

Research Article

Advanced Anti-Jam Indoor Adaptive GNSS Signal Acquisition: Part 1, Normal Distribution—Theory and Simulations

Ilir F. Proгри

Giftet Inc., 5 Euclid Ave. #3, Worcester, MA 01610, USA

ORCID: 0000-0001-5197-1278

Correspondence should be addressed to Ilir Proгри; iproгри@giftet.com

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Abstract—In this paper an advanced anti-jam indoor adaptive GNSS signal acquisition and tracking algorithm is considered.

Initially, we were able to determine that double-dwell structure (DDS) reduces the processing time penalty caused by false alarm. Nevertheless, the DDS is still vulnerable to interference and jamming.

In order to determine a suitable advanced anti-jam indoor adaptive GNSS signal acquisition and tracking algorithm for DDS we first perform the Bayesian parameter estimation; i.e., we analytically compute the posterior Bayes probability density function (pdf) and cumulative distribution function (cdf) by applying the Bayes theorem in three steps.

First, we compute the *complex signal distribution* and *complex matrix variate signal distribution*. This is an original new result never published before.

Second, we provide an introduction of the *equivalence of the maximum likelihood (ML) GNSS parameter estimation based MLE with the Bayes Parameter Estimation* (see Appendix A).

Third, under the assumption of interference as normal distribution, in both the scalar case and complex matrix variate cases we observe that the complex matrix variate Bayesian posterior pdf or cdf is invariant of the observation data or is identical to the prior *complex matrix variate signal distribution* model. This is an original new and very powerful result never published before which leads to the equivalence of the *MLE with the Bayes Parameter Estimation* (see Appendix A).

Why this result is so powerful is because up until now we never had a complete theoretical validation of our GNSS receiver design based on either autocorrelation or cross-correlation properties since, the complex matrix variate Bayesian posterior pdf or cdf is invariant of the observation data or is identical to the prior *complex matrix variate signal distribution* model.

Simulation results illustrate that *the MLE receiver exhibits a 75 dB SINR improvement performance against a Cross-Correlator CC receiver* even under extreme jamming conditions.

Index Terms—Adaptive GNSS signal acquisition; Weak GNSS signal; Constant false alarm rate (CFAR); Double-dwell structure (DDS); Bayesian parameter estimation; Maximum-likelihood estimation, Normal distribution, Complex signal distribution; Complex matrix variate signal distribution; Complex normal matrix variate interference distribution.

1 Introduction

Bayesian probability is an interpretation of the concept of probability, in which, instead of frequency or propensity of some phenomenon, assigned probabilities represent states of knowledge or belief [1].

The Bayesian interpretation of probability can be seen as an extension of propositional logic that enables reasoning with hypotheses, i.e., the propositions whose truth or falsity is uncertain. In the Bayesian view, a probability is assigned to a hypothesis, whereas under frequentist inference, a hypothesis is typically tested without being assigned a probability [1].

Bayesian probability belongs to the category of evidential probabilities; to evaluate the probability of a hypothesis, the Bayesian probabilist specifies some prior probability, which is then updated to a posterior probability in the light of new, relevant data (evidence). The Bayesian interpretation provides a standard set of procedures and formulae to perform this calculation [1].

The term “Bayesian” derives from the 18th century mathematician and theologian Thomas Bayes, who provided the first mathematical treatment of a non-trivial problem of Bayesian inference. Mathematician Pierre-Simon Laplace pioneered and popularized what is now called Bayesian probability [1].

Broadly speaking, there are two views on Bayesian probability that interpret the probability concept in different ways. According to the objectivist view, the rules of Bayesian statistics can be justified by requirements of rationality and consistency and interpreted as an extension of logic. According to the subjectivist view, probability quantifies a “personal belief” [1].

My first impression when I wrote this paper was to do what

I said I was going to do which is produce a Bayesian parameter estimation algorithm combined with Markov Chain, Monte Carlo global search [3]-[6] to defeat interference and/or jamming and significantly improve the performance of a double-dwell structure; i.e., focus more or less on the mechanics of the problem which is more or less what my co-author (Matthew Bromberg and I have done in the past) [3]-[6].

What was the main weakness with this approach; i.e., with the approach that we took in the past?

First, I believe that our approach in the past lacked tremendous insights. Hence, I changed the focus of the approach from focusing on the mechanics of the problem to focusing on obtaining tremendous insights and then evaluating the mechanics and the results of the past approaches [3]-[6].

Second, should the assumptions of the noise distribution change, the previous approach will require the researcher, scholar, scientists under frequentist inference to reproduce new mechanics and interpret the results solely from the point of view of the mechanics of the problem. On the other hand, the new approach presented in this paper, should the assumptions of the noise distribution change, the new approach will provide tremendous insights even before the mechanics have been fully understood and developed. This is even more important when the mechanics of the problem require very difficult numerical computations resulting from extremely difficult analytical derivations.

The main purpose of this paper is to provide insights as to whether a Bayesian parameter estimation algorithm combined with Markov Chain, Monte Carlo global search [3]-[6] is a suitable algorithm to improve the performance of a double-dwell structure (DDS) [7] against interference and/or jamming.

The Bayesian parameter estimation consists of producing a posterior probability density function (pdf) for a scalar argument or a posterior complex matrix variate pdf (or

cumulative distribution function (cdf)) based on certain hypothesis or assumptions of prior distributions [8] in our case the noise interference distribution.

In our previous approach of the Bayesian parameter estimation algorithm, we made the assumption that the interference distribution was *Complex Normal (or Gaussian) Matrix Variate Interference Distribution* [3]-[6]. However, we were unable to produce the complex matrix variate Bayesian posterior distribution because it required full integration based on the distribution of the signal model that we did not have. In this paper we have produced the complex matrix variate signal distribution and then we have produced the complex matrix variate posterior distribution based on the assumption (or hypothesis) of *Complex Gaussian (or Normal) Matrix Variate Interference Distribution*.

Future simulation results will validate the theory and re-interpret the simulation results we produced in the past [3]-[6]. Hopefully, we can publish this paper soon. This is really the end of part 1.

In part 2 of this publication, as part of future work, we will show to how produce the complex matrix variate Bayesian posterior density based on the *Complex Matrix Variate Signal Distribution* that we have produced in this paper and either the *Complex Matrix Variate Bessel Interference Distribution* or *Complex Matrix Variate Parabolic Function Interference Distribution*.

This paper is organized as follows: Brief introduction of DDS is discussed first. Signal model and complex matrix variate signal distribution is present second. Complex Bessel or parabolic function interference distribution is depicted third. Complex normal interference distribution is analyzed fourth. Conclusion is given in afterwards along with a list of references. At the end of the paper Appendix A discusses the equivalence of the MLE with the Bayes parameter estimation and Appendix B contains derivation of (167).

2 Brief introduction of DDS

In this section, a brief introduction of DDS in GPS signal acquisition is discussed because of the obvious advantage of DDS lies in reducing the average acquisition time by lowering “penalty” time caused by false alarm (FA) and tracking loss (TL) [7].

DDS consists of a 2-tier GPS signal detection structure that works simultaneously because the probability of a 2-tier

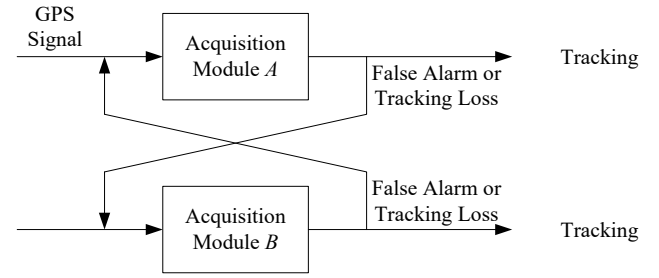


FIGURE 1: Block diagram of GPS signal acquisition based on DDS.

detection structure encountering FA is much smaller than that of a single structure; i.e., one detection structure works effectively even if FA or TL happens in the other structure. The 2-tier detection structures can be combined in parallel or cascaded form.

Using the parallel detection structure, detection module *A* and detection module *B* work simultaneously to search for weak (or highly degraded and/or jammed) GPS signals. If one of the two detection modules (for example module *A*) claims that GPS signal has been detected, the other module (module *B*) exerts as the verification module to test whether FA happen in module *A* [7]. At the same time, module *A* still works to search for possible GPS signals in a GPS receiver. If module *B* finds that FA did not happen, GPS receiver turns to tracking state; if module *B* finds that FA happened in module *A*, GPS receiver can utilize the detection result of module *A*, while it searching [7].

Using the cascade detection structure, detection module *A* searches for weak (or degraded) GPS signals. Before module *A* claims that a weak GPS signal is detected, module *B* stays vacant. Only after module *A* has acquired a GPS signal, module *B* begins to verify [7].

By comparing and contrasting the two combination methods (or subsystems), the hardware resource in module *B* is not utilized. Therefore, parallel method (or subsystem) is preferred in this paper [7].

Figure 1 illustrates the parallel block diagram of a DDS based GPS signal acquisition; i.e., the DDS without interference and/or jamming [7].

The traditional (or conventional) DDS is not a good solution to acquire GPS signal. The unfitness of conventional structure lies in two aspects: considering the discussions on GPS signal pseudorandom code carrier phase domain search in Fig. 1, the carrier frequency domain search was not considered; the cascade form is studied in the existed DDS [7] with coherent accumulation (CAC) structure taken as the first tier and non-

coherent accumulation (NCAC) as the second one. What is the main issue with the conventional DDS? Since, the first tier of the conventional DDS is CAC then conventional CAC is sensitive to both large Doppler shift and data bit transition.

In order to overcome this issue, in this section, the mixed parallel DDS is proposed which consists of differential coherent accumulation (DCAC) and NCAC, which enables small probability of miss detection and a significant reduction of the average acquisition time [7]; because DCAC based GPS signal acquisition scheme possesses better Doppler shift tolerance than NCAC, which means that the detection statistics of DCAC based acquisition scheme outperforms the ones of NCAC: at huge Doppler shift and due to lack of sensitivity to data bit transition or the carrier phase transition. On the other hand, when encountering small Doppler shift, NCAC based acquisition scheme can generate higher accumulation gain in SNR producing a difference in detection performance as illustrated in Fig. 1 [7].

Due to advantages in different aspects of the detection performance, DCAC and NCAC based GPS signal acquisition schemes are employed in the two modules respectively [7].

In the environment with huge Doppler shift, DCAC module can achieve coarse GPS signal acquisition; afterwards, NCAC module performs accurate acquisition on GPS signal in the condition of small Doppler shift [7].

So far we briefly discussed the main advantages of the mixed parallel DDS. What are the main disadvantages of the mixed parallel DDS? There appear to be *two main disadvantages*.

The *first disadvantage* has to do with the *unequal complexity of the noise model* [7]. If we compare and contrast the noise model of NCAC and CAC with DCAC, they vary significantly both in terms of simplicity vs. complexity in notation (or closed form expression) and numerical computation [7]. The noise models of CAC and NCAC can be described very easily by means of closed form expressions of well-known functions such as a normal distribution, Rayleigh, Rician, Lognormal, etc. [9]-[11]; hence, they can be computed easily by means of commercially available software such as MATLAB 2017a or earlier versions. The description of noise models of DCAC; however, requires significantly more complicated closed-form expressions of PFA or PMD by means of Bessel function distribution models [12]-[14] or parabolic function distribution models [15] whose cdf are currently not well understood because they require the computation of functions such as Kampé de Fériet functions of scalar or matrix argument and

even Jack functions of scalar or matrix argument. The level of expertise required performing the noise modeling and processing for DCAC modules is much higher than for both/either NCAC and/or CAC.

Even if we were able to manage the first disadvantage it is the second disadvantage that is even more problematic. The *second disadvantage* has to do with the *unequal susceptibility against interference and/or jamming* [7], [9]-[28]. With this paper we make the first attempt trying to exploit this *unequal susceptibility against interference and/or jamming* which will produce a revolutionary understanding both in terms of the analytically explained and understood mathematics or notation (or closed form expressions) and in terms of fast, numerically or computationally efficient algorithms. Regardless of the advantages in signal acquisition and innovation in architecture, DDS still remains *unequally* vulnerable to interference and/or jamming. If we have an application that requires both high performance signal acquisition by means of DDS and at the same time high performance anti-jam capability then it is *unequally* clear what type of high performance anti-jam algorithm would be suitable for algorithms that employ high performance DDS acquisition algorithms. Under the general assumption that signal acquisition is both/either NCAC and/or CAC DDS [7] we have been able to exploit these noise models and produce fast and efficient anti-jam acquisition algorithms [3]-[6], [11], [22], [28]. On the other hand, under the general assumption that the signal acquisition is DCAC DDS then it is still a mystery or not at all well understood as to what type of fast and efficient anti-jam acquisition algorithms are suitable for DCAC DDS modules.

The obvious question is how do we entirely exploit the *unequal susceptibility* of DDS modules against interference and/or jamming? What type of analytical model do we need to produce to effectively and efficiently assess the *unequal susceptibility* of DDS modules against interference and/or jamming? The obvious choice is the Bayes algorithm because of its unique capability in exploiting a wide range of expertise (or offering unparalleled insights) in constructing the prior and Bayes posterior pdfs/cdfs.

The rest of this paper deals with the mechanics of Bayes algorithm for NCAC or CAC DDS modules. One section of this paper and Progi 2018 [16] in the end of this paper offers some insights as to what necessary and/or sufficient analytical tools are required to employ the Bayes algorithm for DCAC DDS modules.

This concludes a brief introduction of DDS modules advantages and disadvantages. Next, we discuss the signal model and complex matrix variate signal distribution that are required to produce the posterior Bayes pdf/cdf for NCAC or CAC DDS modules.

3 Signal Model

The first step in producing the Bayes algorithm is the formulation of a linearized signal model. There is a perfect analogy between the Bayes algorithm and the Kalman filter algorithm when it comes to algorithms related to signal acquisition, tracking, and state estimation. Both algorithms require: (1) either a linear or non-linear statistical signal model; (2) certain assumptions about the noise or interference covariance matrix; (3). make certain assumptions about the initial conditions.

When Bayes algorithm is married with Motte-Carlo Markov Chain (MCMC) tracking algorithm then the similarities between the Bayes algorithm [3]-[6] and the Kalman filter algorithm [29]-[35], [38]-[42] are exquisite. This might as well be the subject of a future paper or future publications.

When the Bayes algorithm stands by itself; i.e., when Bayes algorithm is based on the complete analytical derivation of Bayes theorem, then Bayes algorithm [3]-[6] is in fact a scaled version of the maximum-likelihood algorithm [22].

The main purpose of the this paper is to produce an understanding from the Bayes theorem by means of constructing a linear signal model and then producing the complex matrix variate signal distribution. Further details of the Bayes theorem and its understanding are discussed later in the paper.

3.1 Linear Signal Model

The statistical linearized DDS signal model of our proposed Bayesian receiver takes the form of [3], [17]

$$\mathbf{Y}(n) = \sum_{k=1}^K \mathbf{a}_k(n) s_k(n) + \mathbf{z}(n) \quad (1)$$

where K is the number of satellite emitters, $\mathbf{a}_k$ is the complex,

$$s_k[\tau_k, f_k, d_k(\tau_k)] = c_k(nT - m_k C - \tau_k) d_k([nT - d_k D - p_k C - \tau_k]_D) e^{j[2\pi L_i(nT - m_k C - \tau_k) + 2\pi f_k(nT - m_k C - \tau_k) + \theta_{k0}]} \quad (8)$$

Or

$$s_k[\tau_k, f_k, d_k(\tau_k)] = c_k(nT - \tau_k) d_k([nT - p_k C - \tau_k]_D) e^{j\theta_{ik}(\tau_k, f_k)} \quad (9)$$

where $\theta_{ik}(\tau_k, f_k)$ is the total phase given by

$$\theta_{ik}(\tau_k, f_k) = \theta_i + \theta_{k0} + \Delta\theta_{ik}(\tau_k, f_k) \quad (10)$$

$M \times 1$ vector channel or steering vector associated with satellite emitter k which is corrupted by multiplicative noise $\varepsilon_*(n)$ as follows

$$\mathbf{a}_k(n) = a_k \varepsilon_{k*}(n) \quad (2)$$

$s_k(n)$ is the transmitted data symbol received at time sample, n , and $\mathbf{z}(n)$ is the residual interference vector received at time sample, n . Our model allows for the possibility of using either polarization diverse or a multiple antennae receiver with M digital feeds, although our techniques do not require receiver diversity.

