

Research Article

VBOC1(α) GMGM Waveforms and ACF PSO: Part 2—Theory and Simulations

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Abstract—This paper examines the $VBOC1(\alpha)$ generalized multidimensional geolocation modulation (GMGM) waveforms, their time domain representation, the autocorrelation function (ACF), power spectral density (PSD) function, and the ACF pure signal optimization (PSO) for the generalized subcarrier parameter frequency, ($p = \{1, \dots, 8\}$); i.e., this paper is an extension of the work that was performed for $VBOC1(\alpha)$ GMGM waveforms and ACF PSO for the generalized subcarrier parameter frequency, ($p = \{1, 2, 3, 4\}$).

There are a number of reasons why the extension of this work is important.

First, it is not really obvious or intuitive how this extension is done. Second, so far no one has produced results to clearly demonstrate that this extension does not change the performance results $VBOC1(\alpha)$ GMGM waveforms and ACF PSO aside from my preliminary investigation. Third, this extension now serves as building block. One, should be able to easily extend this work for values of $p = \{9, 10, \dots, \infty\}$; i.e., this work is really the closes thing to a complete poof that we can give.

Afterwards, by employing any of the optimization algorithms such as sum, product, and MMSE on GPS L1C $BOC(6,1)$ signals such as $VBOC1(6,1,0.5)$ we are able to produce ACFs that are one *hundred percent more efficient* than the corresponding ACFs of the GPS L1C $BOC(6,1)$ signals. Hence, we have the provision for a new L1C signal: $TMVBOC(1,1,29/33,0.5)$ on the data and $TMVBOC(6,1,4/33,0.5)$ on the pilot.

I am also making provision for the new signal in the military code of the current M-code from the current $BOC(2,1)$ to the $TMVBOC(8,1,4/33,0.5)$ in which the data is $TMVBOC(2,1,29/33,0.5)$.

Index Terms—Pulse generation, pulse amplitude modulation, pulse width modulation, multidimensional sequences, signal design, signal analysis, generalized functions, time-frequency analysis, minimization methods, optimization

methods.

1 Introduction

The main objective of this paper is to introduce the first generation $VBOC1(\alpha)$ generalized multidimensional geolocation modulation waveforms so as to fill in substantial signal design [1]-[38] methodology gaps created over the years as a results of incomplete signal design methodologies.

In the past, signal design methodology was mainly motivated on performance metrics, such as sharper ACFs and user equipment performance measured by the signal ability to mitigate multipath, mitigate interference, jamming and product's ability to produce a working system [26]-[30] or based on a symmetric signal design waveform; i.e., symmetric non-return-to-zero (NRZ) [6] (or S-NRZ): the time (or duration) for which (+1) signal amplitude voltage occurs is equal to the time (or duration) for which (−1) signal amplitude voltage occurs.

The achieved level of success is based mostly on user segment performance metrics and very little on improvements from the signal design methodology. Hence, from the system, design, user equipment engineering point of view we have achieved substantial outstanding milestones; one would argue that from the rigorous signal design methodology (asymmetric NRZ or As-NRZ) point of view we have achieved reasonably good intermediate steps; hence, the main objective of this paper.

For example $VBOC1(\alpha)$, which is a type of As-NRZ BOC for $0 \leq \alpha \leq 1$, generalizes the transition from the BPSK waveform $BPSK(t) = VBOC1_{(m,n,\alpha=1)}(t)$ to the current S-NRZ BOC modulation, used extensively in GPS L1 and L2 frequencies, because $BOC_{(m,n)}(t) = VBOC1_{(m,n,\alpha=0)}(t)$; however, in the current GNSS standard the choice $\alpha = 0$ in the context of $VBOC1_{(m,n,\alpha)}(t)$ is made entirely arbitrary [12]-[24]. Because in the interval from zero to one there are an infinite number of α s one cannot arbitrary select $\alpha = 0$ as the best waveform and make $BOC_{(m,n)}(t)$ the standard for all GPS III, IV and other GNSS users without a single explanation whatsoever regardless of integer values of m and n ; i.e., the sub-carrier frequency [12]-[24].

Initially, the signal design approach for pseudolite applications [8], [31]-[38] was primarily driven by the mentality of achieving user performance metrics; however, it was not until very recently that various signal waveforms

articulate new objectives of various signal design teams in the 21st century in *Indoor Geolocation Systems—Theory and Applications* [6], and *Geolocation of RF Signals—Principles and Simulations* [38].

Why is the above discussion so important? First, signal design and optimization parameter, α , exploits a particular asymmetry of the As-NRZ signal coding modulation in order to improve $VBOC1(\alpha)$ signal design and optimization. Second, the derivation of generalized ACFs and PSDs as a function of α and p provides for the first time the opportunity to understand the properties of all individual $VBOC1(\alpha)$ such as for all positive integer values of p without having to analyze all individual $VBOC1(\alpha)$ separately. Third, for the first time ever, optimization theorems show the consistency between the sum and mean-square criteria [1].

$VBOC1(\alpha)$ pure signal design or broad definition of generalized ACF and PSD offers a unique signal design methodology and provides the necessary framework for $VBOC1(\alpha)$ ACF pure signal optimization [1].

This paper is organized as follows: $VBOC1(\alpha)$ pure signal design is discussed next. Next, numerical results are provided; Conclusion is given in afterwards along with a list of references. $VBOC1(\alpha)$ ACF pure signal optimization is discussed in Appendix A.

2 $VBOC1(\alpha)$ Pure Signal Design

Detailed discussion on $VBOC1(\alpha)$ pure signal design includes:

- (1) $VBOC1(\alpha)$ signal definition and discussion; and
- (2) $VBOC1(\alpha)$ generalized ACF definition and discussion; and
- (3) $VBOC1(\alpha)$ generalized PSD definition and discussion.

2.1 $VBOC1(\alpha)$ Signal Definition and Discussion

Definition 1: $VBOC1_{(m,n,\alpha)}(t)$ waveform is a generalized, periodic function of the $BOC_{(m,n)}(t)$ waveform with period $2T_s$, known as the subcarrier period, for all values of $t - \infty < t < \infty$.

$$VBOC1_{(m,n,\alpha)}(t) = VBOC1_{(m,n,\alpha)}(t \pm 2T_s) \quad (1)$$

and the relation of integers m, n is given by

$$2mT_s = nT_c = T \quad (2)$$

where T_c is the defined as the chipping period and T is defined as the integration period.

Definition 2: One sub-carrier period of $VBOC1_{(m,n,\alpha)}(t)$ is the superposition of two pulses: a rising pulse and a falling pulse with amplitude/pulse widths, $+1/w_r$ and $-1/w_f$ respectively, or a falling pulse and a rising pulse amplitude/pulse widths $-1/w_f$ and $+1/w_r$ respectively that satisfy the following:

$$w_r = T_s(1 \pm \alpha) = \frac{nT_c(1 \pm \alpha)}{2m} = \frac{T_c(1 \pm \alpha)}{2p} \quad (3)$$

$$w_f = T_s(1 \mp \alpha) = \frac{nT_c(1 \mp \alpha)}{2m} = \frac{T_c(1 \mp \alpha)}{2p} \quad (4)$$

$$VBOC1_{(m,n,\alpha)}(t) = \sum_{q=-\infty}^{\infty} [p_{w_r}(t_1, w_r) - p_{w_f}(t_2, w_f)] \quad (5)$$

where

$$t_1 = t - 0.5w_r - qT_s \quad (6)$$

$$t_2 = t - 0.5w_f - w_r - qT_s \quad (7)$$

and $p_w(t - t_0, w)$ is a unit rectangular pulse function with width w centered at t_0 with amplitude $+1$.

Although, definitions 1 and 2 completely define the waveform $VBOC1_{(m,n,\alpha)}(t)$; however, the acceptable range of values of α can be derived from the following theorem.

Theorem 1: Prove that α cannot be smaller than zero and greater than one; i.e., based on *Definitions 1* and *2*, the only acceptable range of α is given by

$$0 \leq \alpha \leq 1^i \quad (8)$$

The *proof of theorem 1* is straightforward. Since, based on *definitions 1* and *2* the following holds

Fourier Transform and it is given by

$$w_r + w_f = 2T_s = \frac{nT_c}{m} = \frac{T_c}{p} \quad (9)$$

$$w_f = T_s(1 + \alpha) = \frac{T_c(1 + \alpha)}{2p} \quad (10)$$

Hence, we can prove that α cannot be smaller than zero or greater than one in two different ways as follows

$$0 \leq w_r \leq T_s \quad (11)$$

and

$$T_s \leq w_f \leq 2T_s \quad (12)$$

Either solution of (11)/(12) leads to the desired range of acceptable values of α (8).

This concludes $VBOC1(\alpha)$ signal definition and discussion; next we continue with $VBOC1(\alpha)$ generalized ACF definition

and discussion.

3 $VBOC1(\alpha)$ Generalized ACF Definition and Discussion

I have already developed generalized ACF definition and discussion in [1] for values of $p = \{1, \dots, 4\}$; hence, it remains to extend that discussion here for values of $p = \{5, \dots, 8\}$ which is accomplished by means of the following theorem.

Theorem 2: Show that, for values of $p = \{5, 6, 7, 8\}$, $R_{VBOC1_{(m=np,n,\alpha)}}(\tau)$ is the generalized ACF with respect to p ; i.e., (1) $R_{VBOC1_{(n,n,\alpha)}}(\tau) = R_{VBOC1_{(pn=n,n,\alpha)}}(\tau)$; $p = 5$; (2) $R_{VBOC1_{(2n,n,\alpha)}}(\tau) = R_{VBOC1_{(pn=2n,n,\alpha)}}(\tau)$; $p = 6$; (3) $R_{VBOC1_{(3n,n,\alpha)}}(\tau) = R_{VBOC1_{(pn=3n,n,\alpha)}}(\tau)$; $p = 7$; and (4) $R_{VBOC1_{(4n,n,\alpha)}}(\tau) = R_{VBOC1_{(pn=4n,n,\alpha)}}(\tau)$; $p = 8$; assuming that $i = \{0, 1, \dots, 7\}$

$$\tau_{1+3i(\alpha)} = \frac{T_c(1+2i-\alpha)}{2p} = T_s(1+2i-\alpha) \quad (13)$$

$$\tau_{2+3i(\alpha)} = \frac{T_c(1+2i+\alpha)}{2p} = T_s(1+2i+\alpha) \quad (14)$$

$$\tau_{3+3i} = \frac{(i+1)T_c}{p} = 2T_s(1+i) \quad (15)$$

then $R_1(\tau) = R_{VBOC1_{(m=np,n,\alpha)}}(\tau)$ is given by the expression below:

$$R_1(\tau) = \begin{cases} \frac{R_1(\tau)}{p=\{1,2,\dots,8,\dots,\infty\}} & , \quad 0 \leq |\tau| \leq \tau_{12} \\ \frac{R_1(\tau)}{p=\{5,\dots,8,\dots,\infty\}} & , \quad \tau_{12} \leq |\tau| \leq \tau_{24} \\ 0, & |\tau| \geq T_c \end{cases} \quad (16)$$

The first part of $R_1(\tau)$ is exactly the same as (22) in [1] which was proved to be correct for values of $p = \{1, \dots, 4\}$. The second part of is the $R_1(\tau)$ extension of the $p = \{5, \dots, 8\}$.

The *proof of theorem 2* is straightforward. However, one always wonders how the ACF of $VBOC1(\alpha)$ was produced. The ACF is the result of integration. In previous publications [1], [2], [7], [8] we have shown how to produce the ACF of $VBOC1(\alpha)$ for special cases. Because of the periodicity of the ACF is it straightforward to extend the ACF by just plugging in the $p = \{5, \dots, 8\}$ values. The details of this extension will be provided in a separate journal paper when we discuss the most general case for $p = \{1, 2, 3, \dots, \infty\}$.

