psi(k+1) + \psi(p+k+1)] \gamma(2p+2k+1, dx)}{2^{2k} k! (p+k)!} \quad (204)$$

Where, here we use the continued fraction expansion of the incomplete gamma function (see Proгри 2018 [2], (1))

$$\gamma(2p_1, dx) = \frac{x^{2p_1} d^{2p_1} e^{-dx}}{2p_1} \Phi(1, 2p_2; dx) \quad (205)$$

$$\gamma(\alpha, dx) = \frac{x^\alpha d^\alpha e^{-dx}}{\alpha \equiv 2p+2k+1} \Phi(1, \alpha+1; dx) \quad (206)$$

Employing the continued fraction expansion for $\gamma(2p_1, dx)$ (205), and $\gamma(\alpha, dx)$ (206) we have,

$$\mathcal{F}_{22}(x; a, d, p) = \frac{(-1)^p e^{-dx} [\psi(1) + \psi(p+1)] \frac{x^{2p+1}}{2p+1} \Phi(1, 2+2p; dx) + x^{2p} \sum_{k=1}^{\infty} \frac{[\psi(k+1) + \psi(p+k+1)] \frac{x^k}{k!} \Phi(1, \alpha+1; dx)}{k!(p+k)!}}{C_2(p, d)} \quad (207)$$

Equation (206) can be computed recursively, $\forall k = 1, 2, \dots$

$$\gamma(\alpha, dx) = 2p_{2k} \gamma[2p_{2k}, dx] - (dx)^{2p_{2k}} e^{-dx} \quad (208)$$

Substituting (205) and (208) into (204) produces the final expression of $\mathcal{F}_{22}(x; a, d, p)$

$$x\Phi(1, \alpha+1; dx) = \begin{cases} x & d = 0 \\ \frac{\alpha[\Phi(1, \alpha; dx) - 1]}{d} & d \neq 0 \end{cases} \quad (209)$$

$$\mathcal{F}_{22}(x; a, d, p) = \frac{\frac{(-1)^p x^{2p} e^{-dx}}{2^{p+1} p_1 \Gamma(p_2)} \left\{ [\psi(1) + \psi(p+1)] \Phi(1, 2p_2; dx) + 2p_1 \sum_{k=1}^{\infty} \left[\frac{1}{k} + \psi(k) + \frac{1}{p+k} + \psi(p+k) \right] \frac{(p_1)_k}{(p_3)_k} \frac{x^k}{k!} \left[\frac{x[\Phi(1, \alpha; dx) - 1]}{d} - \frac{[\Phi(1, 1+2(p+k); dx) - 1]}{d} \right] \right\}}{C_2(p, d)} \quad (210)$$

Equation (210) is an original algorithm for computation of $\mathcal{F}_{22}(x; a, d, p)$ but it is several orders of magnitude less accurate than the direct computation via (142). However, in computational performance (210) and (142) are about the same.

hypergeometric series $F^{(3)}[x, y, z]$ [18]-[22]; hence, it is appropriate to introduce this function here and provide a computationally efficient algorithm.

The Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ is given by

12 Appendix E: The Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$

It appears that the cdf of the Bessel function distribution of the second kind involves the use of Srivastava's triple

$$F^{(3)}[x, y, z] \equiv F^{(3)} \left[\begin{matrix} (A) :: (B); (B'); (B''); (C); (C'); (C''); \\ (E) :: (G); (G'); (G''); (H); (H'); (H''); \end{matrix} ; x, y, z \right] = \sum_{l, m, n=0}^{\infty} \Lambda(l, m, n) \frac{x^l y^m z^n}{l! m! n!} \quad (211)$$

where $\Lambda(l, m, n)$ is given by

$$\Lambda(l, m, n) \equiv \frac{((A_a)_{l+m+n})}{((E_e)_{l+m+n})} \cdot \frac{((B_b; B'_b; B''_b))_{l+m; m+n; l+n}}{((G_g; G'_g; G''_g))_{l+m; m+n; l+n}} \cdot \frac{((C_c; C'_c; C''_c))_{l; m; n}}{((H_h; H'_h; H''_h))_{l; m; n}} \quad (212)$$

where $((B_b; B'_b; B''_b))_{l; m; n}$ denotes a triple generalized Pochhammer symbol as defined by

$$((B_b; B'_b; B''_b))_{l; m; n} = ((B_b))_l ((B'_b))_m ((B''_b))_n \quad (213)$$

where $((a))_k$ denotes a single generalized Pochhammer symbol as given by

$$((A_a))_k = \prod_{j=1}^a (A_j)_k = (A_1)_k (A_2)_k \cdots (A_a)_k \quad (214)$$