The GPS C/A-code waveform transmitted from a k th GPS satellite $s_k(t_k)$ is given by Progi 2017 [18]

$$s_k(n, t_k, L_{ik}) = c_k([t_k]_C) d_k(q \equiv [t_k]_D) e^{j(2\pi L_{ik} t_k + \theta_{k0})} \quad (3)$$

where t_k is the time of transmission of the k th GPS signal and is given by [20]

$$t_k \equiv t_r - \zeta_k \cong nT - m_k C - \tau_k \quad (4)$$

t_r is the receiver time and ζ_k is the time of flight, T is the sampling period, $c_k(t_k)$ is pseudorandom code at time t_k , C is pseudorandom sequence repetition period, which for the GPS L1 or L2 [11] C is the C/A code with code repetition sequence period at 1 ms, n, m_k are integers, τ_k is the residual code phase, L_{ik} is the carrier frequency which for GPS is L1 = 1575.42 MHz, L2 = 1227.6 MHz, or L5 = 1176.42 MHz, $d_k([t_k]_D)$ is the data bit transition for satellite k spread by the code $c_k([t_k]_C)$ [18] and D is the period when a data bit transition occurs (for the GPS L1 data case, it is equal to 20 ms), θ_{k0} is some initial carrier phase of the signal; and the index k changes from $\{1, 2, \dots, K\}$. Furthermore $[t_k]_C$ and $[t_k]_D$ are given by [22]

$$[t_k]_C \equiv \left\lfloor \frac{t_k}{C} \right\rfloor = \left\lfloor \frac{nT - \zeta_k}{C} \right\rfloor = \left\lfloor \frac{nT - m_k C - \tau_k}{C} \right\rfloor = \left\lfloor \frac{nT - \tau_k}{C} \right\rfloor \quad (5)$$

$$\begin{aligned} [t_k]_D &\equiv \left\lfloor \frac{t_k}{D} \right\rfloor = \left\lfloor \frac{nT - \zeta_k}{D} \right\rfloor = \left\lfloor \frac{nT - m_k C - \tau_k}{D} \right\rfloor \\ &= \left\lfloor \frac{nT - d_k q C - p_k C - \tau_k}{D} \right\rfloor = \left\lfloor \frac{nT - d_k D - p_k C - \tau_k}{D} \right\rfloor \\ &= \left\lfloor \frac{nT - p_k C - \tau_k}{D} \right\rfloor \end{aligned} \quad (6)$$

$$m_k = d_k q + p_k, \quad p_k < q \text{ integers} \quad (7)$$

Substituting (4) through (7) into (3) then we obtain

$$s_k[\tau_k, f_k, d_k(\tau_k)] = c_k(nT - m_k C - \tau_k) d_k([nT - d_k D - p_k C - \tau_k]_D) e^{j[2\pi L_i(nT - m_k C - \tau_k) + 2\pi f_k(nT - m_k C - \tau_k) + \theta_{k0}]} \quad (8)$$

where θ_i is a component of the total phase due to carrier frequency minus the contribution from the code repetition cycles

$$\theta_i = 2\pi L_i(nT - m_k C) \quad (11)$$

where $\Delta\theta_{ik}(\tau_k, f_k)$ is the fractional phase given by

$$\Delta\theta_{ik}(\tau_k, f_k) = 2\pi(nT f_k - m_k C - L_i \tau_k - f_k \tau_k)^1 \quad (12)$$

This is just to show that what we did in the past was the use of a wrong model because it ignored other contributions. Next, we assume that the carrier frequency L_{ik} can be modeled as

$$L_{ik} = L_i + f_k \quad (13)$$

where f_k is the frequency offset of the k th satellite which is also an unknown parameter.

In this publication we are going to compute the distribution of the phase signal $\theta_{ik}(t_k, L_{ik})$ not the distribution of $\theta_{ik}(\tau_k, f_k)$

$$\theta_{ik}(t_k, L_{ik}) = 2\pi L_{ik} t_k + \theta_{k0} \quad (14)$$

This concludes the discussion of the signal model.

4 Complex Matrix Variate Signal Distribution

In order to produce the *Complex Matrix Variate Signal Distribution* we need first to produce an understanding of the *Complex signal distribution* which is presented next.

4.1 Complex Signal Distribution

The first task is producing a distribution for the complex signal $s_k[\tau_k, f_k, d_k(\tau_k)]$ based on the prior distributions of τ_k , f_k , and $d_k(q)$.

For our prior distributions we choose τ_k to be uniformity distributed between $[0, T_c]$, which symbolically can be written as $\tau_k \in \text{unif}[0, T_c]$, where T_c is the defined as the chipping period and since we can only determine our propagation delay modulo the time duration of a GPS CDMA spreading code [20]; $f_k \in \text{unif}[-f_D, f_D]$ Hz and $d_k(q)$ is uniformly distributed among the members of the QPSK constellation. The coarse propagation delay can be decoded from the information symbols [5]. The Doppler frequency f_D is typically in the range of 5,000 or some other constant value.

Let us produce the distribution of the total phase $\theta_{ik}(t_k, L_{ik})$ based on the prior distributions of $\tau_k \in \text{unif}[0, T_c]$ and $f_k \in \text{unif}[-f_D, f_D]$ or $t_k \in \text{unif}[nT - m_k C - T_c, nT - m_k C]$ and $L_{ik} \in \text{unif}[L_i - f_D, L_i + f_D]$.

Next, if we define a variable a as

$$a = nT - m_k C \quad (15)$$

then we have the distribution of the time of transmission, t_k , and carrier frequency, L_{ik} , of the k th GPS signal as follows

$$t_k \in \text{unif}[a - T_c, a] \Rightarrow \quad (16)$$

$$f_{T_k}(t_k) = \text{rect}\left[\frac{t_k - a + 0.5T_c}{T_c}\right] \equiv \begin{cases} \frac{1}{T_c}, & -T_c \leq t_k - a < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$L_{ik} \in \text{unif}[L_i - f_D, L_i + f_D] \Rightarrow$$

$$f_{L_{ik}}(l_{ik}) = \text{rect}\left[\frac{l_{ik} - L_i}{2f_D}\right] \equiv \begin{cases} \frac{1}{2f_D}, & -f_D \leq l_{ik} - L_i < f_D \\ 0, & \text{otherwise} \end{cases} \quad (17)$$

Furthermore, let φ be defined as

$$\varphi = L_{ik} t_k \quad (18)$$

Based on the derivations of G.R. Cooper, C.L. McGillem 1999 [41] pp 142-145 we have

Second, we recognize that substituting $p = 6$ in (17) we get

$$f_\Phi(\varphi) = \int_{-\infty}^{\infty} \frac{1}{|w|} \text{rect}\left(\frac{w - a + 0.5T_c}{T_c}\right) \text{rect}\left(\frac{\varphi \cdot w^{-1} - L_i}{2f_D}\right) dw \quad (19)$$

One can easily produce a sketch of the integration limits as follows:

$$f_\Phi(z) = \int_{\varphi_a}^{\varphi_b} \frac{1}{w} dw = \ln(\varphi_b) - \ln(\varphi_a) = \ln(\varphi_b \varphi_a^{-1}) \quad (20)$$

$$\varphi_i \leq \varphi \leq \varphi_{i+1}, \quad i = \{0, 1, 2\}$$

where, $\varphi_0, \dots, \varphi_3$, are defined as follows

$$\varphi_0 = (a - T_c)(L_i - f_D) \quad (21)$$

$$\varphi_1 = (a - T_c)(L_i + f_D) \quad (22)$$

$$\varphi_2 = a(L_i - f_D) \quad (23)$$

$$\varphi_3 = a(L_i + f_D) \quad (24)$$

Finally, $f_\Phi(\varphi)$ can be obtained from

$$f_\Phi(\varphi) = \begin{cases} \ln(\varphi_b \varphi_a^{-1}), & \varphi_0 \leq \varphi < \varphi_3 \\ 0, & \text{otherwise} \end{cases} \quad (25)$$

Where φ_a and φ_b take on the following values based on the interval of φ

$$\begin{cases} (1) \varphi_b = \varphi L_{i-}^{-1}; \varphi_a = \varphi_0 L_{i-}^{-1}; & \text{if } \varphi_0 \leq \varphi < \varphi_1 \\ (2) \varphi_b = \varphi L_{i-}^{-1}; \varphi_a = \varphi L_{i+}^{-1}; & \text{if } \varphi_1 \leq \varphi < \varphi_2 \\ (3) \varphi_b = \varphi_3 L_{i+}^{-1}; \varphi_a = \varphi L_{i-}^{-1}; & \text{if } \varphi_2 \leq \varphi < \varphi_3 \end{cases} \quad (26)$$

where

$$L_{i\mp} = L_i \mp f_D \quad (27)$$

Since, L_i is the dominant factor then $f_\Phi(\varphi)$ is really

$$f_\Phi(\varphi) = \ln(\varphi_b \varphi_a^{-1}) \cong \ln(1) = 0 \quad (28)$$

Hence, for most practical purposes the distribution of $\theta_{ik}(t_k, L_{ik})$ it is equal to the distribution of θ_{k0} ; hence, the

¹ In [3]-[6] we have considered special cases of the fractional phase model

given by (12).

distribution of the complex signal is reduced to a distribution of a real signal amplitude. This result is actually revolutionary.

Based on the derivations of G.R. Cooper, C.L. McGillem 1999 [43] pp 95-97 we obtain the pdf of the signal amplitude or the probability of the C code as

$$f_C(x) = p_{-1}([t_k]_C)\delta(x+1) + p_1([t_k]_C)\delta(x-1) \quad (29)$$

where

$$p_{-1}([t_k]_C) = 1 - p_1([t_k]_C) \quad (30)$$

Hence, the probability of the signal is equal to the probability of the signal code $c_k([t_k]_C)$ as we have ignored the probability of the data $d_k([t_k]_D)$ as $D = qC$ and q an integer much greater than one; i.e., $q \gg 1$.

Finally, we are able to produce for the first time the probability of the signal as $s_k(n, t_k, L_{ik})$ as follows:

$$f_S(s) = p_1([t_k]_C)\delta(s-1) + p_{-1}([t_k]_C)\delta(s+1) \quad (31)$$

From where we can compute the cdf of the signal $s_k(n, t_k, L_{ik})$

$$F_S(s) = \begin{cases} 1 & s \geq 1 \\ p_{-1}([t_k]_C) = 1 - p_1([t_k]_C) & -1 \leq s < 1 \\ 0 & s < -1 \end{cases} \quad (32)$$

Or $F_S(s)$ can be written in compact form with the help of the unit-step function Arfken, G.B., and Weber, H.J., 1995, [44], ex. 1.15.13 pg. 89 as follows

$$F_S(s) = p_{-1}([t_k]_C)u(s+1) + p_1([t_k]_C)u(s-1) \quad (33)$$

Where the unit-step function is defined in Arfken, G.B., and

$$f_S(\mathbf{S}) = \prod_{i=1}^N \prod_{j=1}^K f_S(s_{ij}) = \prod_{i=1}^N \prod_{j=1}^K \{p_1([t_k]_C)\delta(s_{ij}-1) + p_{-1}([t_k]_C)\delta(s_{ij}+1)\} \quad (36)$$

Let us define with $\mathbf{P}$ as the following 2×2 diagonal matrix

$$\mathbf{P} = \begin{bmatrix} p_1([t_k]_C) & 0 \\ 0 & p_{-1}([t_k]_C) \end{bmatrix} \quad (37)$$

And Δ_{ij} as the following 2×2 diagonal matrix

$$\Delta_{ij} = \begin{bmatrix} \delta(s_{ij}-1) & 0 \\ 0 & \delta(s_{ij}+1) \end{bmatrix} \quad (38)$$

Hence, substituting (37) and (38) into (36) we obtain

$$\begin{aligned} f_S(\mathbf{S}) &= \prod_{i=1}^N \prod_{j=1}^K f_S(s_{ij}) \\ &= \prod_{i=1}^N \prod_{j=1}^K \text{tr}(\mathbf{P}\Delta_{ij}) \\ &= \text{tr}(\prod_{i=1}^N \prod_{j=1}^K \mathbf{P}\Delta_{ij}) \end{aligned} \quad (39)$$

where $\otimes$ denotes the Kronecker or Tensor Product (see Progni 2018 [16]) and where $\prod_{j=1}^K \mathbf{x}_j$ is defined as

$$\prod_{j=1}^K \mathbf{x}_j = \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \dots \otimes \mathbf{x}_K \quad (40)$$

Using the properties of the Kronecker product we can write (39) as

Weber, H.J., 1995, [42], ex. 1.15.13 pg. 89

$$u(s) = \int_{-\infty}^s \delta(x)dx = \begin{cases} 1 & s > 0 \\ 0 & s < 0 \end{cases} \quad (34)$$

This concludes the derivations of the *complex signal distribution*. Next we continue with the derivations of the *Complex Matrix Variate Signal Distribution*.

4.2 Complex Matrix Variate Signal Distribution

The GPS satellite complex matrix signal $\mathbf{S}$ consists of code division multiple access (CDMA) modulated, quadrature phase shift keying (QPSK) symbols that are assumed to be both time and frequency shifted. In general, we can model this signal can be written in matrix form as [3]-[6]

$$\mathbf{Y} = \mathbf{S}\mathbf{A}^\dagger + \mathbf{Z}^i \quad (35)$$

where $\mathbf{Y}$ is an $N \times M$ element complex matrix whose n th row is given by $\mathbf{y}^\dagger(n)$ from (1), $\mathbf{S}$ is an $N \times K$ signal matrix whose (n, k) th entry is given by $s_k^*(n)$, $\mathbf{A}$ is an $M \times K$ complex matrix whose k th row is $\mathbf{a}_k(n)$ and $\mathbf{Z}$ is the $N \times M$ complex interference matrix whose n th row is given by $\mathbf{z}^\dagger(n)$.

Since, the distribution of the signal values s is real, not complex, then the complex matrix variate signal distribution, $f_S(\mathbf{S})$, is really the real matrix variate signal distribution, $f_{\text{Real}(\mathbf{S})}[\text{Real}(\mathbf{S})]$, whose elements take only two real values $\{-1, +1\}$. Hence, the complex matrix variate signal pdf, $f_S(\mathbf{S})$, is the product of all the signal pdfs of the complex value s , $f_S(s_{ij})$, as follows

$$\begin{aligned} f_S(\mathbf{S}) &= \text{tr}(\prod_{i=1}^N \prod_{j=1}^K \mathbf{P}\Delta_{ij}) \\ &= \text{tr}[(\prod_{i=1}^N \prod_{j=1}^K \mathbf{P})(\prod_{i=1}^N \prod_{j=1}^K \Delta_{ij})] \end{aligned} \quad (41)$$

which is equivalent with

$$f_S(\mathbf{S}) = \text{vec}(\prod_{i=1}^N \prod_{j=1}^K \mathbf{P})\text{vec}(\prod_{i=1}^N \prod_{j=1}^K \Delta_{ij}) \quad (42)$$

Next, let us define $\tilde{\mathbf{P}}$ and $\tilde{\Delta}$ are $I \times I$, $I \equiv 2^{(K-1)+(N-1)+1}$, diagonal matrices

$$\tilde{\mathbf{P}} = \prod_{i=1}^N \prod_{j=1}^K \mathbf{P} \quad (43)$$

$$\tilde{\Delta} = \prod_{i=1}^N \prod_{j=1}^K \Delta_{ij} \quad (44)$$

Substituting (43) and (44) into (42) yields

$$f_S(\mathbf{S}) = \text{vec}(\tilde{\mathbf{P}})\text{vec}(\tilde{\Delta}) \quad (45)$$

Next, we define the Dirac delta function of matrix argument as follows Zhang 2016, pg. 11, [45]:

$$\delta(\mathbf{S}) = \prod_{i=1}^N \prod_{j=1}^K \delta(s_{ij}) \quad (46)$$

Based on the definition of (46) we can easily obtain the following

$$\delta(\mathbf{S} - \mathbf{1}_{N \times K}) = \prod_{i=1}^N \prod_{j=1}^K \delta(s_{ij} - 1), \quad \mathbf{1}_{N \times K} \equiv \mathbf{1} \quad (47)$$

where $\mathbf{1}_{N \times K} \equiv \mathbf{1}$ is a matrix that contains all 1's.

In general we make the following definition

$$\delta(\mathbf{S} - \mathbf{1}_{u \times v}) = \prod_{i=1}^u \prod_{j=1}^v \delta(s_{ij} - 1) \prod_{i=u+1}^N \prod_{j=v+1}^K \delta(s_{ij} + 1) \quad (48)$$

where $\mathbf{1}_{u \times v}$ is an $N \times K$ matrix that contains $u \times v$ $\{+1\}$ elements and $N \times K - u \times v$ $\{-1\}$ elements in rows and columns as defined by the indices u and v .