The closed form expression of $R_1(\tau) = R_{VBOC1_{(m=np,n,\alpha)}}(\tau)$ is given by:

¹ For ease of analysis we assume $' + '$ for w_r and $' - '$ for w_f .

$$\begin{aligned}
R_1(\tau) = & \left[\begin{aligned}
& \underbrace{\frac{1}{a_{1(p,n,\alpha)}} + \frac{-(4p-1)}{T_c} |\tau|}_{r_{1(p,n,\alpha)}; p=\{1,2,\dots,8,\dots\infty\}}, 0 \leq |\tau| \leq \tau_{1(\alpha)} \\
& \underbrace{\frac{-(p-1)+(2p-1)\alpha}{a_{2(p,n,\alpha)}} + \frac{-1}{T_c} |\tau|}_{r_{2(p,n,\alpha)}; p=\{1,2,\dots,8,\dots\infty\}}, \tau_{1(\alpha)} \leq |\tau| \leq \tau_{2(\alpha)} \\
& \underbrace{-\frac{3p-2}{a_{3(p,n,\alpha)}} + \frac{4p-3}{T_c} |\tau|}_{r_{3(p,n,\alpha)}; p=\{1,2,\dots,8,\dots\infty\}}, \tau_{2(\alpha)} \leq |\tau| \leq \tau_3 \\
& \underbrace{\frac{5p-6}{a_{4(p,n,\alpha)}} + \frac{-(4p-5)}{T_c} |\tau|}_{r_{4(p,n,\alpha)}; p=\{2,3,\dots,8,\dots\infty\}}, \tau_3 \leq |\tau| \leq \tau_{4(\alpha)} \\
& \underbrace{\frac{-(p-3)+(2p-3)\alpha}{a_{5(p,n,\alpha)}} + \frac{-1}{T_c} |\tau|}_{r_{5(p,n,\alpha)}; p=\{2,3,\dots,8,\dots\infty\}}, \tau_{4(\alpha)} \leq |\tau| \leq \tau_{5(\alpha)} \\
& \underbrace{-\frac{7p-12}{a_{6(p,n,\alpha)}} + \frac{(4p-7)}{T_c} |\tau|}_{r_{6(p,n,\alpha)}; p=\{2,3,\dots,8,\dots\infty\}}, \tau_{5(\alpha)} \leq |\tau| \leq \tau_6 \\
& \underbrace{\frac{9p-20}{a_{7(p,n,\alpha)}} + \frac{-(4p-9)}{T_c} |\tau|}_{r_{7(p,n,\alpha)}; p=\{3,4,\dots,8,\dots\infty\}}, \tau_6 \leq |\tau| \leq \tau_{7(\alpha)} \\
& \underbrace{\frac{-(p-5)+(2p-5)\alpha}{a_{8(p,n,\alpha)}} + \frac{-1}{T_c} |\tau|}_{r_{8(p,n,\alpha)}; p=\{3,4,\dots,8,\dots\infty\}}, \tau_{7(\alpha)} \leq |\tau| \leq \tau_{8(\alpha)} \\
& \underbrace{-\frac{(11p-30)}{a_{9(p,n,\alpha)}} + \frac{(4p-11)}{T_c} |\tau|}_{r_{9(p,n,\alpha)}; p=\{3,4,\dots,8,\dots\infty\}}, \tau_{8(\alpha)} \leq |\tau| \leq \tau_9 \\
& \underbrace{\frac{13p-42}{a_{10(p,n,\alpha)}} + \frac{-(4p-13)}{T_c} |\tau|}_{r_{10(p,n,\alpha)}; p=\{4,5,\dots,8,\dots\infty\}}, \tau_9 \leq |\tau| \leq \tau_{10(\alpha)} \\
& \underbrace{\frac{-(p-7)+(2p-7)\alpha}{a_{11(p,n,\alpha)}} + \frac{-1}{T_c} |\tau|}_{r_{11(p,n,\alpha)}; p=\{4,5,\dots,8,\dots\infty\}}, \tau_{10(\alpha)} \leq |\tau| \leq \tau_{11(\alpha)} \\
& \underbrace{-\frac{(15p-56)}{a_{12(p,n,\alpha)}} + \frac{(4p-15)}{T_c} |\tau|}_{r_{12(p,n,\alpha)}; p=\{4,5,\dots,8,\dots\infty\}}, \tau_{11(\alpha)} \leq |\tau| \leq \tau_{12}
\end{aligned} \right]
\end{aligned}$$

$$\begin{aligned}
& \left[\begin{aligned}
& \underbrace{\frac{17p-72}{a_{13(p,n,\alpha)}} + \frac{-(4p-17)}{T_c} |\tau|}_{r_{13(p,n,\alpha)}; p=\{5,\dots,8,\dots\infty\}}, \tau_{12} \leq |\tau| \leq \tau_{13(\alpha)} \\
& \underbrace{\frac{-(p-9)+(2p-9)\alpha}{a_{14(p,n,\alpha)}} + \frac{-1}{T_c} |\tau|}_{r_{14(p,n,\alpha)}; p=\{5,\dots,8,\dots\infty\}}, \tau_{13(\alpha)} \leq |\tau| \leq \tau_{14(\alpha)} \\
& \underbrace{-\frac{19p-90}{a_{15(p,n,\alpha)}} + \frac{(4p-19)}{T_c} |\tau|}_{r_{15(p,n,\alpha)}; p=\{5,\dots,8,\dots\infty\}}, \tau_{14(\alpha)} \leq |\tau| \leq \tau_{15} \\
& \underbrace{\frac{21p-110}{a_{16(p,n,\alpha)}} + \frac{-(4p-21)}{T_c} |\tau|}_{r_{16(p,n,\alpha)}; p=\{6,7,8,\dots\infty\}}, \tau_{15} \leq |\tau| \leq \tau_{16(\alpha)} \\
& \underbrace{\frac{-(p-11)+(2p-11)\alpha}{a_{17(p,n,\alpha)}} + \frac{-1}{T_c} |\tau|}_{r_{17(p,n,\alpha)}; p=\{6,7,8,\dots\infty\}}, \tau_{16(\alpha)} \leq |\tau| \leq \tau_{17(\alpha)} \\
& \underbrace{-\frac{23p-132}{a_{18(p,n,\alpha)}} + \frac{(4p-23)}{T_c} |\tau|}_{r_{18(p,n,\alpha)}; p=\{6,7,8,\dots\infty\}}, \tau_{17(\alpha)} \leq |\tau| \leq \tau_{18} \\
& \underbrace{\frac{25p-156}{a_{19(p,n,\alpha)}} + \frac{-(4p-25)}{T_c} |\tau|}_{r_{19(p,n,\alpha)}; p=\{7,8,\dots\infty\}}, \tau_{18} \leq |\tau| \leq \tau_{19(\alpha)} \\
& \underbrace{\frac{-(p-13)+(2p-13)\alpha}{a_{20(p,n,\alpha)}} + \frac{-1}{T_c} |\tau|}_{r_{20(p,n,\alpha)}; p=\{7,8,\dots\infty\}}, \tau_{19(\alpha)} \leq |\tau| \leq \tau_{20(\alpha)} \\
& \underbrace{-\frac{27p-182}{a_{21(p,n,\alpha)}} + \frac{(4p-27)}{T_c} |\tau|}_{r_{21(p,n,\alpha)}; p=\{7,8,\dots\infty\}}, \tau_{20(\alpha)} \leq |\tau| \leq \tau_{21} \\
& \underbrace{\frac{29p-210}{a_{22(p,n,\alpha)}} + \frac{-(4p-29)}{T_c} |\tau|}_{r_{22(p,n,\alpha)}; p=\{8,\dots\infty\}}, \tau_{21} \leq |\tau| \leq \tau_{22(\alpha)} \\
& \underbrace{\frac{-(p-15)+(2p-15)\alpha}{a_{23(p,n,\alpha)}} + \frac{-1}{T_c} |\tau|}_{r_{23(p,n,\alpha)}; p=\{8,\dots\infty\}}, \tau_{22(\alpha)} \leq |\tau| \leq \tau_{23(\alpha)} \\
& \underbrace{-\frac{31p-240}{a_{24(p,n,\alpha)}} + \frac{(4p-31)}{T_c} |\tau|}_{r_{24(p,n,\alpha)}; p=\{8,\dots\infty\}}, \tau_{23(\alpha)} \leq |\tau| \leq \tau_{24}
\end{aligned} \right]
\end{aligned}$$

$$0, |\tau| \geq T_c \quad (17)$$

and T is given by (2).

First, we recognize that substituting $p = 5$ in (17) we get

$$R_1(\tau) = \left[\begin{array}{l} 1 - \frac{(4p-1)}{T_c} |\tau|, 0 \leq |\tau| \leq \tau_{1(\alpha)} \\ -\frac{(p-1)+(2p-1)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{1(\alpha)} \leq |\tau| \leq \tau_{2(\alpha)} \\ -\frac{3p-2}{p} + \frac{(4p-3)}{T_c} |\tau|, \tau_{2(\alpha)} \leq |\tau| \leq \tau_3 \\ \frac{5p-6}{p} - \frac{(4p-5)}{T_c} |\tau|, \tau_3 \leq |\tau| \leq \tau_{4(\alpha)} \\ -\frac{(p-3)+(2p-3)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{4(\alpha)} \leq |\tau| \leq \tau_{5(\alpha)} \\ -\frac{7p-12}{p} + \frac{(4p-7)}{T_c} |\tau|, \tau_{5(\alpha)} \leq |\tau| \leq \tau_6 \\ \frac{9p-20}{p} - \frac{(4p-9)}{T_c} |\tau|, \tau_6 \leq |\tau| \leq \tau_{7(\alpha)} \\ -\frac{(p-5)+(2p-5)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{7(\alpha)} \leq |\tau| \leq \tau_{8(\alpha)} \end{array} \right] \left[\begin{array}{l} -\frac{(11p-30)}{p} + \frac{(4p-11)}{T_c} |\tau|, \tau_{8(\alpha)} \leq |\tau| \leq \tau_9 \\ \frac{13p-42}{p} - \frac{(4p-13)}{T_c} |\tau|, \tau_9 \leq |\tau| \leq \tau_{10(\alpha)} \\ -\frac{(p-7)+(2p-7)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{10(\alpha)} \leq |\tau| \leq \tau_{11(\alpha)} \\ -\frac{(15p-56)}{p} + \frac{(4p-15)}{T_c} |\tau|, \tau_{11(\alpha)} \leq |\tau| \leq \tau_{12} \\ \frac{17p-72}{p} - \frac{(4p-17)}{T_c} |\tau|, \tau_{12} \leq |\tau| \leq \tau_{13(\alpha)} \\ -\frac{(p-9)+(2p-9)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{13(\alpha)} \leq |\tau| \leq \tau_{14(\alpha)} \\ -\frac{19p-90}{p} + \frac{(4p-19)}{T_c} |\tau|, \tau_{14(\alpha)} \leq |\tau| \leq \tau_{15} \\ 0, |\tau| \geq T_c \end{array} \right] \quad (18)$$

Equation (18), for $p = 5$, is an original result not published before.

Second, we recognize that substituting $p = 6$ in (17) we get

$$R_1(\tau) = \left[\begin{array}{l} 1 - \frac{(4p-1)}{T_c} |\tau|, 0 \leq |\tau| \leq \tau_{1(\alpha)} \\ -\frac{(p-1)+(2p-1)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{1(\alpha)} \leq |\tau| \leq \tau_{2(\alpha)} \\ -\frac{3p-2}{p} + \frac{(4p-3)}{T_c} |\tau|, \tau_{2(\alpha)} \leq |\tau| \leq \tau_3 \\ \frac{5p-6}{p} - \frac{(4p-5)}{T_c} |\tau|, \tau_3 \leq |\tau| \leq \tau_{4(\alpha)} \\ -\frac{(p-3)+(2p-3)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{4(\alpha)} \leq |\tau| \leq \tau_{5(\alpha)} \\ -\frac{7p-12}{p} + \frac{(4p-7)}{T_c} |\tau|, \tau_{5(\alpha)} \leq |\tau| \leq \tau_6 \\ \frac{9p-20}{p} - \frac{(4p-9)}{T_c} |\tau|, \tau_6 \leq |\tau| \leq \tau_{7(\alpha)} \\ -\frac{(p-5)+(2p-5)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{7(\alpha)} \leq |\tau| \leq \tau_{8(\alpha)} \\ -\frac{(11p-30)}{p} + \frac{(4p-11)}{T_c} |\tau|, \tau_{8(\alpha)} \leq |\tau| \leq \tau_9 \end{array} \right] \left[\begin{array}{l} \frac{13p-42}{p} - \frac{(4p-13)}{T_c} |\tau|, \tau_9 \leq |\tau| \leq \tau_{10(\alpha)} \\ -\frac{(p-7)+(2p-7)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{10(\alpha)} \leq |\tau| \leq \tau_{11(\alpha)} \\ -\frac{(15p-56)}{p} + \frac{(4p-15)}{T_c} |\tau|, \tau_{11(\alpha)} \leq |\tau| \leq \tau_{12} \\ \frac{17p-72}{p} - \frac{(4p-17)}{T_c} |\tau|, \tau_{12} \leq |\tau| \leq \tau_{13(\alpha)} \\ -\frac{(p-9)+(2p-9)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{13(\alpha)} \leq |\tau| \leq \tau_{14(\alpha)} \\ -\frac{19p-90}{p} + \frac{(4p-19)}{T_c} |\tau|, \tau_{14(\alpha)} \leq |\tau| \leq \tau_{15} \\ \frac{21p-110}{p} - \frac{(4p-21)}{T_c} |\tau|, \tau_{15} \leq |\tau| \leq \tau_{16(\alpha)} \\ -\frac{(p-11)+(2p-11)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{16(\alpha)} \leq |\tau| \leq \tau_{17(\alpha)} \\ -\frac{23p-132}{p} + \frac{(4p-23)}{T_c} |\tau|, \tau_{17(\alpha)} \leq |\tau| \leq \tau_{18} \\ 0, |\tau| \geq T_c \end{array} \right] \quad (19)$$

Equation (19), for $p = 6$, is an original result not published anywhere else.