An efficient implementation of

$$F^{(3)}[x, y, z] = \sum_{l=0}^{\infty} \frac{((C_c))_l}{((H_h))_l} \frac{x^l}{l!} \sum_{m=0}^{\infty} \frac{((B_b))_{l+m}}{((G_g))_{l+m}} \frac{((C'_c))_m}{((H'_h))_m} \frac{y^m}{m!} \sum_{n=0}^{\infty} \frac{((A_a))_{l+m+n}}{((E_e))_{l+m+n}} \frac{((B'_b))_{m+n}}{((G'_g))_{m+n}} \frac{((B''_b))_{l+n}}{((G''_g))_{l+n}} \frac{((C''_c))_n}{((H''_h))_n} \frac{z^n}{n!} \quad (215)$$

If we set $x = 0$ then the $F^{(3)}[x, y, z]$ is reduced to a Kampé de Fériet double hypergeometric series as follows

$$F^{(3)}[0, y, z] = \sum_{m=0}^{\infty} \frac{((B_b))_m}{((G_g))_m} \frac{((C'_c))_m}{((H'_h))_m} \frac{y^m}{m!} \sum_{n=0}^{\infty} \frac{((A_a))_{m+n}}{((E_e))_{m+n}} \frac{((B'_b))_{m+n}}{((G'_g))_{m+n}} \frac{((B''_b))_n}{((G''_g))_n} \frac{((C''_c))_n}{((H''_h))_n} \frac{z^n}{n!} \quad (216)$$

Which can be written in compact form as

$$F^{(3)}[0, y, z] = F_{e,g':g,h':g'',h''}^{a,b':b,c';b'',c''} \left[\begin{matrix} (A)_a, (B')_{b'}, (B)_{b'}, (C')_{c'}, (B'')_{b''}, (C'')_{c''}; \\ (E)_e, (G')_{g'}, (G)_{g'}, (H')_{h'}, (G'')_{g''}, (H'')_{h''}; y, z \end{matrix} \right] \quad (217)$$

This concludes the discussion on the Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$. Next we discuss a super-efficient implementation of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ based on the a supper efficient implementation algorithm of the Kampé de Fériet double hypergeometric series developed by Progi 2018 [2], Appendix A.

13 Appendix F: A Super-Efficient Implementation of the Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$

The main purpose of this section is to develop a super efficient

$$F^{(3)}[x, y, z] = \sum_{l=0}^{\infty} \frac{((C_c))_l}{((H_h))_l} \frac{x^l}{l!} \sum_{m=0}^{\infty} \frac{((B_b))_{l+m}}{((G_g))_{l+m}} \frac{((C'_{c'}))_m}{((H'_{h'}))_m} \frac{y^m}{m!} \sum_{n=0}^{\infty} \frac{((A_a))_{l+m+n}}{((E_e))_{l+m+n}} \cdot \frac{((B'_{b'}))_{m+n}}{((G'_{g'}))_{m+n}} \cdot \frac{((C''_{c''}))_{l+n}}{((H''_{h''}))_{l+n}} \cdot \frac{((C'_{c'}))_n}{((H'_{h'}))_n} \frac{z^n}{n!} \quad (218)$$

A super-efficient implementation of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[23] should be able to take advantage of the super-fast implementation of the generalized hypergeometric function [27], [31]; i.e., reduce the computation of the Srivastava's triple hypergeometric series

$${}_aF_c \left[\begin{matrix} (A)_a \\ (C)_c \end{matrix} ; x \right] = {}_aF_c \left[\begin{matrix} A_1, \dots, A_a \\ C_1, \dots, C_c \end{matrix} ; x \right] = {}_aF_c[A_1, \dots, A_a; C_1, \dots, C_c; x] = \sum_{m=0}^{\infty} \frac{((A_a))_m}{((C_c))_m} \frac{x^m}{m!} = \sum_{m=0}^{\infty} \frac{(A_1)_m \dots (A_a)_m}{(C_1)_m \dots (C_c)_m} \frac{x^m}{m!} \quad (219)$$