Hence, based on the above definition of (48) we can write (45) as follows

$$f_S(\mathbf{S}) = \sum_{i=1}^I p_1^{m(i)} ([t_k]_C) p_{-1}^{n(i)} ([t_k]_C) \delta\{\mathbf{S} - \mathbf{1}_{u(i) \times v(i)}\} \quad (49)$$

such that $m(i)$, $n(i)$ are integers whose relation as a function of integer index i is determined from

$$m(i) + n(i) = N \times K \equiv NK \quad (50)$$

$$m(i = 1) = N \times K, \quad n(i = 1) = 0 \quad (51)$$

$$m(i = I) = 0, \quad n(i = I) = N \times K \equiv NK \quad (52)$$

And $u(i)$, $v(i)$ are also integers whose relation as a function of integer index i is as follows

$$u(i = 1) = N, \quad u(i = I) = 0 \quad (53)$$

$$v(i = 1) = K, \quad v(i = I) = 0 \quad (54)$$

The exact closed form expression of integers $m(i)$, $n(i)$, $u(i)$, and $v(i)$ is not important for now. We can investigate those details later. What is important is that we define the prior complex matrix variate signal cdf as follows:

$$F_S(\mathbf{S}) = \int_{\mathbf{S}} f_S(\mathbf{X}) d\mathbf{X} \quad (55)$$

Substituting (49) into (55) and then changing the order of summation and integration produces

$$F_S(\mathbf{S}) = \sum_{i=1}^I p_1^{m(i)} ([t_k]_C) p_{-1}^{n(i)} ([t_k]_C) \int_{\mathbf{S}} \delta\{\mathbf{X} - \mathbf{1}_{u(i) \times v(i)}\} d\mathbf{X} \quad (56)$$

Next, we define the Heaviside unit step function of matrix argument, $u(\mathbf{S})$, as follows:

$$u(\mathbf{S}) = \int_{\mathbf{S}} \delta(\mathbf{X}) d\mathbf{X} \quad (57)$$

Substituting (46) into (57) and then changing the order of summation and integration produces

$$\begin{aligned} u(\mathbf{S}) &= \int_{\mathbf{S}} \prod_{i=1}^N \prod_{j=1}^K \delta(x_{ij}) d\mathbf{X} \quad \text{iii} \\ &= \prod_{i=1}^N \prod_{j=1}^K \int_{s_{ij}} \delta(x_{ij}) dx_{ij} \\ &= \prod_{i=1}^N \prod_{j=1}^K u(x_{ij}) \end{aligned} \quad (58)$$

Equation (58) is closer to the definition of Heaviside unit step function of matrix argument given by Liu 2016, pg. 12 [46]

$$F_S(\mathbf{S}) = \begin{cases} \sum_{i=1}^I p_1^{m(i)} ([t_k]_C) p_{-1}^{n(i)} ([t_k]_C) = \prod_{i=1}^N \prod_{j=1}^K \{p_1([t_k]_C) + p_{-1}([t_k]_C)\} \equiv 1 & \mathbf{S} \geq \mathbf{1} \\ \sum_{i=1}^I p_1^{m(i)} ([t_k]_C) p_{-1}^{n(i)} ([t_k]_C) u[\mathbf{S} - \mathbf{1}_{u(i) \times v(i)}] & -\mathbf{1} \leq \mathbf{S} < \mathbf{1} \\ 0, & \mathbf{S} < -\mathbf{1} \end{cases} \quad (61)$$

Next, let us explore, $F_S(\mathbf{S})$ more precisely let us exploit $u[\mathbf{S} - \mathbf{1}_{u(i) \times v(i)}]$ for $-\mathbf{1} \leq \mathbf{S} < \mathbf{1}$. From (59) $u[\mathbf{S} - \mathbf{1}_{u(i) \times v(i)}]$ can be written as

$$u[\mathbf{S} - \mathbf{1}_{u(i) \times v(i)}] = \begin{cases} 1, & \text{if } \mathbf{S} > \mathbf{1}_{u(i) \times v(i)} \\ 0, & \text{if } \mathbf{S} < \mathbf{1}_{u(i) \times v(i)} \end{cases} \quad (62)$$

$$F_S(\mathbf{S}) = \begin{cases} 1, & \mathbf{S} \geq \mathbf{1} \\ p_{-1}^{NK} ([t_k]_C) \equiv [1 - p_1([t_k]_C)]^{NK}, & -\mathbf{1} \leq \mathbf{S} < \mathbf{1} \\ 0, & \mathbf{S} < -\mathbf{1} \end{cases} \quad (64)$$

Equation (64) can be written in much the same way as in compact form with the unit-step function of matrix argument as

$$F_S(\mathbf{S}) = p_{-1}^{NK} ([t_k]_C) u(\mathbf{S} + \mathbf{1}) + \{1 - p_{-1}^{NK} ([t_k]_C)\} u(\mathbf{S} - \mathbf{1})^{lvv} \quad (65)$$

if the signal matrix $\mathbf{S}$ is Hermitian [28]. We have generalized the definition for any matrix $\mathbf{S}$ as follows

$$u(\mathbf{S}) = \begin{cases} 1, & \text{if } \mathbf{S} > 0 \\ 0, & \text{if } \mathbf{S} < 0 \end{cases} \quad (59)$$

Finally, substituting (58) into (56) produces for the first time the closed form expression of the prior complex matrix variate signal cdf as follows:

$$F_S(\mathbf{S}) = \sum_{i=1}^I p_1^{m(i)} ([t_k]_C) p_{-1}^{n(i)} ([t_k]_C) u[\mathbf{S} - \mathbf{1}_{u(i) \times v(i)}] \quad (60)$$

Equation (60) can be written much in the same way as we wrote the cdf of the complex signal distribution as follows:

However, since $-\mathbf{1} \leq \mathbf{S} < \mathbf{1}$ then

$$u[\mathbf{S} - \mathbf{1}_{u(i) \times v(i)}] = \begin{cases} 1, & \text{if } i = I \\ 0, & \text{if } i = 1 \dots I - 1 \end{cases} \quad (63)$$

Next, substituting (63) into (61) yields

follows

Equation (65) is the simplified closed form expression of the *Complex Matrix Variate Signal Distribution*.

This concludes the discussion of the *Complex Matrix Variate Signal Distribution*. Next, we provide a brief discussion of the *Complex Normal (or Gaussian) Interference Distribution*.

5 Complex Normal Interference Distribution

In this section we are going to discuss two important topics: (1) *Complex Normal (or Gaussian) Interference Distribution*; i.e., the single variable case; (2) *Complex Normal (or Gaussian) Matrix Variate Interference Distribution*; i.e., the *Matrix Variate* case.

5.1 Complex Normal Interference Distribution

For estimation purposes we model Z as zero mean, complex Gaussian noise, with variance σ_Z^2 . The pdf for the interference random variable Z is given by $f_Z(z)$ [3]-[6], [48]-[50]

$$f_Z(z) = (\pi\sigma_Z^2)^{-1} e^{-\sigma_Z^{-2}|z|^2} \quad (66)$$

This expression is called “circularly-symmetric” normal distribution because the pdf depends only on the magnitude of z or $|z|$ but not on its argument $\arg(z)$. As such, the magnitude $|z|$ of standard complex normal random variable will have the Rayleigh distribution and the squared magnitude $|z|^2$ will have the exponential distribution, whereas the argument will be distributed uniformly on $\arg(z) \sim [-\pi, \pi]$ [51].

Furthermore the relation to the real scalar normal distribution is as follows

$$Z = \text{Re}[Z] + j\text{Im}[Z] \quad (67)$$

Then if we make the substituting

$$X = \{\text{Re}[Z], \text{Im}[Z]\} \equiv \{\text{real}[Z], \text{imag}[Z]\} \quad (68)$$

Hence, we obtain the individual distributions of either the real or imaginary ideally and identically distributed (IID) zero mean normal distribution as follows

$$f_X(x) = (\pi 0.5\sigma_Z^2)^{-\frac{1}{2}} e^{-2\sigma_Z^{-2}x^2} \equiv \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{x^2}{2\sigma_X^2}} \quad (69)$$

Where the relation between the real and complex interference variances is as follows

$$\sigma_X^2 = 0.5\sigma_Z^2 \quad (70)$$

Normalizing to the unity variance of the random variable X

$$\sigma_X^2 \equiv 1 \Rightarrow \sigma_Z^2 = 2\sigma_X^2 = 2 \quad (71)$$

such that the new random variables X' and Z' are obtained

from X and Z as

$$X' \equiv \frac{X}{\sigma_X} \Rightarrow Z' \equiv \frac{Z}{\sigma_Z} \quad (72)$$

which produces the following pdfs

$$f_{X'}(x') \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} \quad (73)$$

$$f_{Z'}(z') = \frac{1}{2\pi} e^{-\frac{|z'|^2}{2}} \quad (74)$$

To guarantee Constant false alarm rate (CFAR), the probability of false alarm, P_{fa} , the following should be satisfied [7]

$$P_{fa} \equiv \int_{\varrho_X}^{\infty} f_{X'}(x'|H_0) dx' = \frac{\int_{\varrho_X}^{\infty} e^{-\frac{x'^2}{2}} dx'}{\sqrt{2\pi}} = \text{erfc}(\varrho_X) \quad (75)$$

which is equivalent with

$$P_{fa} \equiv \int_{\varrho_Z}^{\infty} f_{Z'}(z'|H_0) dz' = \frac{\int_{\varrho_Z}^{\infty} e^{-\frac{z'^2}{2}} dz'}{2\pi} = \frac{\text{erfc}(\varrho_Z)}{\sqrt{2\pi}} \quad (76)$$

where H_0 is the hypothesis that no GPS signal is present or the received signal contains only noise; and the threshold, ϱ_X , can be obtained by means of the inverse complementary error function, $\text{erfc}^{-1}(x)$, as

$$\varrho_X \equiv \text{erfc}^{-1}(P_{fa}) \quad (77)$$

$$\varrho_Z \equiv \text{erfc}^{-1}(\sqrt{2\pi}P_{fa}) \quad (78)$$

Next, we employ the following pdf of the conjugate prior for the steering parameter, A , with unknown variance σ_A^2 as follows

$$f_{A|\sigma_A^2}(a|\sigma_A^2) = \frac{\sigma_A^2}{\pi\sigma_A^2} e^{-\frac{(a-\bar{a})^* \sigma_A^2 (a-\bar{a})}{\sigma_A^2}} = \frac{1}{\pi\sigma_A^2} e^{-\frac{|a-\bar{a}|^2}{\sigma_A^2}} \quad (79)$$

where the superscript $*$ denotes the conjugate and $\sigma_{A'}^2$ is given by

$$\sigma_{A'}^2 = \frac{\sigma_A^2}{\sigma_A^2}, \Rightarrow \sigma_{A'} = \frac{\sigma_Z}{\sigma_A} \quad (80)$$

If $\sigma_A^2 = 0$ that means $\sigma_{A'}^2 = \infty$ and $f_A(a) = 0$, which means that the received signal is entirely dominated by noise during the initialization phase.

This concludes the derivations of the *Complex Normal (or Gaussian) Interference Distribution*; i.e., the single variable case. Next we present the *Complex Normal (or Gaussian) Matrix Variate Interference Distribution*; i.e., the *Matrix Variate* case.

5.2 Complex Normal Matrix Variate Interference Distribution

For estimation purposes we model $\mathbf{Z}$ as zero mean, complex Gaussian noise, with autocovariance matrix $\mathbf{\Sigma}$: $M \times M$. The pdf for the interference matrix $\mathbf{Z}$ is given by [3]-[6], [48]-[50]

$$f_Z(\mathbf{Z}) = \det(\pi\mathbf{\Sigma})^{-N} \text{etr}(-\mathbf{\Sigma}^{-1}\mathbf{Z}^\dagger\mathbf{Z})^{\text{viii}} \quad (81)$$

where $\text{etr}(y) \equiv \exp[\text{tr}(y)]$ [3]-[6], [58], [59] and

$$\mathbf{\Sigma} = \text{cov}[\mathbf{Z}] = E[\mathbf{Z}^\dagger\mathbf{Z}] = E[\mathbf{z}^\dagger(n)\mathbf{z}(n)] \quad (82)$$

$$f_{A|\Sigma}(\mathbf{A}|\mathbf{\Sigma}) = \det(\pi^{-1}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}_{AA})^K \text{etr}[-\mathbf{\Sigma}^{-1}(\mathbf{A} - \bar{\mathbf{A}})\mathbf{\Sigma}_{AA}(\mathbf{A} - \bar{\mathbf{A}})^\dagger]^{\text{viii}} \quad (83)$$

where $\bar{\mathbf{A}}: M \times K$ is a Bayesian nuisance hyper parameter which represents the mean matrix of the steering matrix $\mathbf{A}$.

$$E(\mathbf{A}) = \bar{\mathbf{A}} \quad (84)$$

and $\mathbf{\Sigma}_{AA}: K \times K$

$$\mathbf{\Sigma}_{AA} = \text{cov}[\mathbf{A}] = E[(\mathbf{A} - \bar{\mathbf{A}})(\mathbf{A} - \bar{\mathbf{A}})^\dagger] \quad (85)$$

The two-sided density for the steering vectors is needed in order to model spatial cross correlation via $\mathbf{\Sigma}$ and inter-emitter correlation via $\mathbf{\Sigma}_{AA}$. The covariance matrix $\mathbf{\Sigma}$ is deliberately chosen to be the same as the interference covariance in (85), since the steering vector spatial covariance always converges to a multiple of this, as the collect time $N \rightarrow \infty$. $\mathbf{\Sigma}_{AA}$ on the other hand, can be initialized to 0, but converges to N times the joint GPS satellite covariance matrix.

This concludes the discussion on the Complex Normal Matrix Variate Interference Distribution. Next we investigate the *Bayesian Estimation* for both the *single variable* and *Complex Matrix Variate* cases.

If we allow $\mathbf{\Sigma}$ differing from the identity matrix, then it is possible to derive algorithms that excise interference, including un-modeled satellite emitters, or hostile jammers. For this study, we initially consider white noise interference, and model $\mathbf{\Sigma}$ as a scalar multiple of the identity matrix [3].

Since we will largely employ Bayesian estimation to estimate, $d_k(q)$, τ_k and f_k , we provide the following pdf of the conjugate prior for the steering matrix $\mathbf{A}$: [3]-[6]

6 Bayesian Estimation

Bayesian Estimation consists of *Bayesian Estimation for: (1) Complex Normal Interference Distribution for the Single Variable case*; and (2) *Complex Normal Matrix Variate Interference Distribution*.

6.1 Complex Normal Interference Distribution

Initially we rewrite (35) for computing the interference random variable Z as

$$Z = Y - SA^* \quad (86)$$

and substitute (86) in (66), which yields the conditional pdf for the observed data, Y , conditioned on the random signal, S , as

$$f_{Z|S}(y|s) = (\pi\sigma_Z^2)^{-1} e^{-\sigma_Z^{-2}(y-sa^*)^*(y-sa^*)} \quad (87)$$

Taking the product of (79) with (87), which yields the conditional pdf for the observed data, Y , conditioned on random signal, S , and unknown interference (or noise) variance, σ_Z^2 , as follows

$$f_{Z,A|S,\sigma_Z^2}(y,a|s,\sigma_Z^2) = f_{Z|S}(y|s)f_{A|\sigma_Z^2}(a|\sigma_Z^2) = (\pi\sigma_Z^2)^{-2}\sigma_A^2 e^{-\sigma_Z^{-2}[(y-sa^*)^*(y-sa^*)+(a-\bar{a})\sigma_A^2(a-\bar{a})^*]} \quad (88)$$

Which is equivalent with

$$f_{Z,A|S,\sigma_Z^2}(y,a|s,\sigma_Z^2) = (\pi\sigma_Z^2)^{-2}\sigma_A^2 e^{-\sigma_Z^{-2}[|y|^2 - y^*sa^* - (sa^*)^*y + (sa^*)^*(sa^*) + a\sigma_A^2a^* - a\sigma_A^2\bar{a}^* - \bar{a}\sigma_A^2a^* + \bar{a}\sigma_A^2\bar{a}^*]} \quad (89)$$

Which is identical to

$$f_{Z,A|S,\sigma_Z^2}(y,a|s,\sigma_Z^2) = (\pi\sigma_Z^2)^{-2}\sigma_A^2 e^{-\sigma_Z^{-2}[|y|^2 - y^*sa^* - s^*ay + (sa^*)^*(sa^*) + a\sigma_A^2a^* - a\sigma_A^2\bar{a}^* - \bar{a}\sigma_A^2a^* + \bar{a}\sigma_A^2\bar{a}^*]} \quad (90)$$

Or performing the factorization of

$$f_{Z,A|S,\sigma_Z^2}(y,a|s,\sigma_Z^2) = (\pi\sigma_Z^2)^{-2}\sigma_A^2 e^{-\sigma_Z^{-2}[|y|^2 - (y^*s + \bar{a}\sigma_A^2)a^* - a(s^*y + \sigma_A^2\bar{a}^*) + a(|s|^2 + \sigma_A^2)a^* + \bar{a}\sigma_A^2\bar{a}^*]} \quad (91)$$