Third, we recognize that that for up to values $p = 7$ in (17) we get

$$R_1(\tau) = \left[\begin{array}{l} 1 - \frac{(4p-1)}{T_c} |\tau|, 0 \leq |\tau| \leq \tau_{1(\alpha)} \\ -\frac{(p-1)+(2p-1)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{1(\alpha)} \leq |\tau| \leq \tau_{2(\alpha)} \\ -\frac{3p-2}{p} + \frac{(4p-3)}{T_c} |\tau|, \tau_{2(\alpha)} \leq |\tau| \leq \tau_3 \\ \frac{5p-6}{p} - \frac{(4p-5)}{T_c} |\tau|, \tau_3 \leq |\tau| \leq \tau_{4(\alpha)} \\ -\frac{(p-3)+(2p-3)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{4(\alpha)} \leq |\tau| \leq \tau_{5(\alpha)} \\ -\frac{7p-12}{p} + \frac{(4p-7)}{T_c} |\tau|, \tau_{5(\alpha)} \leq |\tau| \leq \tau_6 \\ \frac{9p-20}{p} - \frac{(4p-9)}{T_c} |\tau|, \tau_6 \leq |\tau| \leq \tau_{7(\alpha)} \\ -\frac{(p-5)+(2p-5)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{7(\alpha)} \leq |\tau| \leq \tau_{8(\alpha)} \\ -\frac{11p-30}{p} + \frac{(4p-11)}{T_c} |\tau|, \tau_{8(\alpha)} \leq |\tau| \leq \tau_9 \\ \frac{13p-42}{p} - \frac{(4p-13)}{T_c} |\tau|, \tau_9 \leq |\tau| \leq \tau_{10(\alpha)} \\ -\frac{(p-7)+(2p-7)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{10(\alpha)} \leq |\tau| \leq \tau_{11(\alpha)} \end{array} \right] \left[\begin{array}{l} -\frac{(15p-56)}{p} + \frac{(4p-15)}{T_c} |\tau|, \tau_{11(\alpha)} \leq |\tau| \leq \tau_{12} \\ \frac{17p-72}{p} - \frac{(4p-17)}{T_c} |\tau|, \tau_{12} \leq |\tau| \leq \tau_{13(\alpha)} \\ -\frac{(p-9)+(2p-9)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{13(\alpha)} \leq |\tau| \leq \tau_{14(\alpha)} \\ -\frac{19p-90}{p} + \frac{(4p-19)}{T_c} |\tau|, \tau_{14(\alpha)} \leq |\tau| \leq \tau_{15} \\ \frac{21p-110}{p} - \frac{(4p-21)}{T_c} |\tau|, \tau_{15} \leq |\tau| \leq \tau_{16(\alpha)} \\ -\frac{(p-11)+(2p-11)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{16(\alpha)} \leq |\tau| \leq \tau_{17(\alpha)} \\ -\frac{23p-132}{p} + \frac{(4p-23)}{T_c} |\tau|, \tau_{17(\alpha)} \leq |\tau| \leq \tau_{18} \\ \frac{25p-156}{p} - \frac{(4p-25)}{T_c} |\tau|, \tau_{18} \leq |\tau| \leq \tau_{19(\alpha)} \\ -\frac{(p-13)+(2p-13)\alpha}{p} - \frac{1}{T_c} |\tau|, \tau_{19(\alpha)} \leq |\tau| \leq \tau_{20(\alpha)} \\ -\frac{27p-182}{p} + \frac{(4p-27)}{T_c} |\tau|, \tau_{20(\alpha)} \leq |\tau| \leq \tau_{21} \\ 0, |\tau| \geq T_c \end{array} \right] \quad (20)$$

Equation (20), for $p = 7$, is an original result not published anywhere else. Equation (17), for $p = 8$, is an original result not published anywhere else. This completed the *proof of theorem 2*.

Corollary 1: From *corollary 1* $VBOC1_{(m,n,\alpha=0)}(t) = BOC_{(m,n)}(t)$ [1] and from *theorem 2*, the ACF $R_{VBOC1_{(m,n,\alpha=0)}}(\tau) = R_{BOC_{(m,n)}}(\tau)$; hence, prove that from the generalized ACF with respect to $p = \{5,6,7,8\}$ and $\alpha = 0$ is

equal to the ACF of the BOC modulation with respective coefficients; i.e.,

$$(1) R_{BOC(n,n)}(\tau) = R_{VBOC1(pn=n,n,\alpha=0)}(\tau); p = 5;$$

$$(2) R_{BOC(2n,n)}(\tau) = R_{VBOC1(pn=2n,n,\alpha=0)}(\tau); p = 6;$$

$$(3) R_{BOC(3n,n)}(\tau) = R_{VBOC1(pn=3n,n,\alpha=0)}(\tau); p = 7;$$

$$(4) R_{BOC(4n,n)}(\tau) = R_{VBOC1(pn=4n,n,\alpha=0)}(\tau); p = 8.$$

$$R_{BOC(pn,n)}(\tau) = \begin{cases} \underbrace{\frac{1}{a_{1(p,n)}} + \frac{-(4p-1)}{T_c}}_{b_{1(p,n)}} |\tau|, & |\tau| \leq \frac{T_c}{2p} = T_s \\ \underbrace{r_{1(pn,n)}; p=\{1,\dots,8,\dots\}}_{\tau_1} \\ \underbrace{-\frac{3p-2}{p} + \frac{4p-3}{T_c}}_{b_{3(p,n)}} |\tau|, & \frac{T_c}{2p} \leq |\tau| \leq \frac{T_c}{p} \\ \underbrace{a_{3(p,n)}}_{\tau_3} \\ \underbrace{r_{3(pn,n)}; p=\{1,\dots,8,\dots\}}_{\tau_3} \\ \underbrace{\frac{5p-6}{p} + \frac{-(4p-5)}{T_c}}_{b_{4(p,n)}} |\tau|, & \frac{T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ \underbrace{a_{4(p,n)}}_{\tau_4} \\ \underbrace{r_{4(pn,n)}; p=\{2,\dots,8,\dots\}}_{\tau_4} \\ \underbrace{-\frac{7p-12}{p} + \frac{(4p-7)}{T_c}}_{b_{6(p,n)}} |\tau|, & \frac{3T_c}{2p} \leq |\tau| \leq \frac{2T_c}{p} \\ \underbrace{a_{6(p,n)}}_{\tau_6} \\ \underbrace{r_{6(pn,n)}; p=\{2,\dots,8,\dots\}}_{\tau_6} \\ \underbrace{\frac{9p-20}{p} + \frac{-(4p-9)}{T_c}}_{b_{7(p,n)}} |\tau|, & \frac{2T_c}{p} \leq |\tau| \leq \frac{5T_c}{2p} \\ \underbrace{a_{7(p,n)}}_{\tau_7} \\ \underbrace{r_{7(pn,n)}; p=\{3,\dots,8,\dots\}}_{\tau_7} \\ \underbrace{-\frac{(11p-30)}{p} + \frac{(4p-11)}{T_c}}_{b_{9(p,n)}} |\tau|, & \frac{5T_c}{2p} \leq |\tau| \leq \frac{3T_c}{p} \\ \underbrace{a_{9(p,n)}}_{\tau_9} \\ \underbrace{r_{9(pn,n)}; p=\{3,\dots,8,\dots\}}_{\tau_9} \\ \underbrace{\frac{13p-42}{p} + \frac{-(4p-13)}{T_c}}_{b_{10(p,n)}} |\tau|, & \frac{3T_c}{p} \leq |\tau| \leq \frac{7T_c}{2p} \\ \underbrace{a_{10(p,n)}}_{\tau_{10}} \\ \underbrace{r_{10(pn,n)}; p=\{4,\dots,8,\dots\}}_{\tau_{10}} \\ \underbrace{-\frac{(15p-56)}{p} + \frac{(4p-15)}{T_c}}_{b_{12(p,n)}} |\tau|, & \frac{7T_c}{2p} \leq |\tau| \leq \frac{4T_c}{p} \\ \underbrace{a_{12(p,n)}}_{\tau_{12}} \\ \underbrace{r_{12(pn,n)}}_{\tau_{12}} \end{cases} \quad \begin{cases} \underbrace{\frac{17p-72}{p} + \frac{-(4p-17)}{T_c}}_{b_{13(p,n)}} |\tau|, & \frac{4T_c}{p} \leq |\tau| \leq \frac{9T_c}{2p} \\ \underbrace{a_{13(p,n)}}_{\tau_{13}} \\ \underbrace{r_{13(pn,n)}; p=\{5,\dots,8,\dots\}}_{\tau_{13}} \\ \underbrace{-\frac{19p-90}{p} + \frac{(4p-19)}{T_c}}_{b_{15(p,n)}} |\tau|, & \frac{9T_c}{2p} \leq |\tau| \leq \frac{5T_c}{p} \\ \underbrace{a_{15(p,n)}}_{\tau_{15}} \\ \underbrace{r_{15(pn,n)}; p=\{5,\dots,8,\dots\}}_{\tau_{15}} \\ \underbrace{\frac{21p-110}{p} + \frac{-(4p-21)}{T_c}}_{b_{16(p,n)}} |\tau|, & \frac{5T_c}{p} \leq |\tau| \leq \frac{11T_c}{2p} \\ \underbrace{a_{16(p,n)}}_{\tau_{16}} \\ \underbrace{r_{16(pn,n)}; p=\{6,\dots,8,\dots\}}_{\tau_{16}} \\ \underbrace{-\frac{23p-132}{p} + \frac{(4p-23)}{T_c}}_{b_{18(p,n)}} |\tau|, & \frac{11T_c}{2p} \leq |\tau| \leq \frac{6T_c}{p} \\ \underbrace{a_{18(p,n)}}_{\tau_{18}} \\ \underbrace{r_{18(pn,n)}; p=\{6,\dots,8,\dots\}}_{\tau_{18}} \\ \underbrace{\frac{25p-156}{p} + \frac{-(4p-25)}{T_c}}_{b_{19(p,n)}} |\tau|, & \frac{6T_c}{p} \leq |\tau| \leq \frac{13T_c}{2p} \\ \underbrace{a_{19(p,n)}}_{\tau_{19}} \\ \underbrace{r_{19(pn,n)}; p=\{7,8,\dots\}}_{\tau_{19}} \\ \underbrace{-\frac{27p-182}{p} + \frac{(4p-27)}{T_c}}_{b_{21(p,n)}} |\tau|, & \frac{13T_c}{2p} \leq |\tau| \leq \frac{7T_c}{p} \\ \underbrace{a_{21(p,n)}}_{\tau_{21}} \\ \underbrace{r_{21(pn,n)}; p=\{7,8,\dots\}}_{\tau_{21}} \\ \underbrace{\frac{29p-210}{p} + \frac{-(4p-29)}{T_c}}_{b_{22(p,n)}} |\tau|, & \frac{7T_c}{p} \leq |\tau| \leq \frac{15T_c}{2p} \\ \underbrace{a_{22(p,n)}}_{\tau_{22}} \\ \underbrace{r_{22(pn,n)}; p=\{8,\dots\}}_{\tau_{22}} \\ \underbrace{-\frac{31p-240}{p} + \frac{(4p-31)}{T_c}}_{b_{24(p,n)}} |\tau|, & \frac{15T_c}{2p} \leq |\tau| \leq \frac{4T_c}{p} \\ \underbrace{a_{24(p,n)}}_{\tau_{24}} \\ \underbrace{r_{24(pn,n)}}_{\tau_{24}} \end{cases} \quad \begin{cases} 0, & |\tau| \geq T_c \end{cases} \quad (21)$$

And T is given by (2).

Proof of corollary 1: The *proof of corollary 1* is

straightforward.

First, we recognize that substituting $p = 5$ in (21) we get

$$R_{BOC(pn=5n,n)}(\tau) = \begin{cases} \underbrace{1 - \frac{4p-1}{T_c}}_{b_{1(p,n)}} |\tau|, & |\tau| \leq \frac{T_c}{2p} = T_s \\ \underbrace{-\frac{3p-2}{p} + \frac{4p-3}{T_c}}_{b_{3(p,n)}} |\tau|, & \frac{T_c}{2p} \leq |\tau| \leq \frac{T_c}{p} \\ \underbrace{\frac{5p-6}{p} - \frac{4p-5}{T_c}}_{b_{4(p,n)}} |\tau|, & \frac{T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ \underbrace{-\frac{7p-12}{p} + \frac{4p-7}{T_c}}_{b_{6(p,n)}} |\tau|, & \frac{3T_c}{2p} \leq |\tau| \leq \frac{2T_c}{p} \\ \underbrace{\frac{9p-20}{p} - \frac{4p-9}{T_c}}_{b_{7(p,n)}} |\tau|, & \frac{2T_c}{p} \leq |\tau| \leq \frac{5T_c}{2p} \end{cases} \quad \begin{cases} \underbrace{-\frac{11p-30}{p} + \frac{4p-11}{T_c}}_{b_{13(p,n)}} |\tau|, & \frac{5T_c}{2p} \leq |\tau| \leq \frac{3T_c}{p} \\ \underbrace{\frac{13p-42}{p} - \frac{4p-13}{T_c}}_{b_{15(p,n)}} |\tau|, & \frac{3T_c}{p} \leq |\tau| \leq \frac{7T_c}{2p} \\ \underbrace{-\frac{15p-56}{p} + \frac{4p-15}{T_c}}_{b_{16(p,n)}} |\tau|, & \frac{7T_c}{2p} \leq |\tau| \leq \frac{4T_c}{p} \\ \underbrace{\frac{17p-72}{p} - \frac{4p-17}{T_c}}_{b_{18(p,n)}} |\tau|, & \frac{4T_c}{p} \leq |\tau| \leq \frac{9T_c}{2p} \\ \underbrace{-\frac{19p-90}{p} + \frac{4p-19}{T_c}}_{b_{19(p,n)}} |\tau|, & \frac{9T_c}{2p} \leq |\tau| \leq \frac{5T_c}{p} \end{cases} \quad \begin{cases} 0, & |\tau| \geq T_c \end{cases} \quad (22)$$

Equation (22), for $p = 5$, is an original result not published anywhere else.