Employing a well-known identity of the Pochhammer symbol [29] into the formulation of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[23] the latter one can be written as

$$F^{(3)}[x, y, z] = \sum_{l,m,n \geq 0}^{\infty} \frac{((C_c))_l ((B'_{b'}))_l}{((H_h))_l ((G'_{g'}))_l} \frac{x^l}{l!} \frac{((B_b))_{l+m} ((A_a))_{l+m} ((C'_{c'}))_m ((B'_{b'}))_m}{((G_g))_{l+m} ((E_e))_{l+m} ((H'_{h'}))_m ((G'_{g'}))_m} \frac{y^m}{m!} \frac{((A_a^{lm}))_n ((B'_{b'}^{lm}))_n ((B'_{b'}^{lm}))_n ((C'_{c'}^{lm}))_n}{((E_e^{lm}))_n ((G'_{g'}^{lm}))_n ((G'_{g'}^{lm}))_n ((H'_{h'}^{lm}))_n} \frac{z^n}{n!} \quad (220)$$

Where the following notation is used

$$A_a^{lm} = A_a + l + m \quad (221)$$

$$B_{b'}^{lm} = B_{b'} + m \quad (222)$$

Equation (219) can be split into the product of three summations

$$F^{(3)}[x, y, z] = \sum_{l \geq 0}^{\infty} \frac{((C_c))_l ((B'_{b'}))_l}{((H_h))_l ((G'_{g'}))_l} \sum_{m \geq 0}^{\infty} \frac{((B_b))_{l+m} ((A_a))_{l+m} ((C'_{c'}))_m ((B'_{b'}))_m}{((G_g))_{l+m} ((E_e))_{l+m} ((H'_{h'}))_m ((G'_{g'}))_m} \sum_{n \geq 0}^{\infty} \frac{((A_a^{lm}))_n ((B'_{b'}^{lm}))_n ((B'_{b'}^{lm}))_n ((C'_{c'}^{lm}))_n}{((E_e^{lm}))_n ((G'_{g'}^{lm}))_n ((G'_{g'}^{lm}))_n ((H'_{h'}^{lm}))_n} \frac{z^n}{n!} \frac{y^m}{m!} \frac{x^l}{l!} \quad (223)$$

The inner hypergeometric series can be written as

$${}_aF_b \left[\begin{matrix} (A^{lm})_a (B'^{lm})_{b'} (B'^{lm})_{b''} (C'')_{c''} \\ (E^{lm})_e (G'^{lm})_{g'} (G'^{lm})_{g''} (H'')_{h''} \end{matrix} ; z \right] = \sum_{n \geq 0}^{\infty} \frac{((A_a^{lm}))_n ((B'_{b'}^{lm}))_n ((B'_{b'}^{lm}))_n ((C'_{c'}^{lm}))_n}{((E_e^{lm}))_n ((G'_{g'}^{lm}))_n ((G'_{g'}^{lm}))_n ((H'_{h'}^{lm}))_n} \frac{z^n}{n!} \quad (224)$$

implementation algorithm of the Srivastava's Triple Hypergeometric Series $F^{(3)}[x, y, z]$ [18]-[23] that accomplishes the following three criteria: it is numerically very stable, computationally very efficient, and numerically very accurate; i.e., the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[23] function numerical evaluation in MATLAB is no longer impacted by the numerical stability, efficiency, and accuracy. I hope that the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[23] function, presented in this appendix, can be implemented in MATLAB in like manner as the generalized hypergeometric series function `hypergeom(a,b,z)` [30].

The Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[23] is defined as

$F^{(3)}[x, y, z]$ [18]-[23] into the computation of a double summation.