Next we define

$$\sigma_A'^2(s) = |s|^2 + \sigma_A^2 \quad (92)$$

$$b(s) = y^*s + \bar{a}\sigma_A^2 \quad (93)$$

Substituting (92) and (93) in (91) we obtain

$$f_{Z,A|S,\sigma_Z^2}(y, a|s, \sigma_Z^2) = (\pi\sigma_Z^2)^{-2} \sigma_A^2 e^{-\sigma_Z^{-2}[|y|^2 - ba^* - ab^* + \sigma_A'^2|a|^2 + \sigma_A^2|\bar{a}|^2]} \quad (94)$$

which can be further written as

$$f_{Z,A|S,\sigma_Z^2}(y, a|s, \sigma_Z^2) = (\pi\sigma_Z^2)^{-2} \sigma_A^2 e^{-\sigma_Z^{-2}[|y|^2 + \sigma_A^2|\bar{a}|^2 - ba^* - ab^* + \sigma_A'^2|a|^2]} \quad (95)$$

Typically we make the linear substitution

$$b(s) = \bar{a}'\sigma_A'^2(s); \Rightarrow \bar{a}'(s) = b(s)\sigma_A'^{-2}(s) \quad (96)$$

$$b(s)a^* = \bar{a}'\sigma_A'^2(s)a^* \quad (97)$$

$$ab^*(s) = a\sigma_A'^2(s)\bar{a}^* \quad (98)$$

Hence, the products ba^* and ab^* can be written as

Hence, the following quantity is equal to

$$b(s)a^* + ab^*(s) - \sigma_A'^2(s)|a|^2 = \bar{a}'(s)\sigma_A'^2(s)a^* + a\sigma_A'^2(s)\bar{a}^*(s) - a\sigma_A'^2(s)a^* \quad (99)$$

Next, we add and subtract $\sigma_A'^2(s)|\bar{a}'|^2(s)$ on the right side

of (99) and then after we perform factorization we obtain

$$\sigma_A'^2(s)|\bar{a}'|^2 + \bar{a}'\sigma_A'^2(s)a^* + a\sigma_A'^2(s)\bar{a}^* - a\sigma_A'^2(s)a^* - \sigma_A'^2(s)|\bar{a}'|^2 \equiv \sigma_A'^2(s)|\bar{a}'|^2 + (a - \bar{a}')\sigma_A'^2(s)\bar{a}^* - (a - \bar{a}')\sigma_A'^2(s)a^* \quad (100)$$

Equation (100) is equivalent with

$$\sigma_A'^2(s)|\bar{a}'(s)|^2 + \bar{a}'(s)\sigma_A'^2(s)a^* + a\sigma_A'^2(s)\bar{a}^*(s) - a\sigma_A'^2(s)a^* - \sigma_A'^2(s)|\bar{a}'(s)|^2 \equiv \sigma_A'^2(s)|\bar{a}'(s)|^2 - \sigma_A'^2(s)|a - \bar{a}'(s)|^2 \quad (101)$$

Finally, the joint conditional pdf between the received random data symbol, Y , and the random steering scalar, A , conditioned on random signal, S , and unknown interference (or

noise) variance, σ_Z^2 , can be written in terms of the posterior parameters as

$$f_{Z,A|S,\sigma_Z^2}(y, a|s, \sigma_Z^2) = (\pi\sigma_Z^2)^{-2} \sigma_A^2 e^{-\sigma_Z^{-2}[|y|^2 + \sigma_A^2|\bar{a}|^2 - \sigma_A'^2(s)|\bar{a}'|^2 + \sigma_A'^2(s)|a - \bar{a}'|^2]} \quad (102)$$

Next, integrating all over the steering parameter a (i.e., eliminated the effect of the random steering scalar, A , we obtain

$\mathbf{Z}$ as

$$\mathbf{Z} = \mathbf{Y} - \mathbf{S}\mathbf{A}^\dagger \quad (105)$$

and substituting (105) in (81), which yields the conditional pdf for the observed data conditioned on the complex random signal matrix, $\mathbf{S}$, and unknown interference covariance matrix, $\mathbf{\Sigma}$, as

$$f_{Z|S,\Sigma}(\mathbf{Y}|\mathbf{S}, \mathbf{\Sigma}) = \frac{\text{etr}[-\Sigma^{-1}(\mathbf{Y} - \mathbf{S}\mathbf{A}^\dagger)^\dagger(\mathbf{Y} - \mathbf{S}\mathbf{A}^\dagger)]}{\det(\pi\Sigma)^N} \quad (106)$$

Equation (103) represents an essential quantity for computing conditional pdf, $f_{Z|S}(y|s)$, of the observed data, Y , given the random signal, S .

Multiplying the pdf's from (106) and (83) we can write the joint conditional pdf between the received data matrix $\mathbf{Y}$ and the steering matrix $\mathbf{A}$ conditioned on the complex random signal matrix, $\mathbf{S}$, and interference covariance matrix, $\mathbf{\Sigma}$, as

$$f_{Z,A|S,\Sigma}(\mathbf{Y}, \mathbf{A}|\mathbf{S}, \mathbf{\Sigma}) = f_{Z|S,\Sigma}(\mathbf{Y}|\mathbf{S}, \mathbf{\Sigma})f_{A|\Sigma}(\mathbf{A}) \quad (107)$$

Equation (107) can be written in expanded form as follows

$$f_{Z|S}(y|s) = \int_{\sigma_Z^2} f_{Z|S,\sigma_Z^2}(y|s, \sigma_Z^2) f_{\sigma_Z^2|\sigma_W^2}(\sigma_Z^{-2}|\sigma_W^2) \quad (104)$$

In order to compute conditional pdf, $f_{Z|S}(y|s)$ we need to make a reasonably good assumption about the interference variance σ_Z^2 ; i.e., $f_{\sigma_Z^2|\sigma_W^2}(\sigma_Z^{-2}|\sigma_W^2)$ which is unknown and then perform the integration all over σ_Z^2 as depicted in (104).

6.2 Complex Normal Matrix Variate Interference Distribution

Initially we rewrite (35) for computing the interference matrix

$$f_{Z,A|S,\Sigma}(\mathbf{Y}, \mathbf{A}|\mathbf{S}, \mathbf{\Sigma}) = \frac{\text{etr}[-\Sigma^{-1}(\mathbf{Y} - \mathbf{S}\mathbf{A}^\dagger)^\dagger(\mathbf{Y} - \mathbf{S}\mathbf{A}^\dagger)]}{\det(\pi\Sigma)^N} \frac{\det(\Sigma_{AA})^K \text{etr}[-\Sigma^{-1}(\mathbf{A} - \bar{\mathbf{A}})\Sigma_{AA}(\mathbf{A} - \bar{\mathbf{A}})^\dagger]}{\det(\pi\Sigma)^K} \quad (108)$$

which is equivalent with

$$f_{Z,A|S,\Sigma}(\mathbf{Y}, \mathbf{A}|\mathbf{S}, \Sigma) = \frac{\det(\Sigma_{AA})^K \text{etr}[-\Sigma^{-1}(\mathbf{Y}-\mathbf{S}\mathbf{A}^\dagger)^\dagger(\mathbf{Y}-\mathbf{S}\mathbf{A}^\dagger) - \Sigma^{-1}(\mathbf{A}-\bar{\mathbf{A}})\Sigma_{AA}(\mathbf{A}-\bar{\mathbf{A}})^\dagger]}{\det(\pi\Sigma)^{N+K}} \quad (109)$$

Next, if we factor Σ^{-1} inside the $\text{etr}(\cdot)$ and then expand the multiplication in terms we further obtain

$$f_{Z,A|S,\Sigma}(\mathbf{Y}, \mathbf{A}|\mathbf{S}, \Sigma) = \frac{\det(\Sigma_{AA})^K \text{etr}[-\Sigma^{-1}(\mathbf{Y}^\dagger\mathbf{Y} - \mathbf{Y}^\dagger\mathbf{S}\mathbf{A}^\dagger - \mathbf{A}\mathbf{S}^\dagger\mathbf{Y} + \mathbf{A}\mathbf{S}^\dagger\mathbf{S}\mathbf{A}^\dagger + \mathbf{A}\Sigma_{AA}\mathbf{A}^\dagger - \mathbf{A}\Sigma_{AA}\bar{\mathbf{A}}^\dagger - \bar{\mathbf{A}}\Sigma_{AA}\mathbf{A}^\dagger + \bar{\mathbf{A}}\Sigma_{AA}\bar{\mathbf{A}}^\dagger)]}{\det(\pi\Sigma)^{N+K}} \quad (110)$$

After performing a factorization of terms we obtain

$$f_{Z,A|S,\Sigma}(\mathbf{Y}, \mathbf{A}|\mathbf{S}, \Sigma) = \frac{\det(\Sigma_{AA})^K \text{etr}\{-\Sigma^{-1}[\mathbf{Y}^\dagger\mathbf{Y} - (\mathbf{Y}^\dagger\mathbf{S} + \bar{\mathbf{A}}\Sigma_{AA})\mathbf{A}^\dagger - \mathbf{A}(\mathbf{S}^\dagger\mathbf{Y} + \Sigma_{AA}\bar{\mathbf{A}}^\dagger) + \mathbf{A}(\mathbf{S}^\dagger\mathbf{S} + \Sigma_{AA})\mathbf{A}^\dagger + \bar{\mathbf{A}}\Sigma_{AA}\bar{\mathbf{A}}^\dagger]\}}{\det(\pi\Sigma)^{N+K}} \quad (111)$$

Next we define

$$\mathbf{B}(\mathbf{S}) = \mathbf{Y}^\dagger\mathbf{S} + \bar{\mathbf{A}}\Sigma_{AA} \quad (112)$$

$$\Sigma'_{AA}(\mathbf{S}) = \mathbf{S}^\dagger\mathbf{S} + \Sigma_{AA}$$

(112)

Substituting (112) and (113) in (111) we obtain

$$f_{Z,A|S,\Sigma}(\mathbf{Y}, \mathbf{A}|\mathbf{S}, \Sigma) = \frac{\det(\Sigma_{AA})^K \text{etr}[-\Sigma^{-1}(\mathbf{Y}^\dagger\mathbf{Y} - \mathbf{B}\mathbf{A}^\dagger - \mathbf{A}\mathbf{B}^\dagger + \mathbf{A}\Sigma'_{AA}\mathbf{A}^\dagger + \bar{\mathbf{A}}\Sigma_{AA}\bar{\mathbf{A}}^\dagger)]}{\det(\pi\Sigma)^{N+K}} \quad (114)$$

Which can be further written as

$$f_{Z,A|S,\Sigma}(\mathbf{Y}, \mathbf{A}|\mathbf{S}, \Sigma) = \frac{\det(\Sigma_{AA})^K \text{etr}[-\Sigma^{-1}(\mathbf{Y}^\dagger\mathbf{Y} + \bar{\mathbf{A}}\Sigma_{AA}\bar{\mathbf{A}}^\dagger - \mathbf{B}\mathbf{A}^\dagger - \mathbf{A}\mathbf{B}^\dagger + \mathbf{A}\Sigma'_{AA}\mathbf{A}^\dagger)]}{\det(\pi\Sigma)^{N+K}} \quad (115)$$

Typically we make the linear substitution

$$\mathbf{B}(\mathbf{S})\mathbf{A}^\dagger = \bar{\mathbf{A}}'(\mathbf{S})\Sigma'_{AA}(\mathbf{S})\mathbf{A}^\dagger \quad (117)$$

$$\mathbf{B}(\mathbf{S}) = \bar{\mathbf{A}}'(\mathbf{S})\Sigma'_{AA}(\mathbf{S}) \Rightarrow \bar{\mathbf{A}}'(\mathbf{S}) = \mathbf{B}(\mathbf{S})\Sigma'^{-1}_{AA}(\mathbf{S}) \quad (116)$$

$$\mathbf{A}\mathbf{B}^\dagger(\mathbf{S}) = \mathbf{A}\Sigma'_{AA}(\mathbf{S})\bar{\mathbf{A}}'^\dagger(\mathbf{S}) \quad (118)$$

Hence,

Hence, the term can be written as

$$\mathbf{B}(\mathbf{S})\mathbf{A}^\dagger + \mathbf{A}\mathbf{B}^\dagger - \mathbf{A}\Sigma'_{AA}(\mathbf{S})\mathbf{A}^\dagger = \bar{\mathbf{A}}'(\mathbf{S})\Sigma'_{AA}(\mathbf{S})\mathbf{A}^\dagger + \mathbf{A}\Sigma'_{AA}(\mathbf{S})\bar{\mathbf{A}}'^\dagger(\mathbf{S}) - \mathbf{A}\Sigma'_{AA}(\mathbf{S})\mathbf{A}^\dagger \quad (119)$$

Next, we add and subtract $\bar{\mathbf{A}}'\Sigma'_{AA}\bar{\mathbf{A}}'^\dagger$ and then after we perform factorization, we obtain

$$\bar{\mathbf{A}}'\Sigma'_{AA}\bar{\mathbf{A}}'^\dagger + \bar{\mathbf{A}}'\Sigma'_{AA}\mathbf{A}^\dagger + \mathbf{A}\Sigma'_{AA}\bar{\mathbf{A}}'^\dagger - \mathbf{A}\Sigma'_{AA}\mathbf{A}^\dagger - \bar{\mathbf{A}}'\Sigma'_{AA}\bar{\mathbf{A}}'^\dagger \equiv \bar{\mathbf{A}}'\Sigma'_{AA}\bar{\mathbf{A}}'^\dagger + (\mathbf{A} - \bar{\mathbf{A}}')\Sigma'_{AA}\bar{\mathbf{A}}'^\dagger - (\mathbf{A} - \bar{\mathbf{A}}')\Sigma'_{AA}\mathbf{A}^\dagger \quad (120)$$

which is equivalent with

$$\bar{\mathbf{A}}'\Sigma'_{AA}\bar{\mathbf{A}}'^\dagger + (\mathbf{A} - \bar{\mathbf{A}}')\Sigma'_{AA}\bar{\mathbf{A}}'^\dagger - (\mathbf{A} - \bar{\mathbf{A}}')\Sigma'_{AA}\mathbf{A}^\dagger \equiv \bar{\mathbf{A}}'\Sigma'_{AA}\bar{\mathbf{A}}'^\dagger - (\mathbf{A} - \bar{\mathbf{A}}')\Sigma'_{AA}(\mathbf{A} - \bar{\mathbf{A}}')^\dagger \quad (121)$$

Finally, the joint pdf between the received data matrix $\mathbf{Y}$ and parameters as the steering matrix $\mathbf{A}$ as can be written in terms of the posterior

$$f_{Z,A|S,\Sigma}(\mathbf{Y}, \mathbf{A}|\mathbf{S}, \Sigma) = \frac{\det(\Sigma_{AA})^K \text{etr}[-\Sigma^{-1}(\mathbf{Y}^\dagger\mathbf{Y} + \bar{\mathbf{A}}\Sigma_{AA}\bar{\mathbf{A}}^\dagger - \bar{\mathbf{A}}'\Sigma'_{AA}\bar{\mathbf{A}}'^\dagger + (\mathbf{A} - \bar{\mathbf{A}}')\Sigma'_{AA}(\mathbf{A} - \bar{\mathbf{A}}')^\dagger)]}{\det(\pi\Sigma)^{N+K}} \quad (122)$$

Next, integrating all over the random steering matrix $\mathbf{A}$ we obtain

$$f_{Z|S,\Sigma}(\mathbf{Y}|\mathbf{S}, \Sigma) = \frac{\det(\Sigma_{AA})^K \text{etr}\{-\Sigma^{-1}[\mathbf{Y}^\dagger\mathbf{Y} + \bar{\mathbf{A}}\Sigma_{AA}\bar{\mathbf{A}}^\dagger - \bar{\mathbf{A}}'(\mathbf{S})\Sigma'_{AA}(\mathbf{S})\bar{\mathbf{A}}'^\dagger(\mathbf{S})]\}}{\det(\pi\Sigma)^N \det[\Sigma'_{AA}(\mathbf{S})]^K} \quad (123)$$

This concludes the derivations of the Bayesian Estimation. Next, we proceed with the *Computation of Interference parameters*.

variable case; and (2) *Computation of interference auto-covariance matrix* $\Sigma: M \times M$ for the *Complex Normal Matrix Variate* case.