Second, we recognize that substituting $p = 6$ in (21) we get

$$R_{BOC(pn=6n,n)}(\tau) = \begin{cases} 1 - \frac{4p-1}{T_c}|\tau|, & |\tau| \leq \frac{T_c}{2p} = T_s \\ -\frac{3p-2}{p} + \frac{4p-3}{T_c}|\tau|, & \frac{T_c}{2p} \leq |\tau| \leq \frac{T_c}{p} \\ \frac{5p-6}{p} - \frac{4p-5}{T_c}|\tau|, & \frac{T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ -\frac{7p-12}{p} + \frac{4p-7}{T_c}|\tau|, & \frac{3T_c}{2p} \leq |\tau| \leq \frac{2T_c}{p} \\ \frac{9p-20}{p} - \frac{4p-9}{T_c}|\tau|, & \frac{2T_c}{p} \leq |\tau| \leq \frac{5T_c}{2p} \\ -\frac{11p-30}{p} + \frac{4p-11}{T_c}|\tau|, & \frac{5T_c}{2p} \leq |\tau| \leq \frac{3T_c}{p} \\ \frac{13p-42}{p} - \frac{4p-13}{T_c}|\tau|, & \frac{3T_c}{p} \leq |\tau| \leq \frac{7T_c}{2p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (23)$$

Equation (23), for $p = 6$, is the same as (71) in [7].

Third, we recognize that substituting $p = 7$ in (21) we get

$$R_{BOC(pn=7n,n)}(\tau) = \begin{cases} 1 - \frac{4p-1}{T_c}|\tau|, & |\tau| \leq \frac{T_c}{2p} = T_s \\ -\frac{3p-2}{p} + \frac{4p-3}{T_c}|\tau|, & \frac{T_c}{2p} \leq |\tau| \leq \frac{T_c}{p} \\ \frac{5p-6}{p} - \frac{4p-5}{T_c}|\tau|, & \frac{T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ -\frac{7p-12}{p} + \frac{4p-7}{T_c}|\tau|, & \frac{3T_c}{2p} \leq |\tau| \leq \frac{2T_c}{p} \\ \frac{9p-20}{p} - \frac{4p-9}{T_c}|\tau|, & \frac{2T_c}{p} \leq |\tau| \leq \frac{5T_c}{2p} \\ -\frac{11p-30}{p} + \frac{4p-11}{T_c}|\tau|, & \frac{5T_c}{2p} \leq |\tau| \leq \frac{3T_c}{p} \\ \frac{13p-42}{p} - \frac{4p-13}{T_c}|\tau|, & \frac{3T_c}{p} \leq |\tau| \leq \frac{7T_c}{2p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (24)$$

Equation (24), for $p = 7$, is an original result not published anywhere else. Equation (19), for $p = 8$, is also an original result not published anywhere else. This completed the *proof of corollary 1*.

Corollary 2: From *corollary 2* $VBOC1_{(m,n,\alpha=1)}(t) = BPSK_0(t)$ [1] and from *theorem 2*, the ACF $R_{VBOC1_{(m,n,\alpha=1)}}(\tau) = R_{BPSK_0}(\tau)$ hence, prove that the

generalized ACF with respect to $p = \{5,6,7,8\}$ and $\alpha = 1$ is equal to the ACF of the BPSK₀ modulation; i.e.,

- (1) $R_{BPSK_0}(\tau) = R_{VBOC1_{(pn=n,n,\alpha=1)}}(\tau)$; $p = 5$;
- (2) $R_{BPSK_0}(\tau) = R_{VBOC1_{(pn=2n,n,\alpha=1)}}(\tau)$; $p = 6$;
- (3) $R_{BPSK_0}(\tau) = R_{VBOC1_{(pn=3n,n,\alpha=1)}}(\tau)$; $p = 7$;
- (4) $R_{BPSK_0}(\tau) = R_{VBOC1_{(pn=4n,n,\alpha=1)}}(\tau)$; $p = 8$.

$$R_{BPSK_0}(\tau) = \begin{cases} \underbrace{\frac{1}{a_{14}(p,n)} + \frac{-1}{b_{14}(p,n)}|\tau|}_{r_{14}(pn,n); p=\{5,\dots,8,\dots,\infty\}}, & \frac{4T_c}{p} \leq |\tau| \leq \frac{5T_c}{p} \\ \underbrace{\frac{1}{a_{17}(p,n)} + \frac{-1}{b_{17}(p,n)}|\tau|}_{r_{17}(pn,n); p=\{6,\dots,8,\dots,\infty\}}, & \frac{4T_c}{p} \leq |\tau| \leq \frac{5T_c}{p} \\ \underbrace{\frac{1}{a_{20}(p,n)} + \frac{-1}{b_{20}(p,n)}|\tau|}_{r_{20}(pn,n); p=\{7,8,\dots,\infty\}}, & \frac{4T_c}{p} \leq |\tau| \leq \frac{5T_c}{p} \\ \underbrace{\frac{1}{a_{23}(p,n)} + \frac{-1}{b_{23}(p,n)}|\tau|}_{r_{23}(pn,n); p=\{8,\dots,\infty\}}, & \frac{4T_c}{p} \leq |\tau| \leq \frac{5T_c}{p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (25)$$

And T is given by (2).

The *proof of corollary 2* is straightforward so we leave that as an exercise to the reader.

This concludes $VBOC1(\alpha)$ generalized PSD definition and

discussion for values of $p = \{5 \dots 8\}$. Next, we continue with $VBOC1(\alpha)$ generalized PSD definition and discussion for values of $p = \{5 \dots 8\}$.

4 $VBOC1(\alpha)$ Generalized PSD Definition and Discussion

We have already developed $VBOC1(\alpha)$ generalized PSD definition and discussion in [1] for values of $p = \{1, \dots, 4\}$; hence, it remains to extend that discussion here for values of $p = \{5 \dots, 8\}$ which is accomplished by means of the following theorem.

Theorem 3: Show that, $G_{VBOC1(m=np,n,\alpha)}(f)$ is the generalized ACF with respect to $p = \{5, 6, 7, 8\}$; i.e., show that

- (1) $G_{VBOC1(n,n,\alpha)}(f) = G_{VBOC1(pn=n,n,\alpha)}(f)$; $p = 5$;
- (2) $G_{VBOC1(2n,n,\alpha)}(f) = G_{VBOC1(pn=2n,n,\alpha)}(f)$; $p = 6$;
- (3) $G_{VBOC1(3n,n,\alpha)}(f) = G_{VBOC1(pn=3n,n,\alpha)}(f)$; $p = 7$;
- (4) $G_{VBOC1(4n,n,\alpha)}(f) = G_{VBOC1(pn=4n,n,\alpha)}(f)$; $p = 8$.

Utilizing the generalized PSD of $G_{VBOC1(m=np,n,\alpha)}(f)$ definition given by

$$G_{VBOC1(pn,n,\alpha)}(f) = \mathcal{F} \left\{ R_{VBOC1(pn,n,\alpha)}(\tau) \right\} = \frac{V(f)V(f)=V^2(f)}{T} \quad (26)$$

$$G_{VBOC1(pn,n,\alpha)}(\omega) = 2 \int_0^\infty R_{VBOC1(pn,n,\alpha)}(\tau) \cos(\omega\tau) d\tau \equiv 2 \sum_{i=1}^{24} \left[\int_{\tau_{i-1}}^{\tau_i} r_{i(pn,n,\alpha)}(\tau) \cos(\omega\tau) d\tau \equiv g_{i(pn,n,\alpha)}(\omega) \right] \quad (29)$$

where t_i and $r_{i(pn,n,\alpha)}(\tau)$ are given in the expanded expression of $R_{VBOC1(pn,n,\alpha)}(\tau)$ in (15).

Based on the observation of (29) and (17) the computation of $G_{VBOC1(pn,n,\alpha)}(\omega)$ is in general a laborious process because it involves the computation of *twenty-four integrals* to obtain the generalized expression of $G_{VBOC1(pn,n,\alpha)}(\omega)$.

$$G_{VBOC1(pn,n,\alpha)}(\omega) = 2 \left[\underbrace{\sum_{i=1}^3 g_{i(pn,n,\alpha)}(\omega)}_{G_1; p=\{1, \dots, 8, \dots\}} + \underbrace{\sum_{i=4}^6 g_{i(pn,n,\alpha)}(\omega)}_{G_2; p=\{2, \dots, 8, \dots\}} + \underbrace{\sum_{i=7}^9 g_{i(pn,n,\alpha)}(\omega)}_{G_3; p=\{3, \dots, 8, \dots\}} + \underbrace{\sum_{i=10}^{12} g_{i(pn,n,\alpha)}(\omega)}_{G_4; p=\{4, \dots, 8, \dots\}} + \right. \\ \left. \underbrace{\sum_{i=13}^{15} g_{i(pn,n,\alpha)}(\omega)}_{G_4; p=\{5, \dots, 8, \dots\}} + \underbrace{\sum_{i=16}^{18} g_{i(pn,n,\alpha)}(\omega)}_{G_4; p=\{6, 7, 8, \dots\}} + \underbrace{\sum_{i=19}^{21} g_{i(pn,n,\alpha)}(\omega)}_{G_4; p=\{7, 8, \dots\}} + \underbrace{\sum_{i=22}^{24} g_{i(pn,n,\alpha)}(\omega)}_{G_4; p=\{8, \dots\}} \right] \quad (30)$$

The final expression of the generalized PSD of $G_{VBOC1(m=np,n,\alpha)}(f)$ as

$$G_{VBOC1(pn,n,\alpha)}(\omega) = 2(G_S + G_C) \quad (31)$$

where

$$G_S = \frac{\underbrace{\frac{(p-1)\sin(x)}{p} - \frac{(p-1)\sin(x)}{p} + \frac{(p-2)\sin(2x)}{p} - \frac{(p-2)\sin(2x)}{p} + \frac{(p-3)\sin(3x)}{p} - \frac{(p-3)\sin(3x)}{p} + \frac{(p-4)\sin(4x)}{p}}_{p=\{1, \dots, 8, \dots\}} + \underbrace{\frac{(p-4)\sin(4x)}{p} - \frac{(p-5)\sin(5x)}{p} + \frac{(p-5)\sin(5x)}{p} - \frac{(p-6)\sin(6x)}{p} + \frac{(p-6)\sin(6x)}{p} - \frac{(p-7)\sin(7x)}{p} + \frac{(p-7)\sin(7x)}{p} - \frac{(p-8)\sin(8x)}{p}}_{p=\{5, \dots, 8, \dots\}} + \underbrace{\frac{(p-8)\sin(8x)}{p} - \frac{(p-9)\sin(9x)}{p} + \frac{(p-9)\sin(9x)}{p} - \frac{(p-10)\sin(10x)}{p} + \frac{(p-10)\sin(10x)}{p} - \frac{(p-11)\sin(11x)}{p} + \frac{(p-11)\sin(11x)}{p} - \frac{(p-12)\sin(12x)}{p}}_{p=\{6, \dots, 8, \dots\}} + \underbrace{\frac{(p-12)\sin(12x)}{p} - \frac{(p-13)\sin(13x)}{p} + \frac{(p-13)\sin(13x)}{p} - \frac{(p-14)\sin(14x)}{p} + \frac{(p-14)\sin(14x)}{p} - \frac{(p-15)\sin(15x)}{p} + \frac{(p-15)\sin(15x)}{p} - \frac{(p-16)\sin(16x)}{p}}_{p=\{7, 8, \dots\}} + \underbrace{\frac{(p-16)\sin(16x)}{p} - \frac{(p-17)\sin(17x)}{p} + \frac{(p-17)\sin(17x)}{p} - \frac{(p-18)\sin(18x)}{p} + \frac{(p-18)\sin(18x)}{p} - \frac{(p-19)\sin(19x)}{p} + \frac{(p-19)\sin(19x)}{p} - \frac{(p-20)\sin(20x)}{p}}_{p=\{8, \dots\}}}{\omega} = 0 \quad (33)$$

where

$$x = \frac{\omega T_c}{p} \quad (34)$$

where $V(\omega \equiv 2\pi f) = \mathcal{F}\{v(\tau) \equiv VBOC1_{(m,n,\alpha)}(t)\}$ is the Fourier Transform (FT) of $f(\tau)$ (see (31) and (32) in [1])

$$G_{VBOC1(pn,n,\alpha)}(\omega) = 2 \int_0^\infty R_{VBOC1(pn,n,\alpha)}(\tau) \cos(\omega\tau) d\tau \quad (27)$$

or from (53) in [1] show that

$$G_{VBOC1(pn,n,\alpha)}(\omega) = 2G_C \quad (28)$$

where $R_{VBOC1(pn,n,\alpha)}(\tau)$ is given by (15), T , is given by (2).