For that we need to write the definition of the generalized hypergeometric series as follows [31]

Where

$$\tilde{a} = a, b', b'', c'' \quad \text{and} \quad \tilde{b} = e, g', g'', h'' \quad (225)$$

Substituting (225) into (224) yields a 2-dimensional supper fast and accurate summation of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[23] as follows

$$F^{(3)}[x, y, z] = \sum_{l \geq 0} \frac{((A_a))_l ((B_b))_l ((B_{b''}))_l ((C_c))_l}{((E_e))_l ((G_g))_l ((G_{g''}))_l ((H_h))_l} \sum_{m \geq 0} \frac{((A_a))_m ((B_b))_m ((B_{b'})_m (C_{c'})_m}{((E_e))_m ((G_g))_m ((G_{g'})_m (H_{h'})_m} \tilde{a} \tilde{b} \left[\frac{(A^{lm})_a (B^{lm})_{b'} (B^{lm})_{b''} (C^{lm})_{c''}}{(E^{lm})_e (G^{lm})_{g'} (G^{lm})_{g''} (H^{lm})_{h''}} \right] \frac{y^m x^l}{m! l!} \quad (226)$$

Next, we need a computationally efficient algorithm of

$$((A_a))_m, ((B_b))_m, ((B_{b'})_m), ((C_{c'})_m), ((E_e))_m, ((G_g))_m, ((G_{g'})_m), \text{ and } (H_{h'})_m.$$

Let us consider for example the computation of $((A_a))_m$

$$((A_a))_m = \prod_{i=1}^a (A_i)_m = \prod_{i=1}^a \frac{\Gamma(A_i^{lm})}{\Gamma(A_i)} \quad (227)$$

For large values of $m \gg 1$ and $A_i \gg 1$, $\Gamma(A_i^{lm}) \rightarrow \infty$; hence, an efficient computation of $((A_a))_m / ((E_e))_m$ is required.

From the definition of the Pochhammer symbols $(A_i)_{l+m}$ [29] we have the following

$$(A_i)_m = \prod_{k=0}^{m-1} (A_i^k) \quad (228)$$

Substituting (228) into (227) yields

$$((A_a))_m = \prod_{i=1}^a \prod_{k=0}^{m-1} (A_i^k) = \prod_{k=0}^{m-1} \prod_{i=1}^a (A_i^k) \quad (229)$$

Before we can perform the computation of (226) can we

Similarly we can write exactly the same formulas for $((B_b))_m, ((B_{b'})_m), ((C_{c'})_m), ((E_e))_m, ((G_g))_m, ((G_{g'})_m$, and $(H_{h'})_m$ as follows

$$((B_b))_m = \prod_{l=1}^b \prod_{k=0}^{m-1} (B_b^{lk}) = \prod_{k=0}^{m-1} \prod_{l=1}^b (B_b^{lk}) \quad (230)$$

$$((B_{b'}))_m = \prod_{l=1}^{b'} \prod_{k=0}^{m-1} (B_{b'}^{lk}) = \prod_{k=0}^{m-1} \prod_{l=1}^{b'} (B_{b'}^{lk}) \quad (231)$$

$$((C_{c'}))_m = \prod_{l=1}^{c'} \prod_{k=0}^{m-1} (C_{c'}^{lk}) = \prod_{k=0}^{m-1} \prod_{l=1}^{c'} (C_{c'}^{lk}) \quad (232)$$

$$((E_e))_m = \prod_{l=1}^e \prod_{k=0}^{m-1} (E_e^{lk}) = \prod_{k=0}^{m-1} \prod_{l=1}^e (E_e^{lk}) \quad (233)$$

$$((G_g))_m = \prod_{l=1}^g \prod_{k=0}^{m-1} (G_g^{lk}) = \prod_{k=0}^{m-1} \prod_{l=1}^g (G_g^{lk}) \quad (234)$$

$$((G_{g'}))_m = \prod_{l=1}^{g'} \prod_{k=0}^{m-1} (G_{g'}^{lk}) = \prod_{k=0}^{m-1} \prod_{l=1}^{g'} (G_{g'}^{lk}) \quad (235)$$

$$((H_{h'}))_m = \prod_{l=1}^{h'} \prod_{k=0}^{m-1} (H_{h'}^{lk}) = \prod_{k=0}^{m-1} \prod_{l=1}^{h'} (H_{h'}^{lk}) \quad (236)$$