7.1 Computation of Interference Variance σ_Z^2

The best and easiest assumption we can make for the unknown interference variance, σ_Z^2 , is the exponential distribution as follows

7 Computation of Interference Parameters

Computation of interference parameters includes: (1) *Computation of interference variance* σ_Z^2 for the single

$$f_{\sigma_Z^2|\sigma_W^2}(\sigma_Z^{-2}|\sigma_W^2) = \sigma_W^2 e^{-\sigma_Z^{-2}\sigma_W^2} \quad (124)$$

This result is very significant for two reasons. First, when samples are complex normal I.I.D. then the magnitude of a random variable for a single sample is exponentially distributed^{ix}. Since, the mean of the exponential distribution is

$$f_{Z|S,\sigma_Z^2}(y|s,\sigma_Z^2)f_{\sigma_Z^2|\sigma_W^2}(\sigma_Z^{-2}|\sigma_W^2) = \frac{\sigma_A^2}{(\pi\sigma_Z^2)^2\sigma_A^2} e^{-\sigma_Z^{-2}(|y|^2+\sigma_A^2|\bar{a}|^2-\sigma_A'^2|\bar{a}'|^2)} \sigma_W^2 e^{-\sigma_Z^{-2}\sigma_W^2} \quad (126)$$

Equation (126) is equivalent with

$$f_{Z|S,\sigma_Z^2}(y|s,\sigma_Z^2)f_{\sigma_Z^2|\sigma_W^2}(\sigma_Z^{-2}|\sigma_W^2) = \frac{\sigma_A^2\sigma_W^2}{(\pi\sigma_Z^2)^2\sigma_A^2} e^{-\sigma_Z^{-2}(|y|^2+\sigma_A^2|\bar{a}|^2-\sigma_A'^2|\bar{a}'|^2+\sigma_W^2)} \quad (127)$$

Next, we define the following quantity $c(s)$

$$c(s) = |y|^2 + \sigma_A^2|\bar{a}|^2 - \sigma_A'^2(s)|\bar{a}'|^2 \quad (128)$$

then we have

$$f_{Z|S,\sigma_Z^2}(y|s,\sigma_Z^2)f_{\sigma_Z^2|\sigma_W^2}(\sigma_Z^{-2}|\sigma_W^2) = \frac{\sigma_A^2\sigma_W^2 e^{-\sigma_Z^{-2}[c+\sigma_W^2]}}{(\pi\sigma_Z^2)^2\sigma_A^2} \quad (129)$$

Next, if we integrate over σ_Z^2 then we obtain the following

$$\begin{aligned} f_{Z|S}(y|s) &= \int_{\sigma_Z^2} f_{Z|S,\sigma_Z^2}(y|s,\sigma_Z^2)f_{\sigma_Z^2|\sigma_W^2}(\sigma_Z^{-2}|\sigma_W^2)d\sigma_Z^2 \quad (130) \\ &= \frac{\sigma_A^2\sigma_W^2}{\pi^2\sigma_A'^2} \int_{\sigma_Z^2} \sigma_Z^{-4} e^{-\sigma_Z^{-2}[c+\sigma_W^2]} d\sigma_Z^{-2} \end{aligned}$$

Next, employing the table of integrals (see [45] pg. 337, ex. 3.326 1.) we have

$$\int_0^\infty x^m e^{-ax^n} dx = \frac{\Gamma(\gamma)}{n\beta^\gamma}, \quad \gamma = \frac{m+1}{n}; \quad \text{Re}(\beta; m; n) > 0 \quad (131)$$

Therefore, if we substitute $\sigma_Z^{-2} = x$, and $\beta = c + \sigma_W^2$ then we have

$$\int_0^\infty \frac{x^2 dx}{e^{\beta x}} = \frac{\Gamma(2) \equiv 1}{(\beta \equiv c + \sigma_W^2)^2}; \quad \gamma = \frac{m+1 \equiv 2}{n \equiv 1} = 2; \quad \left| \begin{array}{l} \text{Re}(u) > 0 \\ u = \beta; m; n \end{array} \right. \quad (132)$$

Finally, substituting (132) into (130) we obtain the following

$$f_{Z|S}(y|s) = \frac{\sigma_A^2\sigma_W^2}{\pi^2\sigma_A'^2(c+\sigma_W^2)^2} x \quad (133)$$

We re-write from (133) as

$$f_{Z|S}(y|s) = \eta(s)[c(s) + p]^{-2} = \eta(s)p'^{-2} \quad (134)$$

Where $\eta(s)$ is an integration constant that we need never compute as follows

$$\eta(s) = \frac{\sigma_A^2\sigma_W^2}{\pi^2\sigma_A'^2(s)} \quad (135)$$

And p is the initial variance given by

$$p = \sigma_W^2 \quad (136)$$

$$E[\sigma_Z^{-2}] = \frac{1}{\sigma_W^2} = \sigma_W^{-2} \quad (125)$$

which makes a lot of sense. The second reason is given later in the paper.

Next, taking the product of (103) with (124) yields

Hence, the formulas for updating the Bayesian hyper-parameters can then be written as

$$\sigma_A'^2(s) = \sigma_A^2 + |s|^2 \quad (137)$$

$$\bar{a}'(s) = b(s)\sigma_A'^{-2}(s) = (\bar{a}\sigma_A^2 + y^*s)\sigma_A'^{-2}(s) \quad (138)$$

This concludes the derivations of the *computation of interference variance* σ_Z^2 .

7.2 Computation of Interference Auto-Covariance Matrix $\Sigma: M \times M$

The form of (123) admits a conjugate prior for Σ , suitable for Bayesian parameter estimation. Assume therefore that Σ is distributed according to the complex inverse Wishart, distribution [53], [54], for a block of L samples, this can be written as [4], [51], [55]-[57]

$$f_{\Sigma|\Sigma_W}(\Sigma^{-1}|\Sigma_W) = \frac{\det(L\Sigma_W)^L \det(\Sigma^{-1})^{(L-M)} \text{etr}[-\Sigma^{-1}\Sigma_W L]}{c\Gamma_M(L)} \quad (139)$$

where $\Sigma: M \times M$ and hyper-parameter $\Sigma_W: M \times M$ are CPDM [Chap. 5 of Progni [28]], and $c\Gamma_M(L)$ is the complex multivariate gamma function [56] which is defined as

$$c\Gamma_M(L) = \pi^{\frac{M(M-1)}{2}} \prod_{m=1}^M \Gamma(L - m + 1) \quad (140)$$

The complex matrix variate [inverse] Wishart distribution [48], [53], [54], [58]-[60] is a generalization of the matrix variate exponential distribution which also makes a lot of sense. The second reason that we mentioned in (124) and (125) gave us tremendous insights into the integrity of (139) but at the same time into the physical interpretation of the complex matrix variate [inverse] Wishart distribution [48], [53], [54], [58]-[60] not found anywhere else.

We now multiply the prior (123) into the right hand side of (139) and we obtain

$$f_{Z|S,E}(\mathbf{Y}|\mathbf{S}, \mathbf{\Sigma}) f_{\Sigma|\Sigma_W}(\mathbf{\Sigma}^{-1}|\mathbf{\Sigma}_W) = \frac{\det[\mathbf{\Sigma}_{AA}\mathbf{\Sigma}_{AA}'^{-1}(\mathbf{S})]^K}{\pi^{NM}} \frac{\det(\mathbf{\Sigma}^{-1})^{(L+N-M)} \text{etr}[-\mathbf{\Sigma}^{-1}(\mathbf{Y}^\dagger\mathbf{Y} + \mathbf{\bar{A}}\mathbf{\Sigma}_{AA}\mathbf{\bar{A}}^\dagger - \mathbf{\bar{A}}'(\mathbf{S})\mathbf{\Sigma}'_{AA}(\mathbf{S})\mathbf{\bar{A}}'^\dagger(\mathbf{S}) + \mathbf{\Sigma}_W L)]}{\det(L\mathbf{\Sigma}_W)^{-L} \mathcal{C}\Gamma_M(L)} \quad (141)$$

Next, we define

$$\mathbf{C}(\mathbf{S}) = \mathbf{Y}^\dagger\mathbf{Y} + \mathbf{\bar{A}}\mathbf{\Sigma}_{AA}\mathbf{\bar{A}}^\dagger - \mathbf{\bar{A}}'(\mathbf{S})\mathbf{\Sigma}'_{AA}(\mathbf{S})\mathbf{\bar{A}}'^\dagger(\mathbf{S}) \quad (142)$$

Then, substituting (142) into (141) becomes

$$f_{Z|S,E}(\mathbf{Y}|\mathbf{S}, \mathbf{\Sigma}) f_{\Sigma|\Sigma_W}(\mathbf{\Sigma}^{-1}|\mathbf{\Sigma}_W) = \frac{\det[\mathbf{\Sigma}_{AA}\mathbf{\Sigma}_{AA}'^{-1}(\mathbf{S})]^K}{\pi^{NM}} \frac{\det(\mathbf{\Sigma}^{-1})^{(L+N-M)}}{\det(L\mathbf{\Sigma}_W)^{-L} \mathcal{C}\Gamma_M(L)} \text{etr}\{-\mathbf{\Sigma}^{-1}[\mathbf{C}(\mathbf{S}) + \mathbf{\Sigma}_W L]\} \quad (143)$$

Next, if we integrate over $\mathbf{\Sigma}$ then we obtain the following pdf $f_{Z|S}(\mathbf{Y}|\mathbf{S})$

$$f_{Z|S}(\mathbf{Y}|\mathbf{S}) = \int_{\mathbf{\Sigma}} f_{X|S,E}(\mathbf{Y}|\mathbf{S}, \mathbf{\Sigma}) f_{\Sigma|\Sigma_W}(\mathbf{\Sigma}^{-1}|\mathbf{\Sigma}_W) d\mathbf{\Sigma} = \frac{\det[\mathbf{\Sigma}_{AA}\mathbf{\Sigma}_{AA}'^{-1}(\mathbf{S})]^K}{\pi^{NM} \det(L\mathbf{\Sigma}_W)^{-L} \mathcal{C}\Gamma_M(L)} \int_{\mathbf{\Sigma}} \det(\mathbf{\Sigma}^{-1})^{(L+N-M)} \text{etr}\{-\mathbf{\Sigma}^{-1}[\mathbf{C}(\mathbf{S}) + \mathbf{\Sigma}_W L]\} d\mathbf{\Sigma} \quad (144)$$

Which is equal to

$$f_{Z|S}(\mathbf{Y}|\mathbf{S}) = \int_{\mathbf{\Sigma}} f_{X|S,E}(\mathbf{Y}|\mathbf{S}, \mathbf{\Sigma}) f_{\Sigma|\Sigma_W}(\mathbf{\Sigma}^{-1}|\mathbf{\Sigma}_W) d\mathbf{\Sigma} = \frac{\det(\mathbf{\Sigma}_{AA}\mathbf{\Sigma}_{AA}'^{-1})^K \mathcal{C}\Gamma_M(L+N)}{\pi^{NM} \det(L\mathbf{\Sigma}_W)^{-L} \mathcal{C}\Gamma_M(L) \det(\mathbf{C} + \mathbf{\Sigma}_W L)^{L+N}} \quad (145)$$

We wish to choose, initial hyper-parameters that minimize the effect of the priors on the resulting estimator. Hence, we assume $\mathbf{\bar{A}} = \mathbf{0}$, $L = M$, $\mathbf{\Sigma}_W = w\mathbf{I}$, and $\mathbf{\Sigma} = \varepsilon\mathbf{I}$ (145) can be written as,

$$f_{Z|S}(\mathbf{Y}|\mathbf{S}) = \eta \det[\mathbf{P} + \mathbf{C}(\mathbf{S})]^{-(L+N)} = \eta(\mathbf{S}) \det(\mathbf{P}')^{-L'} \quad (146)$$

Where η is an integration constant that we need never compute as follows

$$\eta(\mathbf{S}) = \frac{\det[\mathbf{\Sigma}_{AA}\mathbf{\Sigma}_{AA}'^{-1}(\mathbf{S})]^K \mathcal{C}\Gamma_M(L+N)}{\pi^{NM} \det(L\mathbf{\Sigma}_W)^{-L} \mathcal{C}\Gamma_M(L)} \quad (147)$$

And $\mathbf{P}$ is the initial covariance matrix given by

$$\mathbf{P} = L\mathbf{\Sigma}_W \quad (148)$$

Hence, the formulas for updating the Bayesian hyper-parameters can then be written as

$$\mathbf{\Sigma}'_{AA}(\mathbf{S}) = \mathbf{\Sigma}_{AA} + \mathbf{S}^\dagger\mathbf{S} \quad (149)$$

$$\mathbf{\bar{A}}'(\mathbf{S}) = \mathbf{B}(\mathbf{S})\mathbf{\Sigma}'_{AA}^{-1}(\mathbf{S}) = (\mathbf{\bar{A}}\mathbf{\Sigma}_{AA} + \mathbf{Y}^\dagger\mathbf{S})\mathbf{\Sigma}'_{AA}^{-1}(\mathbf{S}) \quad (150)$$

$$\begin{aligned} \mathbf{P}'(\mathbf{S}) &= \mathbf{P} + \mathbf{C}(\mathbf{S}) \\ &= \mathbf{P} + \mathbf{Y}^\dagger\mathbf{Y} + \mathbf{\bar{A}}\mathbf{\Sigma}_{AA}\mathbf{\bar{A}}^\dagger - \mathbf{\bar{A}}'(\mathbf{S})\mathbf{\Sigma}'_{AA}(\mathbf{S})\mathbf{\bar{A}}'^\dagger(\mathbf{S}) \end{aligned} \quad (151)$$

This concludes the derivations of the *Computation of*

Interference Auto-Covariance Matrix $\mathbf{\Sigma}: M \times M$.

This concludes the derivations of the *Computation of Interference Parameters*. Next, we discuss the *Bayes Theorem*.

8 Bayes Theorem

Bayes Theorem includes: (1) *Complex Normal Interference Distribution* for the single variable case; and (2) *Complex Normal Matrix Variate Interference Distribution* for the *Complex Normal Matrix Variate* case.

8.1 Complex Normal Interference Distribution

We will use *Bayes Theorem* [3]-[6] and a set of uniform priors for our unknown parameters, t_k , f_k and $d_k(q)$ to obtain the posterior density

$$f_{S|Z}(s|y) = \frac{f_{Z|S}(y|s) f_S(s)}{\int_{-\infty}^{\infty} f_{Z|S}(y|s) f_S(s) ds} \quad (152)$$

Substituting (31) and (134) into (152) and performing the integration via the property of the delta function [42] we obtain the posterior density in closed form expression

$$f_{S|Z}(s|y) = \frac{\eta(s)[c(s)+p]^{-2}\{p_1([t_k]_C)\delta(s-1)+p_{-1}([t_k]_C)\delta(s+1)\}}{\int_{-\infty}^{\infty} \eta(s)[c(s)+p]^{-2}\{p_1([t_k]_C)\delta(s-1)+p_{-1}([t_k]_C)\delta(s+1)\} ds} \eta(1)[c(1)+p]^{-2} p_1([t_k]_C) + \eta(-1)[c(-1)+p]^{-2} p_{-1}([t_k]_C) \quad \text{xii} \quad (153)$$

Since, it turns out that

$$\eta(1) = \frac{\sigma_A^2 \sigma_W^2}{\pi^2 \sigma_A'^2(1)} = \frac{\sigma_A^2 \sigma_W^2}{\pi^2 \sigma_A'^2(-1)} = \eta(-1) \quad (154)$$

Hence, substituting (154) and (155) into (153) we obtain the closed form expression of the posterior Bayesian pdf $f_{S|Z}(s|z)$ as follows

And

$$\begin{aligned} c(1) &= |y|^2 + \sigma_A^2 |\bar{a}|^2 - \sigma_A'^2(1) |\bar{a}'|^2 \\ &= |y|^2 + \sigma_A^2 |\bar{a}|^2 - \sigma_A'^2(-1) |\bar{a}'|^2 = c(-1) \end{aligned} \quad (155)$$

$$f_{S|Z}(s|y, [t_k]_C) = \frac{\eta(s)[c(s)+p]^{-2}\{p_1([t_k]_C)\delta(s-1)+p_{-1}([t_k]_C)\delta(s+1)\}}{\eta(1)[c(1)+p]^{-2}\{p_1([t_k]_C)+p_{-1}([t_k]_C)\}} = \frac{\eta(s)[c(s)+p]^{-2}\{p_1([t_k]_C)\delta(s-1)+p_{-1}([t_k]_C)\delta(s+1)\}}{\eta(1)[c(1)+p]^{-2}} \quad (156)$$

The posterior cdf, $F_{S|Z}(s|y)$, can be easily obtained from integrating (156) as follows

$$F_{S|Z}(s|y, [t_k]_C) = \frac{\int_{-\infty}^s \eta(x)[c(x)+p]^{-2}\{p_1([t_k]_C)\delta(x-1)+p_{-1}([t_k]_C)\delta(x+1)\}dx}{\eta(1)[c(1)+p]^{-2}} \quad (157)$$

This result shows that the posterior cdf, $F_{S|Z}(s|y)$, is actually normal or Gaussian identical to the prior cdf as long as the noise (or interference) is

$$F_{S|Z}(s|y, [t_k]_C) = \begin{cases} 1 & s \geq 1 \\ p_{-1}([t_k]_C) = 1 - p_1([t_k]_C) & -1 \leq s < 1 \equiv F_S(s, [t_k]_C) \\ 0, & s < -1 \end{cases} \quad (158)$$

Equation (158) can also be written in compact form as

$$F_{S|Z}(s|y, [t_k]_C) = p_{-1}([t_k]_C)u(s+1) + p_1([t_k]_C)u(s-1) \equiv F_S(s, [t_k]_C) \quad (159)$$

This result does not contradict the results of the prior publications [3]-[6]; in fact, it enhances the understanding that we did not have prior to this result, which is that we do not gain any new information on obtain the posterior cdf $F_{S|Z}(s|z)$. Hence, the best way of obtaining the unknown parameter $[t_k]_C$ and then of the τ_k is via the estimation of the autocorrelation function or via the maximum likelihood estimation that employs the use of the joint optimization of the autocorrelation function [20].