Proof of theorem 3: The proof of theorem 3 is straightforward and is very similar to the proof of theorem 2.

Assuming that the reader is familiar with the derivation in theorem 6 in [1]; we compute the generalized PSD of $VBOC1_{(pn,n,\alpha)}$, or $G_{VBOC1(pn,n,\alpha)}(\omega)$, as a FT of the ACF such as by extending the result in (40) in [1]

Let us provide the details for obtaining the *twenty four integrals* that lead to the generalized expression of $G_{VBOC1(pn,n,\alpha)}(\omega)$ given by (29) by considering the derivations (41)-(47), in [1] we can obtain the expression for generalized PSD of $VBOC1_{(pn,n,\alpha)}$ or $G_{VBOC1(pn,n,\alpha)}(\omega)$ as follows

$$G_{S/C} = \sum_{i=1}^8 G_{IS/C} \quad (32)$$

Or by performing the computation of (30) and (32) and separating the sine terms yields

And ditto separating the cosine terms produces

$$G_C = \frac{\begin{aligned} & \underbrace{(2-4p)\left\{\cos\left[\frac{x(1-\alpha)}{2}\right] + \cos\left[\frac{x(1+\alpha)}{2}\right]\right\} + (4p-3)\cos(x) + 4p-1 + (6-4p)\left\{\cos\left[\frac{x(3-\alpha)}{2}\right] + \cos\left[\frac{x(3+\alpha)}{2}\right]\right\} + (4p-7)\cos(2x) + (4p-5)\cos(x) +}_{p=\{1,\dots,8,\dots\}} \\ & \underbrace{(10-4p)\left\{\cos\left[\frac{x(5-\alpha)}{2}\right] + \cos\left[\frac{x(5+\alpha)}{2}\right]\right\} + (4p-11)\cos(3x) + (4p-9)\cos(2x) + (14-4p)\left\{\cos\left[\frac{x(7-\alpha)}{2}\right] + \cos\left[\frac{x(7+\alpha)}{2}\right]\right\} + (4p-15)\cos(4x) + (4p-13)\cos(3x) +}_{p=\{3,\dots,8,\dots\}} \\ & \underbrace{(18-4p)\left\{\cos\left[\frac{x(9-\alpha)}{2}\right] + \cos\left[\frac{x(9+\alpha)}{2}\right]\right\} + (4p-19)\cos(5x) + (4p-17)\cos(4x) + (22-4p)\left\{\cos\left[\frac{x(11-\alpha)}{2}\right] + \cos\left[\frac{x(11+\alpha)}{2}\right]\right\} + (4p-23)\cos(6x) + (4p-21)\cos(5x) +}_{p=\{5,\dots,8,\dots\}} \\ & \underbrace{(26-4p)\left\{\cos\left[\frac{x(13-\alpha)}{2}\right] + \cos\left[\frac{x(13+\alpha)}{2}\right]\right\} + (4p-27)\cos(7x) + (4p-25)\cos(6x) + (30-4p)\left\{\cos\left[\frac{x(15-\alpha)}{2}\right] + \cos\left[\frac{x(15+\alpha)}{2}\right]\right\} + (4p-31)\cos(8x) + (4p-29)\cos(7x) +}_{p=\{7,8,\dots\}} \end{aligned}}{\omega^2 T_c} \quad (35)$$

Since, from (31) $G_S = 0$ for any values of p ; hence, (31) and (28) becomes equal to (28).

Now, let us compute the individual expressions of $G_{VBOC1(pn,n,\alpha)}(\omega)$ for values of $p = \{5, \dots, 8\}$ based on (35)

$$G_{VBOC1(5n,n,\alpha)}(\omega) = \frac{\begin{aligned} & 38-36\left\{\cos\left[\frac{x(1-\alpha)}{2}\right] + \cos\left[\frac{x(1+\alpha)}{2}\right]\right\} + 64\cos(x) - 28\left\{\cos\left[\frac{x(3-\alpha)}{2}\right] + \cos\left[\frac{x(3+\alpha)}{2}\right]\right\} + 48\cos(2x) - 20\left\{\cos\left[\frac{x(5-\alpha)}{2}\right] + \cos\left[\frac{x(5+\alpha)}{2}\right]\right\} + 32\cos(3x) \\ & - 12\left\{\cos\left[\frac{x(7-\alpha)}{2}\right] + \cos\left[\frac{x(7+\alpha)}{2}\right]\right\} + 16\cos(4x) - 4\left\{\cos\left[\frac{x(9-\alpha)}{2}\right] + \cos\left[\frac{x(9+\alpha)}{2}\right]\right\} + 2\cos(5x) \end{aligned}}{\omega^2 T_c} \quad (36)$$

Equation (36) is original and it is not published before.

Second, for $p = 6$ from (35) and (28) we obtain

$$G_{VBOC1(6n,n,\alpha)}(\omega) = \frac{\begin{aligned} & 46-44\left\{\cos\left[\frac{x(1-\alpha)}{2}\right] + \cos\left[\frac{x(1+\alpha)}{2}\right]\right\} + 80\cos(x) - 36\left\{\cos\left[\frac{x(3-\alpha)}{2}\right] + \cos\left[\frac{x(3+\alpha)}{2}\right]\right\} + 64\cos(2x) - 28\left\{\cos\left[\frac{x(5-\alpha)}{2}\right] + \cos\left[\frac{x(5+\alpha)}{2}\right]\right\} + 48\cos(3x) \\ & - 20\left\{\cos\left[\frac{x(7-\alpha)}{2}\right] + \cos\left[\frac{x(7+\alpha)}{2}\right]\right\} + 32\cos(4x) - 12\left\{\cos\left[\frac{x(9-\alpha)}{2}\right] + \cos\left[\frac{x(9+\alpha)}{2}\right]\right\} + 16\cos(5x) - 4\left\{\cos\left[\frac{x(11-\alpha)}{2}\right] + \cos\left[\frac{x(11+\alpha)}{2}\right]\right\} + 2\cos(6x) \end{aligned}}{\omega^2 T_c} \quad (37)$$

Equation (37) is original and it is not published before.

Third, for $p = 7$ from (35) and (28) we obtain

$$G_{VBOC1(7n,n,\alpha)}(\omega) = \frac{\begin{aligned} & 54-52\left\{\cos\left[\frac{x(1-\alpha)}{2}\right] + \cos\left[\frac{x(1+\alpha)}{2}\right]\right\} + 96\cos(x) - 44\left\{\cos\left[\frac{x(3-\alpha)}{2}\right] + \cos\left[\frac{x(3+\alpha)}{2}\right]\right\} + 80\cos(2x) - 36\left\{\cos\left[\frac{x(5-\alpha)}{2}\right] + \cos\left[\frac{x(5+\alpha)}{2}\right]\right\} + 64\cos(3x) \\ & - 28\left\{\cos\left[\frac{x(7-\alpha)}{2}\right] + \cos\left[\frac{x(7+\alpha)}{2}\right]\right\} + 48\cos(4x) - 20\left\{\cos\left[\frac{x(9-\alpha)}{2}\right] + \cos\left[\frac{x(9+\alpha)}{2}\right]\right\} + 32\cos(5x) - 12\left\{\cos\left[\frac{x(11-\alpha)}{2}\right] + \cos\left[\frac{x(11+\alpha)}{2}\right]\right\} + 16\cos(6x) \\ & - 4\left\{\cos\left[\frac{x(13-\alpha)}{2}\right] + \cos\left[\frac{x(13+\alpha)}{2}\right]\right\} + 2\cos(7x) \end{aligned}}{\omega^2 T_c} \quad (38)$$

Equation (38) is original and it is not published before.

Fourth and finally, for $p = 8$ from (35) and (28) we obtain

$$G_{VBOC1(8n,n,\alpha)}(\omega) = \frac{\begin{aligned} & 62-60\left\{\cos\left[\frac{x(1-\alpha)}{2}\right] + \cos\left[\frac{x(1+\alpha)}{2}\right]\right\} + 112\cos(x) - 52\left\{\cos\left[\frac{x(3-\alpha)}{2}\right] + \cos\left[\frac{x(3+\alpha)}{2}\right]\right\} + 96\cos(2x) - 44\left\{\cos\left[\frac{x(5-\alpha)}{2}\right] + \cos\left[\frac{x(5+\alpha)}{2}\right]\right\} + 80\cos(3x) \\ & - 36\left\{\cos\left[\frac{x(7-\alpha)}{2}\right] + \cos\left[\frac{x(7+\alpha)}{2}\right]\right\} + 64\cos(4x) - 28\left\{\cos\left[\frac{x(9-\alpha)}{2}\right] + \cos\left[\frac{x(9+\alpha)}{2}\right]\right\} + 48\cos(5x) - 20\left\{\cos\left[\frac{x(11-\alpha)}{2}\right] + \cos\left[\frac{x(11+\alpha)}{2}\right]\right\} + 32\cos(6x) \\ & - 12\left\{\cos\left[\frac{x(13-\alpha)}{2}\right] + \cos\left[\frac{x(13+\alpha)}{2}\right]\right\} + 16\cos(7x) - 4\left\{\cos\left[\frac{x(15-\alpha)}{2}\right] + \cos\left[\frac{x(15+\alpha)}{2}\right]\right\} + 2\cos(8x) \end{aligned}}{\omega^2 T_c} \quad (39)$$

Equation (39) is original and it is not published before.

Equations (26) through (37) complete the proof of *theorem 3*.

Theorem 3 is the most important *theorem* of the PSDs of $VBOC1(\alpha)$ for $p = \{1, \dots, 8\}$. This concludes $VBOC1(\alpha)$ generalized PSD discussion. Next, we continue with $VBOC1(\alpha)$ numerical results or examples $p = \{6, 8\}$.

5 Theoretical, Numerical Results

We discuss two examples: $VBOC1_{(6,1,\alpha)}(t)$ and $VBOC1_{(8,1,\alpha)}(t)$ because $VBOC1_{(6,1,\alpha)}(t)$ is the generalized $BOC_{(61)}(t) = VBOC1_{(6,1,\alpha=0)}(t)$ on GPS L1C pilot signal [1]-[17] and $VBOC1_{(8,1,\alpha)}(t)$ is the generalized $BOC_{(81)}(t) = BOC_{(40,5)}(t) = VBOC1_{(8,1,\alpha=0)}(t)$ is the provision for a new

GPS military M-code [1]-[17] on both GPS L1 and L2 frequencies pilot signal that will enhance the two most important waveforms in the GNSS community at the present time.

5.1 $VBOC1(\alpha)$ Examples

In this subsection we present, numerical results of $VBOC1(\alpha)$ examples signal using the exact closed form expression found in this paper.

Corollary 3: From *definition 1*, $VBOC1_{(6,1,\alpha)}(t)$ is simply

$$VBOC1_{(m=6,n=1,\alpha)}(t) = VBOC1_{(m=6,n=1,\alpha)}(t \pm 2T_s) \quad (40)$$

and the relation between T_s and T_c is given by

$$12T_s = T_c \quad (41)$$

hence,

$$VBOC1_{(m=6,n=1,\alpha)}(t) = VBOC1_{(m=6,n=1,\alpha)}\left(t \pm \frac{T_c}{6}\right) \quad (42)$$

Corollary 4: From *definition 2*, the inequalities of w_r , w_f , and α are as follows

$$w_r = \frac{T_c(1-\alpha)}{12}; w_f = \frac{T_c(1+\alpha)}{12} \quad (43)$$

$$0 \leq w_r \leq \frac{T_c}{12} \text{ or } \frac{T_c}{12} \leq w_f \leq \frac{T_c}{6}; 0 \leq \alpha \leq 1 \quad (44)$$

hence, $VBOC1_{(6,1,\alpha)}(t)$ is simply

$$VBOC1_{(6,1,\alpha)}(t) = VBOC1r_{(6,1,\alpha)}(t) + VBOC1f_{(6,1,\alpha)}(t) \quad (45)$$

Where

$$VBOC1r_{(6,1,\alpha)}(t) = \sum_{n=-\infty}^{\infty} \sum_{i=1}^6 p_{w_r}(t_1, w_r) \quad (46)$$

$$VBOC1f_{(6,1,\alpha)}(t) = \sum_{n=-\infty}^{\infty} \sum_{i=1}^6 p_{w_f}(t_2, w_f) \quad (47)$$

$$R_{VBOC1_{(6,1,\alpha)}}(\tau) = \begin{cases} 1 - \frac{23|\tau|}{T_c}, 0 \leq |\tau| \leq \frac{T_c(1-\alpha)}{12} \\ -\frac{5+11\alpha}{6} - \frac{|\tau|}{T_c}, \frac{T_c(1-\alpha)}{12} \leq |\tau| \leq \frac{T_c(1+\alpha)}{12} \\ -\frac{8}{3} + \frac{21|\tau|}{T_c}, \frac{T_c(1+\alpha)}{12} \leq |\tau| \leq \frac{T_c}{6} \\ 4 - \frac{19|\tau|}{T_c}, \frac{T_c}{6} \leq |\tau| \leq \frac{T_c(3-\alpha)}{12} \\ -\frac{1+3\alpha}{2} - \frac{|\tau|}{T_c}, \frac{T_c(3-\alpha)}{12} \leq |\tau| \leq \frac{T_c(3+\alpha)}{12} \\ -5 + \frac{17|\tau|}{T_c}, \frac{T_c(3+\alpha)}{12} \leq |\tau| \leq \frac{2T_c}{6} \\ \frac{17}{3} - \frac{15|\tau|}{T_c}, \frac{2T_c}{6} \leq |\tau| \leq \frac{T_c(5-\alpha)}{12} \\ -\frac{1+7\alpha}{6} - \frac{|\tau|}{T_c}, \frac{T_c(5-\alpha)}{12} \leq |\tau| \leq \frac{T_c(5+\alpha)}{12} \\ -6 + \frac{13|\tau|}{T_c}, \frac{T_c(5+\alpha)}{12} \leq |\tau| \leq \frac{3T_c}{6} \end{cases}$$

Because $R_{VBOC1_{(6,1,\alpha)}}(\tau)$ is a special case of $R_{VBOC1_{(m,n,\alpha)}}(\tau)$ it is easy to see that it satisfied all conditions of *theorems 2* and *3*.