write the following term as:

$$\frac{((A_a))_m ((B_b))_m ((B_{b'}))_m (C_{c'})_m}{((E_e))_m ((G_g))_m ((G_{g'}))_m (H_{h'})_m} = \frac{\prod_{k=0}^{m-1} \prod_{i=1}^a (A_i^k) \prod_{k=0}^{m-1} \prod_{l=1}^b (B_b^{lk}) \prod_{k=0}^{m-1} \prod_{l=1}^{b'} (B_{b'}^{lk}) \prod_{k=0}^{m-1} \prod_{l=1}^{c'} (C_{c'}^{lk})}{\prod_{k=0}^{m-1} \prod_{l=1}^e (E_e^{lk}) \prod_{k=0}^{m-1} \prod_{l=1}^g (G_g^{lk}) \prod_{k=0}^{m-1} \prod_{l=1}^{g'} (G_{g'}^{lk}) \prod_{k=0}^{m-1} \prod_{l=1}^{h'} (H_{h'}^{lk})} \quad (237)$$

Which can be further simplified as follows:

$$\frac{((A_a))_m ((B_b))_m ((B_{b'}))_m (C_{c'})_m}{((E_e))_m ((G_g))_m ((G_{g'}))_m (H_{h'})_m} = \prod_{k=0}^{m-1} \frac{\prod_{i=1}^a (A_i^k) \prod_{l=1}^b (B_b^{lk}) \prod_{l=1}^{b'} (B_{b'}^{lk}) \prod_{l=1}^{c'} (C_{c'}^{lk})}{\prod_{l=1}^e (E_e^{lk}) \prod_{l=1}^g (G_g^{lk}) \prod_{l=1}^{g'} (G_{g'}^{lk}) \prod_{l=1}^{h'} (H_{h'}^{lk})} \quad (238)$$

If we define q_k as below

$$q_k = \frac{\prod_{i=1}^a (A_i^k) \prod_{l=1}^b (B_b^{lk}) \prod_{l=1}^{b'} (B_{b'}^{lk}) \prod_{l=1}^{c'} (C_{c'}^{lk})}{\prod_{l=1}^e (E_e^{lk}) \prod_{l=1}^g (G_g^{lk}) \prod_{l=1}^{g'} (G_{g'}^{lk}) \prod_{l=1}^{h'} (H_{h'}^{lk})} \quad (239)$$

Similarly, we can write

$$p_k = \frac{\prod_{i=1}^a (A_i^k) \prod_{l=1}^b (B_b^{lk}) \prod_{l=1}^{b'} (B_{b'}^{lk}) \prod_{l=1}^{c'} (C_{c'}^{lk})}{\prod_{l=1}^e (E_e^{lk}) \prod_{l=1}^g (G_g^{lk}) \prod_{l=1}^{g'} (G_{g'}^{lk}) \prod_{l=1}^{h'} (H_{h'}^{lk})} \quad (240)$$

Then we can eliminate the singularity in the computation of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[23] as follows

$$\frac{((A_a))_m ((B_b))_m ((B_{b'}))_m (C_{c'})_m}{((E_e))_m ((G_g))_m ((G_{g'}))_m (H_{h'})_m} = \prod_{k=0}^{m-1} q_k \quad (241)$$

Next, by defining

$$Q_m = \prod_{k=0}^{m-1} q_k; \quad (243)$$

$$Q_m = \prod_{k=0}^{m-1} q_k; \quad \tilde{Q}_m(y) = \prod_{k=1}^m \frac{q_{k-1} y}{k} \quad (244)$$

$$P_l = \prod_{k=0}^{l-1} p_k; \quad (245)$$

$$P_l = \prod_{k=0}^{l-1} p_k; \quad \tilde{P}_l(x) = \prod_{k=1}^l \frac{p_{k-1} x}{k} \quad (246)$$

By substituting (244) and (246) into (226), yields

$$F^{(3)}[x, y, z] = \sum_{l,m \geq 0} \tilde{P}_l(x) \tilde{Q}_m(y) {}_aF_b \left[\begin{matrix} (A^{lm})_a (B'^m)_{b'} (B''^l)_{b''} (C'')_{c''} \\ (E^{lm})_e (G'^m)_{g'} (G''^l)_{g''} (H'')_{h''} \end{matrix} \right] \quad (247)$$