This concludes the derivations of the *Bayes Theorem* for *Complex Normal Interference Distribution*. Next, we investigate the *Bayes Theorem* for *Complex Normal Matrix Variate Interference Distribution*.

8.2 Complex Normal Matrix Variate Interference Distribution

We will use Bayes theorem and a set of uniform priors for our unknown parameters, t_k , f_k and $d_k(q)$ to obtain the posterior density [3]-[6]

$$f_{S|Z}(\mathbf{S}|\mathbf{Y}) = \frac{f_{Z|S}(\mathbf{Y}|\mathbf{S})f_S(\mathbf{S})}{\int_S f_{Z|S}(\mathbf{Y}|\mathbf{S})f_S(\mathbf{S})d\mathbf{S}} \quad (160)$$

Substituting (45) and (145) into (160) and performing the integration via the property of the delta function we obtain the posterior density in closed form expression

$$f_{S|Z}(\mathbf{S}|\mathbf{Y}) = \frac{\frac{\det[\Sigma_{AA}\Sigma_{AA}'^{-1}(\mathbf{S})]^K}{\pi^{NM}\det(L\Sigma_W)^{-L}\det[\mathbf{C}(\mathbf{S})+\Sigma_{WL}]^{L+N}}\text{vec}(\tilde{\mathbf{P}})\text{vec}(\tilde{\mathbf{A}})}{\int_S \frac{\det[\Sigma_{AA}\Sigma_{AA}'^{-1}(\mathbf{S})]^K}{\pi^{NM}\det(L\Sigma_W)^{-L}\det[\mathbf{C}(\mathbf{S})+\Sigma_{WL}]^{L+N}}\text{vec}(\tilde{\mathbf{P}})\text{vec}(\tilde{\mathbf{A}})d\mathbf{S}} = \frac{\frac{\det[\Sigma_{AA}\Sigma_{AA}'^{-1}(\mathbf{S})]^K}{\det[\mathbf{C}(\mathbf{S})+\Sigma_{WL}]^{L+N}}\text{vec}(\tilde{\mathbf{P}})\text{vec}(\tilde{\mathbf{A}})}{\int_S \frac{\det[\Sigma_{AA}\Sigma_{AA}'^{-1}(\mathbf{S})]^K}{\det[\mathbf{C}(\mathbf{S})+\Sigma_{WL}]^{L+N}}\text{vec}(\tilde{\mathbf{P}})\text{vec}(\tilde{\mathbf{A}})d\mathbf{S}} \quad (161)$$

Due to the equivalence of (45) and (49) the denominator of (161) can be written as

$$\int_S \frac{\det[\Sigma_{AA}\Sigma_{AA}'^{-1}(\mathbf{S})]^K}{\det[\mathbf{C}(\mathbf{S})+\Sigma_{WL}]^{L+N}}\text{vec}(\tilde{\mathbf{P}})\text{vec}(\tilde{\mathbf{A}})d\mathbf{S} = \int_S \frac{\det(\Sigma_{AA}\Sigma_{AA}'^{-1}(\mathbf{S}))^K}{\det[\mathbf{C}(\mathbf{S})+\Sigma_{WL}]^{L+N}} \sum_{i=1}^I p_1^{m(i)}([t_k]_C) p_{-1}^{n(i)}([t_k]_C) \delta\{\mathbf{S} - \mathbf{1}_{u(i) \times v(i)}\} d\mathbf{S} \quad (162)$$

By changing the order of summation and integration we obtain

$$\sum_{i=1}^I p_1^{m(i)}([t_k]_C) p_{-1}^{n(i)}([t_k]_C) \int_S \frac{\det(\Sigma_{AA}\Sigma_{AA}'^{-1}(\mathbf{S}))^K}{\det(\mathbf{C}+\Sigma_{WL})^{L+N}} \delta\{\mathbf{S} - \mathbf{1}_{u(i) \times v(i)}\} d\mathbf{S} = \sum_{i=1}^I p_1^{m(i)}([t_k]_C) p_{-1}^{n(i)}([t_k]_C) \frac{\det\{\Sigma_{AA}\Sigma_{AA}'^{-1}(\mathbf{1}_{u(i) \times v(i)})\}^K}{\det\{\mathbf{C}(\mathbf{1}_{u(i) \times v(i)})+\Sigma_{WL}\}^{L+N}} \quad (163)$$

Substituting (163) into (161) produces the closed form expression of the complex normal matrix variate posterior pdf

$$f_{S|Z}(\mathbf{S}|\mathbf{Y}) = \frac{\sum_{i=1}^I p_1^{m(i)}([t_k]_C) p_{-1}^{n(i)}([t_k]_C) \frac{\det[\Sigma_{AA}\Sigma_{AA}'^{-1}(\mathbf{S})]^K}{\det[\mathbf{C}(\mathbf{S})+\Sigma_{WL}]^{L+N}} \delta\{\mathbf{S} - \mathbf{1}_{u(i) \times v(i)}\}}{\sum_{n=1}^I p_1^{m(i)}([t_k]_C) p_{-1}^{n(i)}([t_k]_C) \frac{\det\{\Sigma_{AA}\Sigma_{AA}'^{-1}(\mathbf{1}_{u(i) \times v(i)})\}^K}{\det\{\mathbf{C}(\mathbf{1}_{u(i) \times v(i)})+\Sigma_{WL}\}^{L+N}}} \quad (164)$$

Next, we produce the complex matrix variate posterior cdf by integrating (164) as follows:

$$F_{S|Z}(\mathbf{S}|\mathbf{Y}) = \frac{\sum_{i=1}^I p_1^{m(i)}([t_k]_C) p_{-1}^{n(i)}([t_k]_C) \frac{\det[\Sigma_{AA} \Sigma_{AA}'^{-1}(\mathbf{1}_{u(i) \times v(i)})]^K}{\det[\mathbf{C}(\mathbf{1}_{u(i) \times v(i)}) + \Sigma_W L]^{L+N}} u(\mathbf{S} - \mathbf{1}_{u(i) \times v(i)})}{\sum_{n=1}^I p_1^{m(i)}([t_k]_C) p_{-1}^{n(i)}([t_k]_C) \frac{\det[\Sigma_{AA} \Sigma_{AA}'^{-1}(\mathbf{1}_{u(i) \times v(i)})]^K}{\det[\mathbf{C}(\mathbf{1}_{u(i) \times v(i)}) + \Sigma_W L]^{L+N}}} \quad (165)$$

We can simplify (165), based on the definition of Heaviside

unit step function of matrix argument given by (61), as follows

$$F_{S|Z}(\mathbf{S}|\mathbf{Y}) = \begin{cases} 1 & \mathbf{S} \geq \mathbf{1} \\ \frac{p_{-1}^{NK}([t_k]_C) \equiv [1 - p_1([t_k]_C)]^{NK} \frac{\det[\Sigma_{AA} \Sigma_{AA}'^{-1}(-\mathbf{1})]^K}{\det[\mathbf{C}(-\mathbf{1}) + \Sigma_W L]^{L+N}}}{\sum_{n=1}^I p_1^{m(i)}([t_k]_C) p_{-1}^{n(i)}([t_k]_C) \frac{\det[\Sigma_{AA} \Sigma_{AA}'^{-1}(\mathbf{1}_{u(i) \times v(i)})]^K}{\det[\mathbf{C}(\mathbf{1}_{u(i) \times v(i)}) + \Sigma_W L]^{L+N}}} & -1 \leq \mathbf{S} < \mathbf{1} \\ 0, & \mathbf{S} < -1 \end{cases} \quad (166)$$

It can be easily shown (see Appendix B) that under certain

well based assumptions

$$\frac{\det[\Sigma_{AA} \Sigma_{AA}'^{-1}(\mathbf{1}_{u(i) \times v(i)})]^K}{\det[\mathbf{C}(\mathbf{1}_{u(i) \times v(i)}) + \Sigma_W L]^{L+N}} \equiv \frac{\det[\Sigma_{AA} \Sigma_{AA}'^{-1}(\mathbf{1}_{u(i) \times v(i)})]^K}{\det[\mathbf{C}(\mathbf{1}_{u(i) \times v(i)}) + \Sigma_W L]^{L+N}} \equiv \frac{\det[\Sigma_{AA} \Sigma_{AA}'^{-1}(-\mathbf{1})]^K}{\det[\mathbf{C}(-\mathbf{1}) + \Sigma_W L]^{L+N}}, \quad i \in \{1, 2, \dots, I\} \quad (167)$$

Why? Because the indices matrix $\mathbf{1}_{u(i) \times v(i)}$ does not change the eigen-values of the correlation matrix it only performs a

phase rotation.

Hence, substituting (167) into (166) we obtain the following

$$F_{S|Z}(\mathbf{S}|\mathbf{Y}) = \begin{cases} 1 & \mathbf{S} \geq \mathbf{1} \\ \frac{p_{-1}^{NK}([t_k]_C) \equiv [1 - p_1([t_k]_C)]^{NK}}{\sum_{n=1}^I p_1^{m(i)}([t_k]_C) p_{-1}^{n(i)}([t_k]_C)} \equiv p_{-1}^{NK}([t_k]_C) \equiv [1 - p_1([t_k]_C)]^{NK} & -1 \leq \mathbf{S} < \mathbf{1} \\ 0, & \mathbf{S} < -1 \end{cases} \equiv F_S(\mathbf{S}) \quad (168)$$

Or the above result can be written in compact format as

follows

$$F_{S|Z}(\mathbf{S}|\mathbf{Y}) = p_{-1}^{NK}([t_k]_C) u(\mathbf{S} + \mathbf{1}) + \{1 - p_{-1}^{NK}([t_k]_C)\} u(\mathbf{S} - \mathbf{1}) \equiv F_S(\mathbf{S}) \quad (169)$$

Again, (169) does not contradict the results of the prior publications [3]-[6]; in fact, it enhances the understanding that we did not have prior to this result, which is that we do not gain any new information on obtain the posterior pdf $f_{S|Z}(\mathbf{S}|\mathbf{Y})$ or cdf $F_{S|Z}(\mathbf{S}|\mathbf{Y})$. Hence, the best way of obtaining the unknown parameter, $[t_k]_C$, and from which we derive τ_k , is via the estimation of the autocorrelation function or via the maximum likelihood estimation that employs the use of the joint optimization of the autocorrelation function [20].

9 Simulation

The acquisition process of a MLE GPS receiver is illustrated in Fig.1 (c) (Progri 2018, [23]). As shown in Fig.1 (b) and (c) of [23] we assume that the IF GPS signal is employed to excite a two-dimensional maximum likelihood Doppler and Code estimator. The acquisition process contains the MLE model and the Doppler and delay offset estimation. The reader is reminded that in this paper we use interchangeably the terms Doppler estimation for Doppler search and delay offset estimation for Code search and vice versa [22].

By treating all the signals in the environment jointly, it is possible to greatly outperform the “sliding correlator” technique especially in situations where the satellite signals are received with widely varying powers. For a simple numerical example, consider a simulated environment wherein one satellite signal is received at -20 dB and nine jamming signals at 15, 20, 25, 30, 35, 45, 50, 55, and 75 dB signal to white noise power (SWNR) at -10 dB. The received environment is modeled at complex baseband assuming the reception of a 1 ms, 1023 chip Gold codes from one GPS satellite and three jammers. We have also assumed that a standard deviation on the GPS receiver clock error is half of the chipping period, T_c . We assume that the GPS Doppler frequency is normally distributed with 0 mean and 100 Hz standard deviation [22].

The reason we have assumed very high jamming power is because there are already in place requirements to test anti-jam system in completely jammed environments [24]-[27].

The U.S. Air Force 746th Test Squadron has declared Initial Operational Capability (IOC) for its new truth reference, the Ultra High-Accuracy Reference System (UHARS) at the White Sands Missile Range in New Mexico [26], [27].

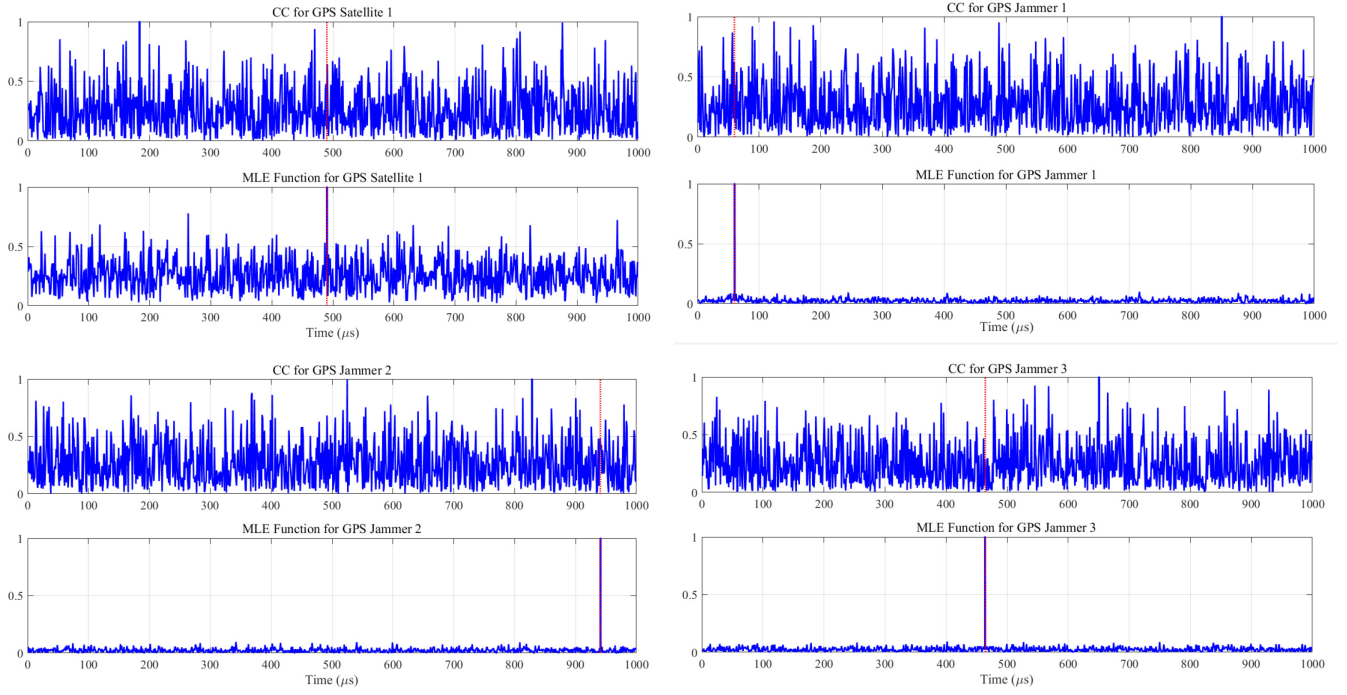


FIGURE 2: The estimated and true $\tau_{1:4}$ using the CC(top) and MLE (bottom) for GPS Satellite 1 and Jammers 1 through 3.