Corollary 5: Prove that the ACF of $VBOC1_{(6,1,\alpha)}(t)$ given by (49) will result in $BOC_{(6,1)}(t)$ for $\alpha = 0$ as indicated in *definition 1* in [1] and $BOC_{(1/2,1)}(t)$ for $\alpha = 1$ as indicated in

$$t_1 = t - 0.5w_r - (n+i) \left[T_s \equiv \frac{T_c}{12} \right]; i = \{1, \dots, 6\} \quad (48)$$

$$t_2 = t - 0.5w_f - w_r - (n+i) \left[T_s \equiv \frac{T_c}{12} \right]; i = \{1, \dots, 6\} \quad (49)$$

If we consider only one chipping period or twelve subcarrier periods of $VBOC1_{(6,1,\alpha)}(t)$; i.e., for $p = 0$ we get

$$VBOC1_{(6,1,\alpha)}(t) = \sum_{i=1}^6 \left[p_{w_r}(t_1, w_r) + p_{w_f}(t_2, w_f) \right] \quad (50)$$

$$t_1 = t - 0.5w_r - (1+i) \left[T_s \equiv \frac{T_c}{12} \right]; i = \{1, \dots, 6\} \quad (51)$$

$$t_2 = t - 0.5w_f - w_r - (1+i) \left[T_s \equiv \frac{T_c}{12} \right]; i = \{1, \dots, 6\} \quad (52)$$

Based on *Theorem 2* (17) we find the ACF for $VBOC1_{(6,1,\alpha)}(t)$, or $R_{VBOC1_{(6,1,\alpha)}}(\tau)$ given by

$$\begin{cases} 6 - \frac{11|\tau|}{T_c}, \frac{3T_c}{6} \leq |\tau| \leq \frac{T_c(7-\alpha)}{12} \\ \frac{1+5\alpha}{6} - \frac{|\tau|}{T_c}, \frac{T_c(7-\alpha)}{12} \leq |\tau| \leq \frac{T_c(7+\alpha)}{12} \\ -\frac{17}{3} + \frac{9|\tau|}{T_c}, \frac{T_c(7+\alpha)}{12} \leq |\tau| \leq \frac{4T_c}{6} \\ 5 - \frac{7|\tau|}{T_c}, \frac{4T_c}{6} \leq |\tau| \leq \frac{T_c(9-\alpha)}{12} \\ \frac{1+\alpha}{2} - \frac{|\tau|}{T_c}, \frac{T_c(9-\alpha)}{12} \leq |\tau| \leq \frac{T_c(9+\alpha)}{12} \\ -4 + \frac{5|\tau|}{T_c}, \frac{T_c(9+\alpha)}{12} \leq |\tau| \leq \frac{5T_c}{6} \\ \frac{8}{3} - \frac{3|\tau|}{T_c}, \frac{5T_c}{6} \leq |\tau| \leq \frac{T_c(11-\alpha)}{12} \\ \frac{5+\alpha}{6} - \frac{|\tau|}{T_c}, \frac{T_c(11-\alpha)}{12} \leq |\tau| \leq \frac{T_c(11+\alpha)}{12} \\ -1 + \frac{|\tau|}{T_c}, \frac{T_c(11+\alpha)}{12} \leq |\tau| \leq T_c \\ 0, |\tau| \geq T_c \end{cases} \quad \text{ii} \quad (53)$$

definition 2 in [1].

Proof of corollary 5: The *proof of corollary 5* is straightforward. First, we substitute values of $\alpha = 0$ in (53) and we get $R_{VBOC1_{(6n,n,\alpha=0)}}(\tau) = R_{BOC_{(6n,n)}}(\tau)$ as indicated in (71) in [7].

$$R_{VBOC1_{(6n,n,\alpha=0)}}(\tau) = \begin{cases} 1 - \frac{23|\tau|}{T_c}, 0 \leq |\tau| \leq \frac{T_c}{12} \\ -\frac{8}{3} + \frac{21|\tau|}{T_c}, \frac{T_c}{12} \leq |\tau| \leq \frac{T_c}{6} \\ 4 - \frac{19|\tau|}{T_c}, \frac{T_c}{6} \leq |\tau| \leq \frac{3T_c}{12} \\ -5 + \frac{17|\tau|}{T_c}, \frac{3T_c}{12} \leq |\tau| \leq \frac{T_c}{3} \\ \frac{17}{3} - \frac{15|\tau|}{T_c}, \frac{T_c}{3} \leq |\tau| \leq \frac{5T_c}{12} \\ -6 + \frac{13|\tau|}{T_c}, \frac{5T_c}{12} \leq |\tau| \leq \frac{T_c}{2} \end{cases} \begin{cases} 6 - \frac{11|\tau|}{T_c}, \frac{T_c}{2} \leq |\tau| \leq \frac{7T_c}{12} \\ -\frac{17}{3} + \frac{9|\tau|}{T_c}, \frac{7T_c}{12} \leq |\tau| \leq \frac{2T_c}{3} \\ 5 - \frac{7|\tau|}{T_c}, \frac{2T_c}{3} \leq |\tau| \leq \frac{3T_c}{4} \\ -4 + \frac{5|\tau|}{T_c}, \frac{3T_c}{4} \leq |\tau| \leq \frac{5T_c}{6} \\ \frac{8}{3} - \frac{3|\tau|}{T_c}, \frac{5T_c}{6} \leq |\tau| \leq \frac{11T_c}{12} \\ -1 + \frac{|\tau|}{T_c}, \frac{11T_c}{12} \leq |\tau| \leq T_c \\ 0, |\tau| \geq T_c \end{cases} = R_{BOC_{(6n,n)}}(\tau) \quad \text{iii} \quad (54)$$

Second, we substitute values of $\alpha = 1$ in (53) and we get in *corollary 2* in [1].
 $R_{VBOC1(6n,n,\alpha=1)}(\tau) = R_{BOC(1/2n,n)}(\tau) = R_{BPSK}(\tau)$ as indicated

$$R_{VBOC1(6n,n,\alpha=1)}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T_c}, & 0 \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c \end{cases} = R_{BPSK_0}(\tau) \quad (55)$$

Equations (54) and (55) complete the *proof of corollary 5*.

Corollary 6: From *definition 1*, $VBOC1_{(8,1,\alpha)}(t)$ is simply

$$VBOC1_{(m=8,n=1,\alpha)}(t) = VBOC1_{(m=8,n=1,\alpha)}(t \pm 2T_s) \quad (56)$$

and the relation between T_s and T_c is given by

$$16T_s = T_c \quad (57)$$

hence,

$$VBOC1_{(m=8,n=1,\alpha)}(t) = VBOC1_{(m=8,n=1,\alpha)}\left(t \pm \frac{T_c}{8}\right) \quad (58)$$

Corollary 7: From *definition 2*, the inequalities of w_r , w_f , and α are as follows

$$w_r = \frac{T_c(1-\alpha)}{16}; w_f = \frac{T_c(1+\alpha)}{16} \quad (59)$$

$$0 \leq w_r \leq \frac{T_c}{16} \text{ or } \frac{T_c}{16} \leq w_f \leq \frac{T_c}{8}; 0 \leq \alpha \leq 1 \quad (60)$$

hence, $VBOC1_{(8,1,\alpha)}(t)$ is simply

$$VBOC1_{(8,1,\alpha)}(t) = VBOC1r_{(8,1,\alpha)}(t) + VBOC1f_{(8,1,\alpha)}(t) \quad (61)$$

Where

$$R_{VBOC1(8n,n,\alpha=0)}(\tau) = \begin{cases} 1 - \frac{31|\tau|}{T_c}, & 0 \leq |\tau| \leq \frac{T_c(1-\alpha)}{16} \\ -\frac{7+15\alpha}{8} - \frac{|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{16} \leq |\tau| \leq \frac{T_c(1+\alpha)}{16} \\ -\frac{11}{4} + \frac{29|\tau|}{T_c}, & \frac{T_c(1+\alpha)}{16} \leq |\tau| \leq \frac{T_c}{8} \\ \frac{17}{4} - \frac{27|\tau|}{T_c}, & \frac{T_c}{8} \leq |\tau| \leq \frac{T_c(3-\alpha)}{16} \\ -\frac{5+13\alpha}{8} - \frac{|\tau|}{T_c}, & \frac{T_c(3-\alpha)}{16} \leq |\tau| \leq \frac{T_c(3+\alpha)}{16} \\ -\frac{11}{2} + \frac{25|\tau|}{T_c}, & \frac{T_c(3+\alpha)}{16} \leq |\tau| \leq \frac{T_c}{4} \\ \frac{13}{2} - \frac{23|\tau|}{T_c}, & \frac{T_c}{4} \leq |\tau| \leq \frac{T_c(5-\alpha)}{16} \\ -\frac{3+11\alpha}{8} - \frac{|\tau|}{T_c}, & \frac{T_c(5-\alpha)}{16} \leq |\tau| \leq \frac{T_c(5+\alpha)}{16} \\ -\frac{29}{4} + \frac{21|\tau|}{T_c}, & \frac{T_c(5+\alpha)}{16} \leq |\tau| \leq \frac{3T_c}{8} \\ \frac{31}{4} - \frac{19|\tau|}{T_c}, & \frac{3T_c}{8} \leq |\tau| \leq \frac{T_c(7-\alpha)}{16} \\ -\frac{1+9\alpha}{8} - \frac{|\tau|}{T_c}, & \frac{T_c(7-\alpha)}{16} \leq |\tau| \leq \frac{T_c(7+\alpha)}{16} \\ -8 + \frac{17|\tau|}{T_c}, & \frac{T_c(7+\alpha)}{16} \leq |\tau| \leq \frac{T_c}{2} \\ 0, & |\tau| \geq T_c \end{cases} \quad \text{iv} \quad (69)$$

Because $R_{VBOC1(8,1,\alpha)}(\tau)$ is a special case of $R_{VBOC1(m,n,\alpha)}(\tau)$ it is easy to see that it satisfied all conditions of *theorems 2* and *3*.

$$VBOC1r_{(8,1,\alpha)}(t) = \sum_{n=-\infty}^{\infty} \sum_{i=1}^8 p_{w_r}(t_1, w_r) \quad (62)$$

$$VBOC1f_{(8,1,\alpha)}(t) = \sum_{n=-\infty}^{\infty} \sum_{i=1}^8 p_{w_f}(t_2, w_f) \quad (63)$$

$$t_1 = t - 0.5w_r - (n+i) \left[T_s \equiv \frac{T_c}{16} \right]; i = \{1, \dots, 8\} \quad (64)$$

$$t_2 = t - 0.5w_f - w_r - (n+i) \left[T_s \equiv \frac{T_c}{16} \right]; i = \{1, \dots, 8\} \quad (65)$$