Finally, let us try to re-compute (247) or (225) using the technique that we just demonstrated in (223).

$$F^{(3)}[x, y, z] = \sum_{l,m,n \geq 0} \frac{(A^l_a)_m (B^l_b)_m (B'^l_{b'})_m (C^l_{c'})_m}{(E^l_e)_m (G^l_{g'})_m (G''^l_{g''})_m (H^l_{h'})_m} x^l \frac{(A^l_a)_m (B^l_b)_m (B'^l_{b'})_m (C^l_{c'})_m}{(E^l_e)_m (G^l_{g'})_m (G''^l_{g''})_m (H^l_{h'})_m} y^m \frac{(A^l_a)_m (B^l_b)_m (B'^l_{b'})_m (C^l_{c'})_m}{(E^l_e)_m (G^l_{g'})_m (G''^l_{g''})_m (H^l_{h'})_m} z^n \quad (248)$$

This concludes the super-efficient computation of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[23] function. It is numerically very stable, computationally very efficient, and extremely accurate. I have performed an efficient

implementation of the Srivastava's triple hypergeometric series $F^{(3)}[x, y, z]$ [18]-[23] function in MATLAB in like manner as the generalized hypergeometric series function `hypergeom(a,b,z)` [30].

ⁱ Dr. Progi has implemented the Kampé de Fériet function (or double hypergeometric series [1]-[16]) in MATLAB in like manner as the generalized hypergeometric series function `MATLAB implementation hypergeom(a,b,z)` [30]. This function is called `k=kamdefer(a,b,z,op)`; where *a* and *b* are 3-element cell vectors each, *z* is either a vector/matrix of two rows, *op* is an optional vector of two elements. In the current implementation *op*(1) can only contain four values, 1:4 corresponding to four implementation options of the Kampé de Fériet function; *op*(2) is the maximum number of terms in each summation. If the user calls `k=kamdefer(a,b,z)` then *op*(1)=4, and *op*(2)=30. The MATLAB function `kamdefer(a,b,z)` is proprietary information of Giftet Inc. and it is copyrighted by Giftet Inc. 2018.

ⁱⁱ Dr. Progi has implemented the Srivastava's triple hypergeometric series function [18]-[22] in MATLAB in like manner as the generalized hypergeometric series function `MATLAB implementation hypergeom(a,b,z)` [30]. This function is called `s=srivastava(a,b,z,op)`; where *a* and *b* are 7-element cell vectors each, *z* is either a vector/matrix of two rows, *op* is an optional vector of two elements. In the current implementation *op*(1) can only contain four values, 1:4 corresponding to three implementation options of the Srivastava's function; *op*(2) is the maximum number of terms in each summation. If the user calls `s=srivastava(a,b,z)` then *op*(1)=4, and *op*(2)=30. The MATLAB function `srivastava(a,b,z)` is proprietary information of Giftet Inc. and it is copyrighted by Giftet Inc. 2019.

ⁱⁱⁱ Although $K_p(|x|) = K_{-p}(|x|)$ for $p \leq -1/2$; however, in a future publication we will show the required modifications of the Generalized Bessel distribution form as given by Progi 2016 [1] and 2018 [2] to produce a valid pdf and cdf. The main reason for this is that the material of this journal paper is challenging enough as is. These set of journal papers will set the stage for this future journal paper.

^{iv} In implementation we will use Progi implementation of the hypergeometric function `phypergeom(a,b,z,op)`, as it is as shown in this journal paper, about twenty-five times faster than the MATLAB `hypergeom(a,b,z)`. This function is proprietary information of Giftet Inc and it is part of Giftet Inc. Indoor Geolocation Systems—Theory and Applications [17] library.

^v This result cannot be obtained from (62) by setting $p = 0$.

^{vi} Ditto iv.