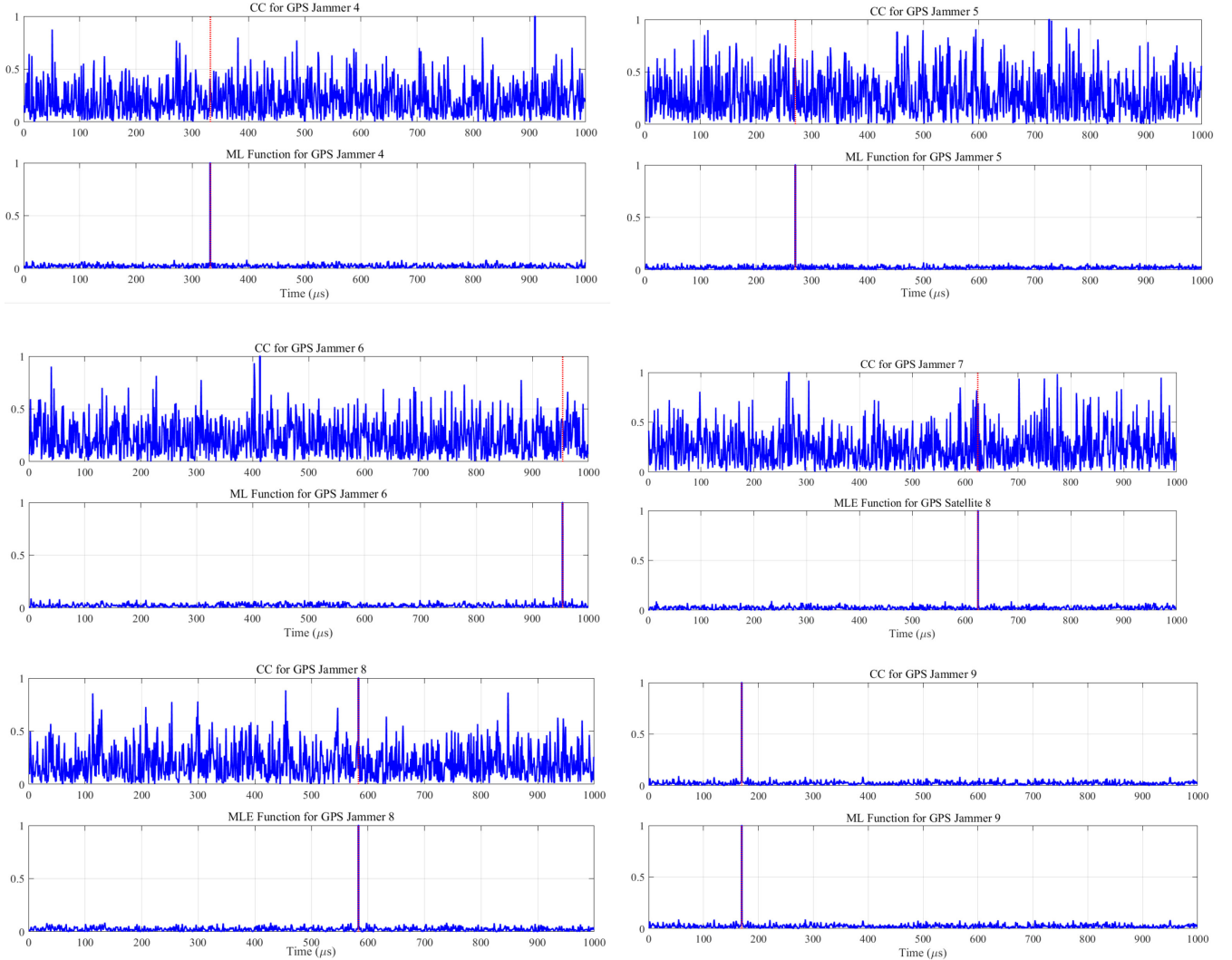


FIGURE 3: The estimated and true $\tau_{5:10}$ using the CC (top) and MLE (bottom) for GPS Jammers 4 through 9.

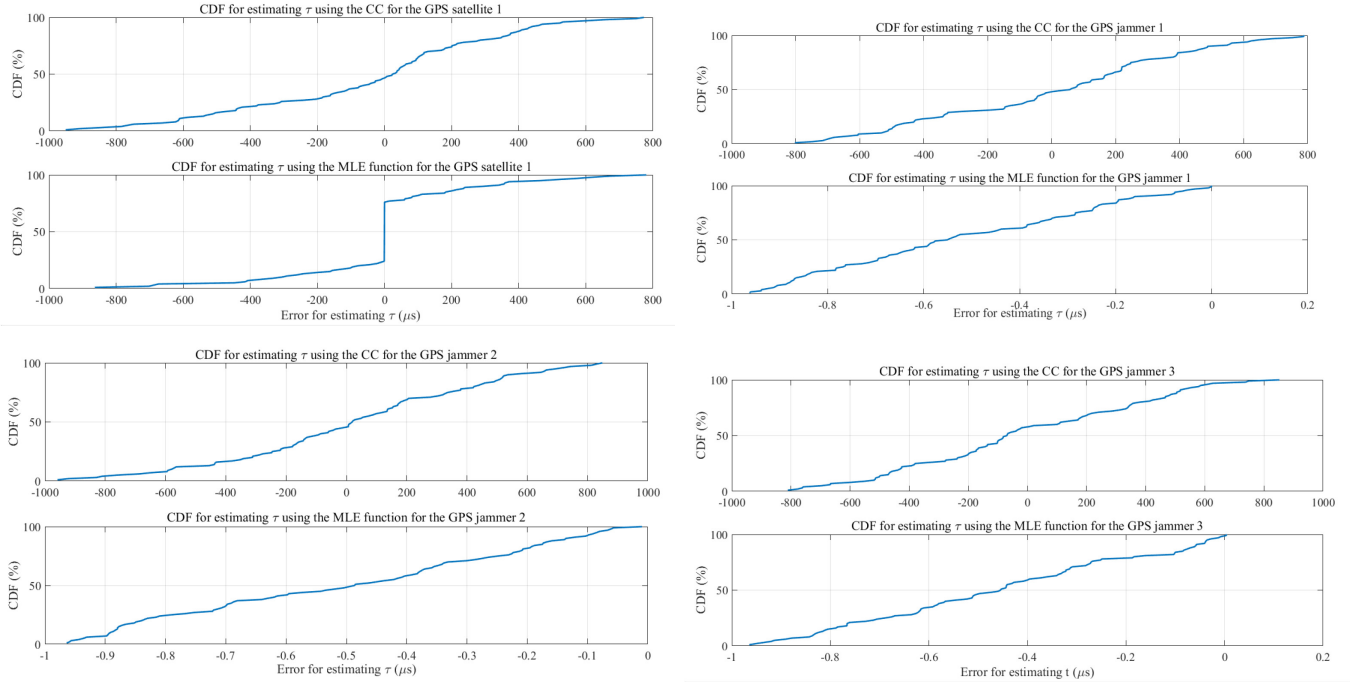


FIGURE 4: CDF of the estimated $\tau_{1:4}$ using the CC(top) and MLE (bottom) for GPS Satellite 1 and Jammers 1 through 3.

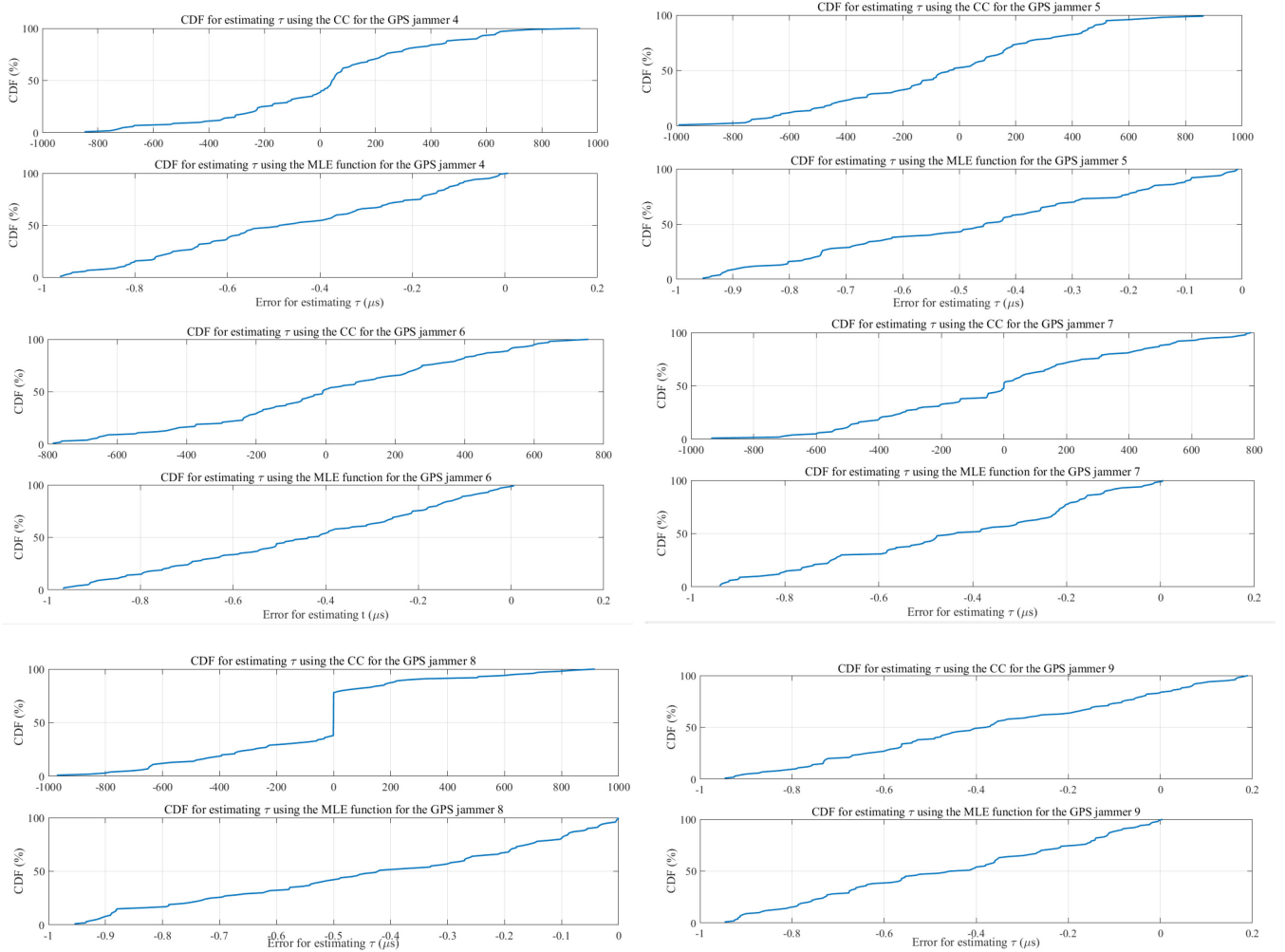


FIGURE 5: CDF of the estimated $\tau_{5:10}$ using the CC (top) and MLE (bottom) for GPS Jammers 4 through 9.

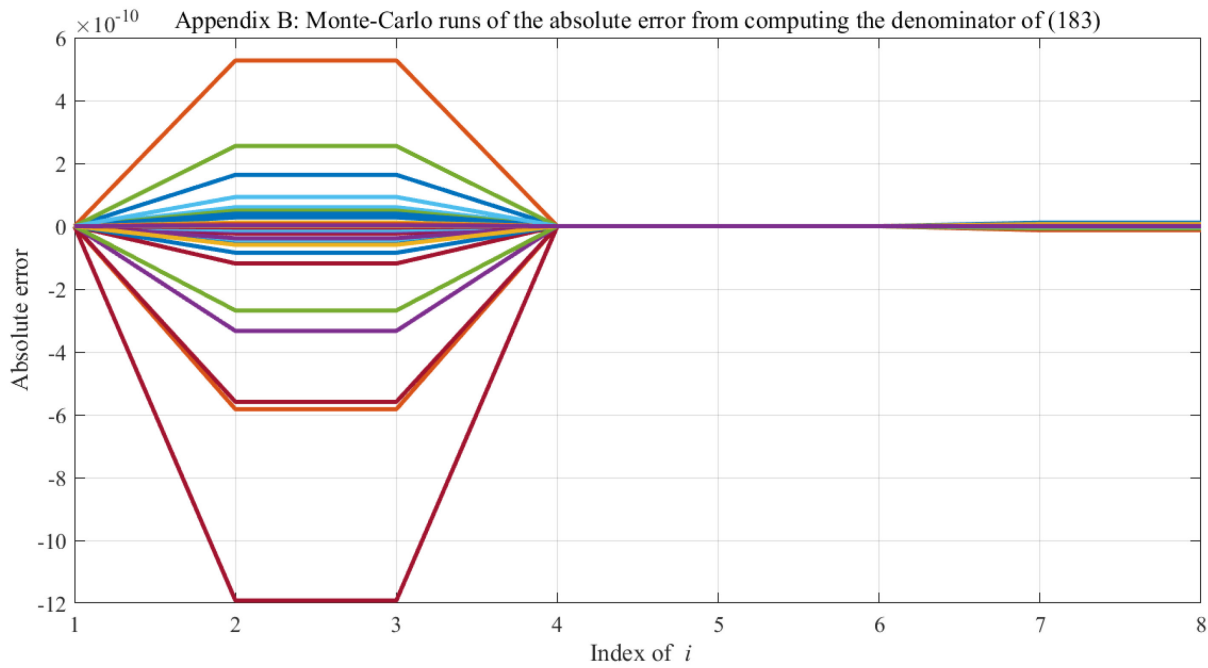


FIGURE 6: Appendix B: Monte-Carlo runs of the absolute error from computing the denominator of (183).

Even when GPS — or any other GNSS system — is being completely jammed, UHARS [25] provides extremely accurate positioning, navigation and time (PNT) over the large area that the system was designed to cover.

Therefore the assumptions made in this scenario are completely realistic and I believe that Giftet Inc. Indoor Geolocation System MATLAB Library will further enhance these already tighter requirements for system to operate in completely jammed environments.

The normalized CC for the weakest signal, at -20 dB is shown in Fig. 2 (top left). The simple cross-correlator cannot detect the weakest signal at -20 dB because the strongest jamming signals at 15-75 dB can jam the weakest signal. This result is in total agreement with the results reported in literature using real data and employing either a Tong search detector or a FFT search detector for 1 ms integration time. In contrast the MLE (i.e., objective function), shown in Fig. 2 (bottom left), can detect the weakest signal more than fifty percent of the time (see Fig. 4 (top left)). Because the strong GPS satellite codes have been cancelled, the MLE can still pick up a clear peak at the correct delay more than fifty percent of the time [22].

This same receive scenario was run over 100 Monte Carlo trials while randomly varying the thermal noise, delay offsets gold sequence seeds, and Doppler frequencies. The standard moving CC achieved a median absolute delay estimation error of $\{225.3077, 325.3015, 296.5697, 340.1617, 224.9457, 317.5196, 267.0810, 307.4821, 90.6868, 0.3745\}$ μs and was never really able to detect the second strongest signal and rarely

able to detect the remaining weaker signals. The MLE was successful in all 100 trials; however, it achieved a median absolute delay error of $\{941.3, 548.8, 485.7, 454.0, 479.6, 453.3, 427.0, 435.7, 420.9, 435.8\}$ ns [22].

Figures 2, 3 present the normalized CC for the jamming signals, at 15, 20, 25, 30, 35, 45, 50, 55, and 75 dB [22]. As shown from Figs. 2-3 the CC type of receiver is only able to correctly estimate the delay for the 75 dB jammer. It fails to estimate the delay for the remaining eight jammers. The MLE GPS receiver is always able to correctly estimate the delay for all nine jammers and the GPS signal.

Figures 4, 5 depict the cdf of both the normalized CC and MLE for all the signals in the environment. *As seen from Figs. 5-7 the CC function fails when the SINR reaches -20 dB as opposed to the MLE which can still reliably detect a signal even at SINR -95 dB; hence, the MLE receiver exhibits a 75 dB SINR improvement performance against a CC receiver.*

This example perfectly illustrates that jamming is no longer an issue for the MLE GPS receiver regardless of the frequency, signal shape waveform, spreading modulation waveform etc.; i.e., the MLE GPS receiver will perform as good as if there was only one signal and noise; i.e., the ideal case. In order to get better performance than you would have to improve the signal via innovative signal design.

We propose a MLE GPS receiver for processing the received GPS signals of the L_1 , L_2 , or L_5 frequencies. The MLE GPS receiver performs a simultaneous, two-dimensional, search of both the Doppler frequencies and GPS Gold codes. The

Doppler bin search size should be not more than 100 Hz. Moreover, we have identified a new approach for improving TOA estimation performance by considering it as multi-user statistical estimation problem and employing MLE techniques. A simple example has been provided showing nearly an order of magnitude improvement in TOA performance. Moreover there are a number of numerical procedures that can be employed to reduce the computational burden of the more powerful estimation technique. It is expected that this approach can yield additional benefits in GPS performance in environments where the “near-far” problem limits acquisition of weak GPS signals by the CC estimation. It is additionally expected to yield further gains as these techniques are extended to environments containing significant multipath [7].

10 Conclusions

This paper is the first complete discussion of the derivation of the Bayesian posterior density of the DDS with interference and/or jamming assuming that the interference is normal or Gaussian distributed.

Either in scalar or matrix formulation, assuming that the interference is normal or Gaussian we do not gain any new information on obtain the posterior pdf $f_{S|Z}(\mathbf{S}|\mathbf{Y})$. Hence, the best way of obtaining the unknown parameter $[t_k]_C$ and then of the τ_k is via the estimation of the autocorrelation function or via the MLE that employs the use of the joint optimization of the autocorrelation function [20]–[22].

As seen from Figs. 4, 5 the CC function fails when the SINR reaches -20 dB as opposed to the MLE which can still reliably detect a signal even at SINR -95 dB; hence, the MLE receiver exhibits a 75 dB SINR improvement performance against a CC receiver.

In part 2 of this publication, as part of future work, we will show to how produce direct approach of the Bayesian posterior density based on the *Complex Matrix Variate Signal Distribution* that we have produced in this paper and either the *Complex Matrix Variate Bessel Interference Distribution* or *Complex Matrix Variate Parabolic Function Interference Distribution* that we will produce later employing the information from Proгри 2018 [16].