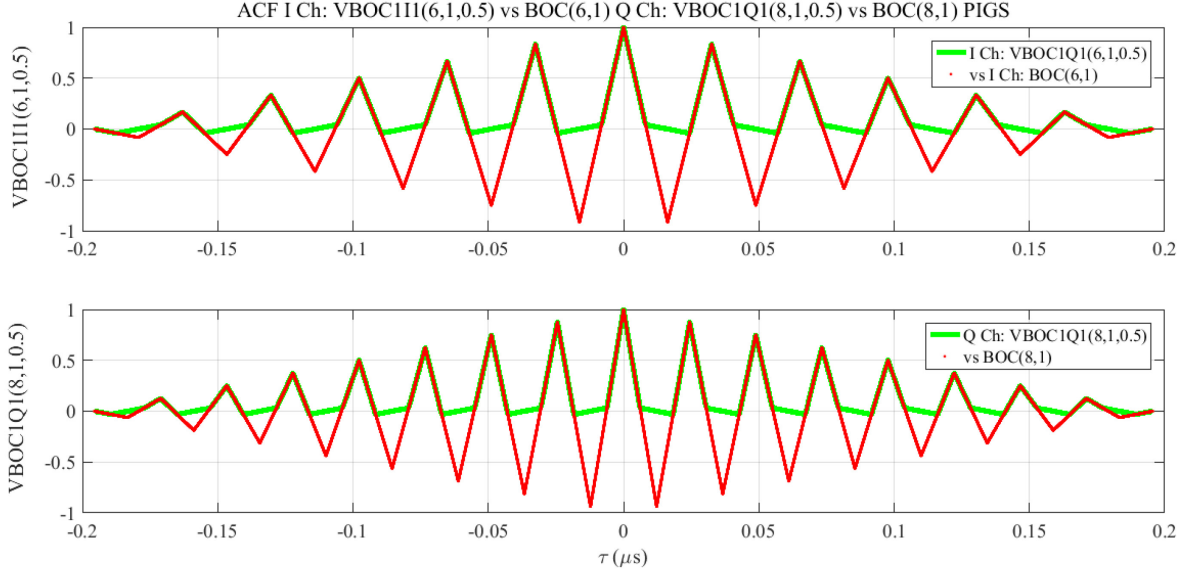
If we consider only one chipping period or twelve subcarrier periods of $VBOC1_{(8,1,\alpha)}(t)$; i.e., for $p = 0$ we get

$$VBOC1_{(8,1,\alpha)}(t) = \sum_{i=1}^8 [p_{w_r}(t_1, w_r) + p_{w_f}(t_2, w_f)] \quad (66)$$

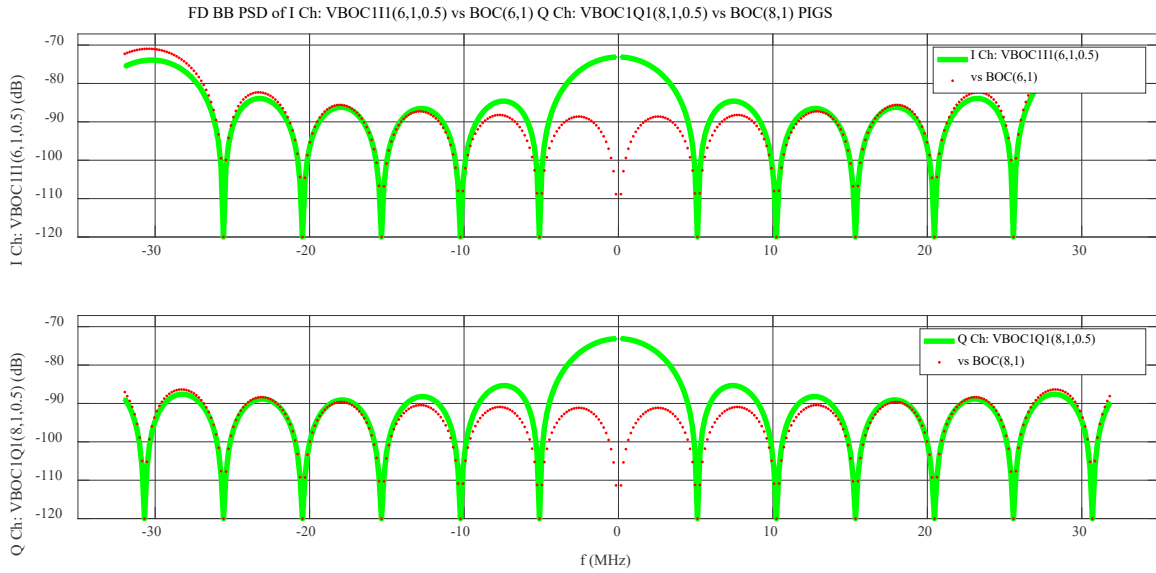
$$t_1 = t - 0.5w_r - (1+i) \left[T_s \equiv \frac{T_c}{16} \right]; i = \{1, \dots, 8\} \quad (67)$$

$$t_2 = t - 0.5w_f - w_r - (1+i) \left[T_s \equiv \frac{T_c}{16} \right]; i = \{1, \dots, 8\} \quad (68)$$

Based on *Theorem 2* (17) we find the ACF for $VBOC1_{(8,1,\alpha)}(t)$, or $R_{VBOC1(8,1,\alpha)}(\tau)$ given by



(a) ACF of $VBOC1_{(6n,n,0.5)}(t)$ vs $BOC_{(6n,n)}(t)$ on I channel and $VBOC1_{(8n,n,0.5)}(t)$ vs $BOC_{(8n,n)}(t)$ on Q channel.



(b) PSD of $VBOC1_{(6n,n,0.5)}(t)$ vs $BOC_{(6n,n)}(t)$ on I channel and $VBOC1_{(8n,n,0.5)}(t)$ vs $BOC_{(8n,n)}(t)$ on Q channel.

FIGURE 1: Simulation results of $VBOC1_{(6n,n,0.5)}(t)$ vs $BOC_{(6n,n)}(t)$ on I channel and $VBOC1_{(8n,n,0.5)}(t)$ vs $BOC_{(8n,n)}(t)$ on Q channel.

Corollary 8: Prove that the ACF of $VBOC1_{(8,1,\alpha)}(t)$ given by (69) will result in $BOC_{(8,1)}(t)$ for $\alpha = 0$ as indicated in definition 1 in [1] and $BOC_{(1/2,1)}(t)$ for $\alpha = 1$ as indicated in definition 2 in [1].

Proof of corollary 8: The proof of corollary 8 is straightforward. First, we substitute values of $\alpha = 0$ in (69) and we get $R_{VBOC1_{(6n,n,\alpha=0)}}(\tau) = R_{BOC_{(6n,n)}}(\tau)$. This is an original result not published anywhere else.

$$R_{VBOC1(8n,n,\alpha=0)}(\tau) = \begin{cases} 1 - \frac{31|\tau|}{T_c}, & 0 \leq |\tau| \leq \frac{T_c}{16} \\ -\frac{11}{4} + \frac{29|\tau|}{T_c}, & \frac{T_c}{16} \leq |\tau| \leq \frac{T_c}{8} \\ \frac{17}{4} - \frac{27|\tau|}{T_c}, & \frac{T_c}{8} \leq |\tau| \leq \frac{3T_c}{16} \\ -\frac{11}{2} + \frac{25|\tau|}{T_c}, & \frac{3T_c}{16} \leq |\tau| \leq \frac{T_c}{4} \\ \frac{13}{2} - \frac{23|\tau|}{T_c}, & \frac{T_c}{4} \leq |\tau| \leq \frac{5T_c}{16} \\ -\frac{29}{4} + \frac{21|\tau|}{T_c}, & \frac{5T_c}{16} \leq |\tau| \leq \frac{3T_c}{8} \\ \frac{31}{4} - \frac{19|\tau|}{T_c}, & \frac{3T_c}{8} \leq |\tau| \leq \frac{7T_c}{16} \\ -8 + \frac{17|\tau|}{T_c}, & \frac{7T_c}{16} \leq |\tau| \leq \frac{T_c}{2} \end{cases} \quad \begin{cases} 8 - \frac{15|\tau|}{T_c}, & \frac{T_c}{2} \leq |\tau| \leq \frac{9T_c}{16} \\ -\frac{31}{84} + \frac{13|\tau|}{T_c}, & \frac{9T_c}{16} \leq |\tau| \leq \frac{5T_c}{8} \\ \frac{29}{4} - \frac{11|\tau|}{T_c}, & \frac{5T_c}{8} \leq |\tau| \leq \frac{11T_c}{16} \\ -\frac{13}{2} + \frac{9|\tau|}{T_c}, & \frac{11T_c}{16} \leq |\tau| \leq \frac{3T_c}{4} \\ \frac{11}{2} - \frac{7|\tau|}{T_c}, & \frac{3T_c}{4} \leq |\tau| \leq \frac{13T_c}{16} \\ -\frac{17}{4} + \frac{5|\tau|}{T_c}, & \frac{13T_c}{16} \leq |\tau| \leq \frac{7T_c}{8} \\ \frac{11}{4} - \frac{3|\tau|}{T_c}, & \frac{7T_c}{8} \leq |\tau| \leq \frac{15T_c}{16} \\ -1 + \frac{|\tau|}{T_c}, & \frac{15T_c}{16} \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c \end{cases} = R_{BOC(8n,n)}(\tau)^v \quad (70)$$

Second, we substitute values of $\alpha = 1$ in (69) and we get in *corollary 2* in [1].
 $R_{VBOC1(6n,n,\alpha=1)}(\tau) = R_{BOC(1/2n,n)}(\tau) = R_{BPSK}(\tau)$ as indicated

$$R_{VBOC1(6n,n,\alpha=1)}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T_c}, & 0 \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c \end{cases} = R_{BPSK_0}(\tau) \quad (71)$$

Equations (70) and (71) complete the *proof of corollary 8*.

Corollary 9: From *theorem 3* the PSD of $VBOC1_{(6,1,\alpha)}(t)$ or $G_{VBOC1(6,1,\alpha)}(\omega)$ is obtained from (37) by substituting in (34) $p = 6$. Prove that $G_{VBOC1(6,1,\alpha=0)}(f) = G_{BOC(6,1)}(f)$ and $G_{VBOC1(6,1,\alpha=1)}(f) = G_{BPSK_0}(f)$.

From *theorem 3* the PSD of $VBOC1_{(8,1,\alpha)}(t)$ or

$$G_{VBOC1(6,1,\alpha=0)}(f) = T_c \text{sinc}^2(fT_c) \sin^2\left(\frac{\pi f T_c}{12}\right) = G_{BOC(6,1)}(f)^{vi} \quad (72)$$

as indicated in (*corollary 1* in [1]) as depicted in Fig. 2, also after some simplifications of (37) and $\alpha = 1$ we obtain

$$G_{VBOC1(6,1,\alpha=1)}(f) = T_c \text{sinc}^2(fT_c) = G_{BPSK_0}(f) \quad (73)$$

$$G_{VBOC1(8,1,\alpha=0)}(f) = T_c \text{sinc}^2(fT_c) \sin^2\left(\frac{\pi f T_c}{16}\right) = G_{BOC(8,1)}(f) \quad (74)$$

as indicated in (*corollary 1* in [1]) as depicted in Fig. 2, also after some simplifications of (37) and $\alpha = 1$ we obtain

$$G_{VBOC1(8,1,\alpha=1)}(f) = T_c \text{sinc}^2(fT_c) = G_{BPSK_0}(f) \quad (75)$$

as indicated in (*corollary 2* in [1]).

Equations (72) through (75) complete the proof of *corollary 9*.

In Fig. 1 (a) the ACF of $VBOC1_{(6,1,\alpha=0.5)}(t)$ or $r_{VBOC1(6,1,\alpha=0.5)}(\tau)$ is shown with solid green has lower peaks than the ACF of $BOC_{(6,1)}(t)$ or $r_{BOC(6,1)}(\tau)$ in dotted red; hence, $VBOC1_{(6,1,\alpha=0.5)}(t)$ should offer better interference protection than $BOC_{(6,1)}(t)$ or the current GPS III L1 data-code. In Fig.1 (a) also the ACF of $VBOC1_{(8,1,\alpha=0.5)}(t)$ or $r_{VBOC1(8,1,\alpha=0.5)}(\tau)$ is illustrated with solid green has lower

$G_{VBOC1(8,1,\alpha)}(\omega)$ is obtained from (39) by substituting in (34) $p = 8$. Prove that $G_{VBOC1(8,1,\alpha=0)}(f) = G_{BOC(8,1)}(f)$ and $G_{VBOC1(8,1,\alpha=1)}(f) = G_{BPSK_0}(f)$.

Proof of corollary 9: The *proof of corollary 9* is straightforward. After some simplifications of (37) and $\alpha = 0$ and if we substitute $\omega = 2\pi f$ it can be shown that

as indicated in (*corollary 2* in [1]).

Next, after some simplifications of (39) and $\alpha = 0$ it can be shown that

peaks than the ACF of $BOC_{(8,1)}(t)$ or $r_{BOC(8,1)}(\tau)$ in dotted red; hence, $VBOC1_{(8,1,\alpha=0.5)}(t)$ should offer better interference protection than $BOC_{(8,1)}(t)$ or GPS M-code. Further details of the new signal in the military code of the current M-code from the current $BOC(2,1)$ to the $TMVBOC(8,1,4/33,0.5)$ in which the data is $TMVBOC(2,1,29/33,0.5)$ will be provided in a new publication.