^{vii} The function $\mathcal{F}_{21}(x; a, d, 0)$ can also be implemented via a recursive algorithm. Substituting (109) into the first summation of (107) yields,

$$\mathcal{F}_{21}(x; a, d, 0) = \frac{\psi(1) \frac{\gamma(1, dx)}{d} + \sum_{k=1}^{\infty} \frac{\psi(k+1) \frac{\gamma(2k+1, dx)}{d^{2k+1}}}{C_2(0, d)} \quad (1)$$

Where

$$\gamma(1, dx) = dx e^{-dx} \left[\Phi(1, 2; dx) \equiv \sum_{n=0}^{\infty} \frac{(1)_n (dx)^n}{(2)_n n!} \right] \quad (2)$$

$$\gamma(2k+1, dx) = \frac{x^{2k+1} d^{2k+1} e^{-dx}}{2^{2k+1}} \Phi(1, 2+2k; dx) \quad (3)$$

Employing the expansion for $\gamma(1, dx)$, and $\gamma(2k+1, dx)$ we have,

$$\mathcal{F}_{21}(x; a, d, 0) = \frac{\psi(1) x e^{-dx} \Phi(1, 2; dx) + e^{-dx} x \sum_{k=1}^{\infty} \left[\frac{1}{k!} \psi(k) \right] \frac{x^{2k} \Phi(1, 2+2k; dx)}{2^{2k} (k!)^2}}{C_2(0, d)} \quad (4)$$

Equation (3) can be computed recursively, $\forall k = 1, 2, \dots$

$$\gamma(2k+1, dx) = 2k\gamma(2k, dx) - (dx)^{2k} e^{-dx} \quad (5)$$

$$\frac{x\Phi(1, 2+2k; dx)}{(1+2k)} = \begin{cases} \frac{x}{1+2k} & d = 0 \\ \frac{\Phi(1, 1+2k; dx) - 1}{d} & d \neq 0 \end{cases} \quad (6)$$

Substituting (2) and (6) into (1) produces the final expression of

$$\mathcal{F}_{21}(x; a, d, 0) = \frac{e^{-dx} \left\{ \psi(1) x \Phi(1, 2; dx) + \sum_{k=1}^{\infty} \left[\frac{1}{k!} \psi(k) \right] \frac{x^k \Phi(1, 2+2k; dx)}{(1+2k)} \right\}}{C_2(0, d)} \quad (7)$$

This recursive approach is fast; but, it is not nearly as accurate as the expansion by means of generalized Kampé de Fériet function.

^{viii} There are a few differences between the generalized Kampé de Fériet function and the normal Kampé de Fériet function [11]-[13]. In the generalized Kampé de Fériet function the outer sum terminates and the Pochhammer exponent may be negative vs. in the normal Kampé de Fériet function [11]-[13] the outer sum goes to infinity and the Pochhammer exponent is always positive. More discussion about the similarities and differences of the will be published in a separate journal paper.

^{ix} Option 1 fails after 50 terms; hence, option 1 is numerically unstable!

^x Option 1 fails after 50 terms; hence, option 1 is numerically unstable!

^{xi} Warning: Infinite or Not-a-Number value encountered at $= 0$.

^{xii} Warning: Reached the limit on the maximum number of intervals in use. Approximate bound on error is 3.9e-06. The integral may not exist, or it may be difficult to approximate numerically to the requested accuracy.

Warning: Infinite or Not-a-Number value encountered.

^{xiii} Ditto xi.

^{xiv} Ditto xii.

^{xv} Exactly the same result can be obtained if (159) was used with Option 3 of the Kampé de Fériet function.

^{xvi} Ditto xi.

^{xvii} Ditto xii.

^{xviii} There are a few differences between the generalized Kampé de Fériet function and the normal Kampé de Fériet function [11]-[13]. In the generalized Kampé de Fériet function the outer sum terminates and the Pochhammer exponent may be negative vs. in the normal Kampé de Fériet function [11]-[13] the outer sum goes to infinity and the Pochhammer exponent is always positive. More discussion about the similarities and differences of the will be published in a separate journal paper.

^{xix} We employ the shorthand notation $p_{2k} \equiv p + k$.

^{xx} I do not believe that I have seen this symbol (or notation rather) introduced anywhere else; hence, this symbol (or notation rather) that denotes the product of three generalized Pochhammer symbols is introduced here for the first time.