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This is without a doubt one of the best analytical, theoretical

papers that I have ever produced. At the same time, this paper would have not been possible without the earlier contributions from my brilliant mathematician co-author Matthew C. Bromberg [3]–[6]. Matthew was able to visualize and derive the initial blue-print of the partial Bayesian algorithm. In this paper I was able to carry out his work even further to show an amazing agreement between the theory, developed in this paper, and the MCMC simulations initially produced in [3]–[6].

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$$\arg \max_{s \in \mathcal{S}} [f_{Z|S}(y|s)] = P(Z \in z) = \int_{-\infty}^{\infty} f_{Z|S}(y|s) f_S(s) ds \quad (171)$$

where $\mathcal{S}$ is the space of the all-possible values of the argument s .

This represents a new or original discovery that brilliantly connects the BPE with MLE not published before. This result is not an accident. This result provides one of the most powerful insights that I was looking for years and it happened because I came to fully develop the prior distribution signal model and the posterior distribution, compute the $P(Z \in z)$ is the probability of the data averaged over all parameters.

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13 Appendix A: Equivalence of the MLE with the Bayes Parameter Estimation

In this appendix we derive the equivalence of the MLE and BPE.

It is already known that a MLE coincides with the most probable Bayesian estimator given a uniform prior distribution on the parameters [2]. Indeed, the maximum a posteriori estimate is the parameter s that maximizes the probability of s given the data, given by Bayes' theorem: [2].

$$f_{S|Z}(s|y) = \frac{f_{Z|S}(y|s) f_S(s)}{P(Z \in z) = \int_{-\infty}^{\infty} f_{Z|S}(y|s) f_S(s) ds} \quad (170)$$

where $f_S(s)$ is the prior distribution for the parameter s and where $P(Z \in z)$ is the probability of the data averaged over all parameters. Since the denominator is independent of s , the Bayesian estimator is obtained by maximizing $f_{Z|S}(y|s) f_S(s)$ with respect to s . If we further assume that the prior $f_S(s)$ is a uniform distribution, the Bayesian estimator is obtained by maximizing the likelihood function $f_{Z|S}(y|s)$. Thus, the Bayesian estimator coincides with the MLE for a uniform prior distribution $f_S(s)$ [2].

What if the prior distribution $f_S(s)$ is identical to the posterior distribution $f_{S|Z}(s|y)$. In this case the Bayesian estimator will maximize the likelihood function $f_{Z|S}(y|s)$ so that it is equal to the $P(Z \in z)$ is the probability of the data averaged over all parameters which can be written as

In application. For the single BPE we have the likelihood function, $f_{Z|S}(y|s)$,

$$f_{Z|S}(y|s) = \eta(s) [c(s) + p]^{-2} = \eta(s) p'(s)^{-2} \quad (172)$$

The probability of the data averaged over all parameters, $P(Z \in z)$, is given by

$$P(Z \in z) = \eta(1) [c(1) + p]^{-2} \quad (173)$$

where z is a subset of space of the realization of the complex random variable Z .

In general, the likelihood function, $f_{Z|S}(y|s)$ is different from the probability of the data averaged over all parameters, $P(Z \in z)$. It is only equal iff for $s \pm 1$ which corresponds with

$$\arg \max_{s \in \mathbb{S}} [f_{Z|S}(y|s) \equiv \eta(s)[c(s) + p]^{-2}] \equiv \eta(1)[c(1) + p]^{-2} \equiv P(Z \in z) = \int_{-\infty}^{\infty} f_{Z|S}(y|s) f_S(s) ds \quad (174)$$

In application. For the matrix variate BPE we have the likelihood function, $f_{Z|S}(\mathbf{Y}|\mathbf{S})$,

$$f_{Z|S}(\mathbf{Y}|\mathbf{S}) = \eta(\mathbf{S}) \det[\mathbf{P} + \mathbf{C}(\mathbf{S}) \equiv \mathbf{P}'(\mathbf{S})]^{-(L+N \equiv L')} \quad (175)$$

$$P(\mathbf{Z} \in \mathbf{z}) = \sum_{n=1}^I p_1^{m(i)} ([t_k]_C) p_{-1}^{n(i)} ([t_k]_C) \frac{\det\{\Sigma_{AA} \Sigma_{AA}'^{-1} [\mathbf{I}_{u(i)v(i)}]\}^K}{\det\{\mathbf{C}[\mathbf{I}_{u(i)v(i)}] + \Sigma_{WL}\}^{L+N}} \quad (176)$$

where $\mathbf{z}$ is a subset of space of the realization of the complex random matrix variate $\mathbf{Z}$.

$$\frac{\det\{\Sigma_{AA} \Sigma_{AA}'^{-1} [\mathbf{I}_{u(i)v(i)}]\}^K}{\det\{\mathbf{C}[\mathbf{I}_{u(i)v(i)}] + \Sigma_{WL}\}^{L+N}} \equiv \frac{\det\{\Sigma_{AA} \Sigma_{AA}'^{-1} [\mathbf{I}_{u(i)v(i)}]\}^K}{\det\{\mathbf{C}[\mathbf{I}_{u(i)v(i)}] + \Sigma_{WL}\}^{L+N}} \equiv \frac{\det\{\Sigma_{AA} \Sigma_{AA}'^{-1} (\mathbf{1})\}^K}{\det\{\mathbf{C}(\mathbf{1}) + \Sigma_{WL}\}^{L+N}}; i \in \{1, 2, \dots, I\} \quad (177)$$

Hence,

$$P(\mathbf{Z} \in \mathbb{Z}) = \frac{\det\{\Sigma_{AA} \Sigma_{AA}'^{-1} (\mathbf{1})\}^K [\sum_{n=1}^I p_1^{m(i)} ([t_k]_C) p_{-1}^{n(i)} ([t_k]_C) \equiv 1]}{\det\{\mathbf{C}(\mathbf{1}) + \Sigma_{WL}\}^{L+N}} \quad (178)$$

In general, the likelihood function, $f_{Z|S}(\mathbf{Y}|\mathbf{S})$ is different

$$\arg \max_{s \in \mathbb{S}} [f_{Z|S}(\mathbf{Y}|\mathbf{S}) \equiv \eta(\mathbf{S}) \det[\mathbf{P}'(\mathbf{S})]^{-L'}] \equiv \frac{\det\{\Sigma_{AA} \Sigma_{AA}'^{-1} (\mathbf{1})\}^K}{\det\{\mathbf{C}(\mathbf{1}) + \Sigma_{WL}\}^{L+N}} \equiv P(\mathbf{Z} \in \mathbf{z}) = \int_{ds} f_{Z|S}(\mathbf{Y}|\mathbf{S}) f_S(s) ds \quad (179)$$

where $\mathbb{S}$ is the multi-dimensional space of the all-possible values of the complex matrix variate argument $\mathbf{s}$.

This represents a new or original discovery that brilliantly connects the BPE with MLE not published before. This result is not an accident. This result provides one of the most powerful insights that I was expecting for years and it happened because I came to fully develop the prior distribution complex signal matrix variate model and the posterior distribution, compute the $P(\mathbf{Z} \in \mathbf{z})$ is the probability of the data averaged over all parameters.

$$\frac{\det\{\Sigma_{AA} \Sigma_{AA}'^{-1} [\mathbf{I}_{u(i)v(i)}]\}^K}{\det\{\mathbf{C}[\mathbf{I}_{u(i)v(i)}] + \Sigma_{WL}\}^{L+N}} \equiv \frac{\det\{\Sigma_{AA} [\Sigma_{AA} + \mathbf{1}_{u(i)v(i)}^\dagger \mathbf{1}_{u(i)v(i)}]^{-1}\}^K}{\det\{\mathbf{Y}^\dagger \mathbf{Y} + \bar{\mathbf{A}} \Sigma_{AA} \bar{\mathbf{A}}^\dagger - (\bar{\mathbf{A}} \Sigma_{AA} + \mathbf{Y}^\dagger \mathbf{1}_{u(i)v(i)}) [\Sigma_{AA} + \mathbf{1}_{u(i)v(i)}^\dagger \mathbf{1}_{u(i)v(i)}]^{-1} (\Sigma_{AA} \bar{\mathbf{A}}^\dagger + \mathbf{1}_{u(i)v(i)}^\dagger \mathbf{Y}) + \Sigma_{WL}\}^{L+N}} \quad (180)$$

Here we exploit the importance of the structure of matrix $\mathbf{1}_{u(i)v(i)}$. The structure of matrix must fulfil the following equality

$$\mathbf{1}_{u(i)v(i)}^\dagger \mathbf{1}_{u(i)v(i)} = \mathbf{K} \mathbf{1}_{K \times K} \equiv \mathbf{K} \mathbf{1} \quad (181)$$

If we were to assume $\Sigma_{AA} = \mathbf{0}$ then $\Sigma_{AA} +$

$$\frac{\det\{\Sigma_{AA} \Sigma_{AA}'^{-1} [\mathbf{I}_{u(i)v(i)}]\}^K}{\det\{\mathbf{C}[\mathbf{I}_{u(i)v(i)}] + \Sigma_{WL}\}^{L+N}} \equiv \frac{\det[\mathbf{I}(\mathbf{I} + \mathbf{K} \mathbf{1})^{-1}]^K}{\det\{\mathbf{Y}^\dagger \mathbf{Y} - \mathbf{Y}^\dagger \mathbf{1}_{u(i)v(i)} (\mathbf{I} + \mathbf{K} \mathbf{1})^{-1} \mathbf{1}_{u(i)v(i)}^\dagger \mathbf{Y} + \Sigma_{WL}\}^{L+N}} \quad (182)$$

Let us assume that for the sake of simplicity that $\Sigma_W = \mathbf{0}$

the values of the parameter for which prior distribution $f_S(s)$ is identical to the posterior distribution $f_{S|Z}(s|y)$; hence,

The probability of the data averaged over all parameters, $P(\mathbf{Z} \in \mathbf{z})$, is given by

It can be easily shown that (see Appendix B)

from the probability of the data averaged over all parameters, $P(\mathbf{Z} \in \mathbf{z})$. It is only equal iff for $s \pm 1$ which corresponds with the values of the parameter for which prior distribution $f_S(s)$ is identical to the posterior distribution $f_{S|Z}(s|y)$; hence,

This concludes the discussion on Appendix A: the equivalence of the MLE with the BPE. Next we discuss Appendix B: the derivations of (167).

14 Appendix B: Derivations of (167)

Since it is not intuitive for most readers; we are providing the details for the derivations of (167).

Let us substitute (142), (149), and (150) into (167) we obtain

$\mathbf{1}_{u(i)v(i)}^\dagger \mathbf{1}_{u(i)v(i)} = \mathbf{K} \mathbf{1}$ does not have an inverse. Hence, as stated earlier, the initialization of $\Sigma_{AA} = \mathbf{0}$ is a bad assumption. Instead we will assume that $\Sigma_{AA} = \mathbf{I}$ is an identity matrix.

If we also assume that $\bar{\mathbf{A}} = \mathbf{0}$ and substituting (181) to (180) we obtain

then the following yields

$$\frac{\det\{\Sigma_{AA}\Sigma'_{AA}^{-1}[\mathbf{I}_{u(i)v(i)}]\}^K}{\det\{\mathbf{C}[\mathbf{I}_{u(i)v(i)}] + \Sigma_W L\}^{L+N}} \equiv \frac{\det[\Sigma_{AA}]^K \det[(\mathbf{I} + K\mathbf{I})^{-1}]^K}{\det\{\mathbf{Y}^\dagger [\mathbf{I} - \mathbf{1}_{u(i)v(i)}(\mathbf{I} + K\mathbf{I})^{-1} \mathbf{1}_{u(i)v(i)}^\dagger] \mathbf{Y}\}^{L+N}} \equiv \frac{\det[\mathbf{I}]^K \det[(\mathbf{I} + K\mathbf{I})^{-1}]^K}{\det\{(\mathbf{Y}^H \mathbf{Y})^{L+N}\} \det\{[\mathbf{I} - \mathbf{1}_{u(i)v(i)}(\mathbf{I} + K\mathbf{I})^{-1} \mathbf{1}_{u(i)v(i)}^\dagger]^{L+N}\}}^{\text{xiii}} \quad (183)$$

Since, the eigen-values of $\mathbf{I} - \mathbf{1}_{u(i)v(i)}(\mathbf{I} + K\mathbf{I})^{-1} \mathbf{1}_{u(i)v(i)}^\dagger$ do not change, hence, the above result is invariant of $\mathbf{1}_{u(i)v(i)}$, which completes the proof.

This concludes the detailed discussion on the derivations of (167). Next, we discuss some example on numerical theoretical results.

$$\mathbf{1}_{u(1:4)v(1:4)} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} -\mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} -\mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}; \quad \mathbf{1}_{u(5:8)v(5:8)} = \begin{bmatrix} -\mathbf{1} & \mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} -\mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & -\mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{bmatrix}; \quad (184)$$

It can be easily shown that (181) holds for $\mathbf{1}_{u(i)v(i)}$ for $i = 1 \dots 8$ given by (184).

Next, we generate $\mathbf{Y} = \text{rand}(N, K)$. Next we define

$$B = \det[(\mathbf{Y}^\dagger \mathbf{Y})^{L+N}] \det\{[\mathbf{I} - \mathbf{1}_{u(i)v(i)}(\mathbf{I} + K\mathbf{I})^{-1} \mathbf{1}_{u(i)v(i)}^\dagger]^{L+N}\} \quad (186)$$

Using $R = 200$ we plot the absolute error of $A - B$ as a function of the index i in Fig. 6. As shown in Fig. 6 the maximum absolute error does not exceed 2×10^{-9} ; hence,

Let us consider a simple numerical calculation for computing the denominator of (183).

For this example, we set $N = 2$, $K = 2$, $M = 2$, $L = 2^{(N-1)+(K-1)+1} = 8$, number of Monte-Carlo runs $R = 200$.

For this example, $\mathbf{1}_{u(i)v(i)}$ for $i = 1 \dots 8$ is as follows:

$$A = \det\{\mathbf{Y}^\dagger [\mathbf{I} - \mathbf{1}_{u(i)v(i)}(\mathbf{I} + K\mathbf{I})^{-1} \mathbf{1}_{u(i)v(i)}^\dagger] \mathbf{Y}\}^{L+N} \quad (185)$$

And

(185) is an identity of (186) and (183) is an identity.

This concludes the proof (167) and the discussion of Appendix B.

ⁱ Although, there is no unified agreement in the literature about the exact value of the unit step function at $s = 0$; the definition presented in this paper is the definition that most scholars agree with and it is the preferred definition.

ⁱⁱ The superscript † denotes complex matrix or vector transposition; i.e., Hermitian transpose (see further https://en.wikipedia.org/wiki/Hermitian_matrix).

ⁱⁱⁱ This is the only definition of the Heaviside step function of matrix argument that currently exists in the literature and it is a generalization of the definition given by Liu 2016, pg. 12, [46].

^{iv} It can be easily shown that for $N = 1$ and $K = 1$ the *Complex Matrix Variate Signal Distribution* is reduced to the complex signal distribution; i.e., $N = 1$ and $K = 1$ (64) is equivalent with (32).

^v We could have used another equivalent definition for obtaining $F_S(\mathbf{S})$ such as $F_S(\mathbf{S}) = \prod_{i=1}^N \prod_{j=1}^K F_S(s_{ij})$. Then we could have used (32) and arrived in exactly the same conclusion as in (64). We leave this as an exercise to the reader to show that in fact this is true.

^{vi} There was one error in (4) in [3], [5], [6]. The first error was that a $-\mathbf{1}$ was missing from the $\text{etr}(x)$ [3], [5], [6] but it was correct in (4) in [4].

^{vii} In (81) the property five of the determinant is used [52].

^{viii} There was an error in (6) in [3]; $-\mathbf{N}$ should have been $-\mathbf{K}$. It is OK in (5) in [4], [5] although the $-\mathbf{1}$ from the $\text{etr}(x)$ was missing in [3], [5], [6].

^{ix} This result was first published by A.T. James, 1964 in "Distributions of matrix variates and latent roots derived from normal samples" [64] where he shows that exponential distribution is a type I or hypergeometric function of matrix argument ${}_pF_q$ for $p = 0$ and $q = 0$. It is, however, very nice to see that I was also able to produce this result and confirm the link between the exponential function distribution and Wishart distribution.

^x Equation (133) could have been obtained directly from (145) if we were to set $N = M = L = 1$, we have $\Sigma = \sigma_Z^2$, $\Sigma_W = \sigma_W^2$.

^{xi} There was a typo in (11) in [4] which resulted from [55]. The " $L + M$ " should have been " $L - M$ ". It is correct in [51].

^{xii} In Appendix A we show the equivalence of the MLE with the BPE and further interpretation of (153).

^{xiii} In (183) we have exploited a property of the determinant of a quadratic form discussed in Progi 2018 [16].