Similarly, in Fig. 1 (b) the PSD of $VBOC1_{(6,1,\alpha=0.5)}(t)$ or $G_{VBOC1(6,1,\alpha=0.5)}(f)$ is depicted with solid green is quasi-flatter and wider than the PSD of $BOC_{(6,1)}(t)$ or $G_{BOC(6,1)}(f)$ in dotted red; hence, $VBOC1_{(6,1,\alpha=0.5)}(t)$ should offer better interference protection than $BOC_{(6,1)}(t)$ or the current GPS III

L1 data-code. In Fig. 1 (b) also the PSD of $VBOC1_{(8,1,\alpha=0.5)}(t)$ or $G_{VBOC1_{(8,1,\alpha=0.5)}}(f)$ is displayed with solid green is quasi-flatter and wider than the PSD of $BOC_{(8,1)}(t)$ or $G_{BOC_{(8,1)}}(f)$ in dotted red; hence, $VBOC1_{(8,1,\alpha=0.5)}(t)$ should offer better interference protection than $BOC_{(8,1)}(t)$ or GPS M-code.

This completes the detailed discussion on $VBOC1(\alpha)$ generalized ACFs and PSDs which contains four definitions, three theorems and nine corollaries.

6 Conclusions

This paper is the first complete discussion on pure signal design for the first generation $VBOC1(p = \{1, \dots, 8\}, \alpha)$ generalized multidimensional geolocation modulation waveforms.

Contrast the results of this paper with previous signal design methodologies, this paper offers for the first time a complete pure signal design methodology subject to both signal design and optimization parameter α and generalized signal design and optimization parameter p .

Signal parameters α and p not only define the waveform $VBOC1(\alpha)$ and generalized ACFs and PSDs but they also play a very important role in the optimization of $VBOC1(\alpha)$ generalized ACFs and PSDs. The computational technique offers a unique and original description of the generalized ACFs and PSDs of $VBOC1(\alpha)$ as functions of both α and p .

In the paper it is argued that the selection of BOC(1,1) on the GPS L1 civil data code and BOC(10,5) (or the military code or M-Code) on both GPS L1 and L2 frequencies is entirely arbitrary because BOC modulation is a special case of $VBOC1(\alpha)$ for $\alpha = 0$ or $\alpha = 1$; hence, all the current state-of-the-art GNSS waveforms exhibit sub-optimal signal design performance even at the end-user when generalized global objective functions are applied.

The above is based on a discussion of $VBOC1(\alpha)$ pure signal optimization in [1]: (1) the criteria for validating the closed form expression of the generalized ACF of $VBOC1(\alpha)$ known as a set of *continuity theorems*; and (2) the criteria for selecting the optimum $0 \leq \alpha \leq 1$ based on a set of criteria known as *optimization theorems* regardless of generalized parameter p (or subcarrier frequency).

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9 Appendix A $VBOC1(\alpha)$ ACF Pure Signal Optimization

Detailed discussion on $VBOC1(\alpha)$ pure signal design is given in the beginning of this paper. $VBOC1(\alpha)$ ACF pure signal optimization includes: (1) $VBOC1(\alpha)$ generalized ACF continuity theorems and (2) generalized ACF optimization theorems.

9.1 Generalized ACF continuity theorems

Theorem 4: Prove that the generalized ACF $R_{VBOC1(m=pn,n,\alpha)}(\tau)$, given by (17), obeys the continuity theorem of the first kind; i.e., it is a continuous function for every value of τ , $p = \{1, \dots, 8\}$, and continuous values of $0 \leq \alpha \leq 1$.

Proof of theorem 4: The proof of theorem 4 is straightforward. In order to prove that the ACF, $R_{VBOC1(m=pn,n,\alpha)}(\tau)$, given by (17), obeys the continuity theorem of the first kind; i.e., it is a continuous function for every values of τ , $p = \{1, \dots, 8\}$, and continuous values of $0 \leq \alpha \leq 1$; it suffices to prove that $R_{VBOC1(pn=m,n,\alpha)}(\tau)$, given by (15), and $p = \{1, \dots, 8\}$ obeys the continuity theorem of the first kind; i.e., it is a continuous function for every values of τ .

Since, ACF $R_{VBOC1(m=pn,n,\alpha)}(\tau)$ was already shown in [2] to obey the continuity theorem of the first kind $p = \{1, \dots, 4\}$ it only remains to extend and prove that the ACF $R_{VBOC1(m=pn,n,\alpha)}(\tau)$ obeys the continuity theorem of the first kind $p = \{5, \dots, 8\}$.

Because the ACF $R_{VBOC1(pn,n,\alpha)}(\tau)$ is an even function of τ it is sufficient to check the continuity of $R_{VBOC1(pn=m,n,\alpha)}(\tau)$ for values of τ given by

$$\tau_{\mp i}(\alpha) = \left\lfloor \frac{T_c(1+2j-\alpha)}{2p} \right\rfloor_{\mp} \equiv \tau_{\mp[1+3j(\alpha)]} \quad \frac{T_c(1+2j+\alpha)}{2p} \Big|_{\mp} \equiv \tau_{\mp[2+3j(\alpha)]} \quad \frac{(j+1)T_c}{p} \Big|_{\mp} \equiv \tau_{\mp[3+3j]} \quad (76)$$

Where

$$i = \{1,2,3\} + 3j = \{\{1,2,3\} \dots \{3p-2, 3p-1, 3p\}\}; j = \{0,1, \dots, 7\}, \text{ and } p = j+1 = \{1, \dots, 8\} \quad (77)$$

First, we substitute values of $\tau = \tau_{\mp 1}(\alpha)$ in $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), and we obtain

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{-1}(\alpha)] = 1 - \frac{(4p-1)(1-\alpha)}{2p} = \frac{1-2p+(4p-1)\alpha}{2p} = \left\{ \frac{-9+19\alpha}{p=5}, \frac{-11+23\alpha}{p=6}, \frac{-13+17\alpha}{p=7}, \frac{-15+31\alpha}{p=8} \right\} \quad (78)$$

On the other hand, we have

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{+1}(\alpha)] = \frac{-(p-1)+(2p-1)\alpha}{p} - \frac{1-\alpha}{2p} = \frac{1-2p+(4p-1)\alpha}{2p} = R_{VBOC1(pn,n,\alpha)}[\tau_{-1}(\alpha)] = \left\{ \frac{-9+19\alpha}{p=5}, \frac{-11+23\alpha}{p=6}, \frac{-13+17\alpha}{p=7}, \frac{-15+31\alpha}{p=8} \right\} \quad (79)$$

Second, we substitute values of $\tau = \tau_{\mp 2}(\alpha)$ in $R_{VBOC1(pn=n,n,\alpha)}(\tau)$, given by (17) we obtain

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{-2}(\alpha)] = \frac{-(p-1)+(2p-1)\alpha}{p} - \frac{1+\alpha}{2p} = \frac{1-2p+(4p-3)\alpha}{2p} = \left\{ \frac{-9+17\alpha}{p=5}, \frac{-11+21\alpha}{p=6}, \frac{-13+25\alpha}{p=7}, \frac{-15+29\alpha}{p=8} \right\} \quad (80)$$

On the other hand, we have

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{+2}(\alpha)] = -\frac{3p-2}{p} + \frac{(4p-3)(1+\alpha)}{2p} = \frac{1-2p+(4p-3)\alpha}{2p} = \left\{ \frac{-9+17\alpha}{p=5}, \frac{-11+21\alpha}{p=6}, \frac{-13+25\alpha}{p=7}, \frac{-15+29\alpha}{p=8} \right\} = R_{VBOC1(pn,n,\alpha)}[\tau_{-2}(\alpha)] \quad (81)$$

Third, we substitute values of $\tau = \tau_{\mp 3}$ in $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), and we get

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{-3}(\alpha)] = \frac{2-3p}{p} + \frac{4p-3}{p} = \frac{p-1}{p} = \left\{ \frac{4}{p=5}, \frac{5}{p=6}, \frac{6}{p=7}, \frac{7}{p=8} \right\} \quad (82)$$

On the other hand, we have

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{+3}(\alpha)] = \frac{5p-6-4p+5}{p} = \frac{p-1}{p} = \left\{ \frac{4}{p=5}, \frac{5}{p=6}, \frac{6}{p=7}, \frac{7}{p=8} \right\} = R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{-3}(\alpha)] \quad (83)$$

Fourth, we substitute values of $\tau = \tau_{\mp 4}(\alpha)$ in $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), and we obtain the following

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{-4}(\alpha)] = \frac{5p-6}{p} - \frac{(4p-5)(3-\alpha)}{2p} = \frac{3-2p+(4p-5)\alpha}{2p} = \left\{ \frac{-7+15\alpha}{p=5}, \frac{-9+19\alpha}{p=6}, \frac{-11+23\alpha}{p=7}, \frac{-13+27\alpha}{p=8} \right\} \quad (84)$$

On the other hand, we have

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{+4}(\alpha)] = \frac{-(p-3)+(2p-3)\alpha}{p} - \frac{3-\alpha}{2p} = \frac{3-2p+(4p-5)\alpha}{2p} = \left\{ \frac{-7+15\alpha}{p=5}, \frac{-9+19\alpha}{p=6}, \frac{-11+23\alpha}{p=7}, \frac{-13+27\alpha}{p=8} \right\} = R_{VBOC1(pn,n,\alpha)}[\tau_{-4}(\alpha)] \quad (85)$$

Fifth, we substitute values of $\tau = \tau_{\mp 5}(\alpha)$ in $R_{VBOC1(np=2n,n,\alpha)}(\tau)$, given by (17), and we get

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{-5}(\alpha)] = \frac{-(p-3)+(2p-3)\alpha}{p} - \frac{3+\alpha}{2p} = \frac{3-2p+(4p-7)\alpha}{2p} = \left\{ \frac{-7+13\alpha}{p=5}, \frac{-9+17\alpha}{p=6}, \frac{-11+21\alpha}{p=7}, \frac{-13+25\alpha}{p=8} \right\} \quad (86)$$

On the other hand, we have

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{+5}(\alpha)] = -\frac{7p-12}{p} + \frac{(4p-7)(3+\alpha)}{2p} = \frac{3-2p+(4p-7)\alpha}{2p} = \left\{ \frac{-7+13\alpha}{p=5}, \frac{-9+17\alpha}{p=6}, \frac{-11+21\alpha}{p=7}, \frac{-13+25\alpha}{p=8} \right\} = R_{VBOC1(pn,n,\alpha)}[\tau_{-5}(\alpha)] \quad (87)$$

Sixth, we substitute values of $\tau = \tau_{\mp 6}$ in $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), yields

$$R_{VBOC1(pn,n,\alpha)}(\tau = \tau_{-6}) = -\frac{7p-12}{p} + \frac{2(4p-7)}{p} = \frac{p-2}{p} = \left\{ \frac{3}{p=5}, \frac{4}{p=6}, \frac{5}{p=7}, \frac{6}{p=8} \right\} \quad (88)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}(\tau = \tau_{+6}) = \frac{9p-20-2(4p-9)}{p} = \frac{p-2}{p} = \left\{ \frac{3}{p=5}, \frac{4}{p=6}, \frac{5}{p=7}, \frac{6}{p=8} \right\} = R_{VBOC1(p,n,\alpha)}(\tau_{-6}) \quad (89)$$

Seventh, we substitute values of $\tau = \tau_{+7}(\alpha)$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), we obtain

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+7}(\alpha)] = \frac{9p-20}{p} - \frac{(4p-9)(5-\alpha)}{2p} = \frac{5-2p+(4p-9)\alpha}{2p} = \left\{ \frac{-5+11\alpha}{p=5}, \frac{-7+15\alpha}{p=6}, \frac{-9+19\alpha}{p=7}, \frac{-11+23\alpha}{p=8} \right\} \quad (90)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+7}(\alpha)] = \frac{-(p-5)+(2p-5)\alpha}{p} - \frac{(5-\alpha)}{2p} = \frac{5-2p+(4p-9)\alpha}{2p} = R_{VBOC1(p,n,\alpha)}[\tau_{-7}(\alpha)] = \left\{ \frac{-5+11\alpha}{p=5}, \frac{-7+15\alpha}{p=6}, \frac{-9+19\alpha}{p=7}, \frac{-11+23\alpha}{p=8} \right\} \quad (91)$$

Eighth, we substitute values of $\tau = \tau_{+8}(\alpha)$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), and we get

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+8}(\alpha)] = \frac{-(p-5)+(2p-5)\alpha}{p} - \frac{5+\alpha}{2p} = \frac{5-2p+(4p-11)\alpha}{2p} = \left\{ \frac{-5+9\alpha}{p=5}, \frac{-7+13\alpha}{p=6}, \frac{-9+17\alpha}{p=7}, \frac{-11+21\alpha}{p=8} \right\} \quad (92)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+8}(\alpha)] = -1 + \frac{5+\alpha}{2p} = \frac{-2p+5+\alpha}{2p} = R_{VBOC1(p,n,\alpha)}[\tau_{-8}(\alpha)] = \left\{ \frac{-5+9\alpha}{p=5}, \frac{-7+13\alpha}{p=6}, \frac{-9+17\alpha}{p=7}, \frac{-11+21\alpha}{p=8} \right\} \quad (93)$$

Ninth, we substitute values of $\tau = \tau_{+9}$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), we get

$$R_{VBOC1(p,n,\alpha)}(\tau = \tau_{+9}) = \frac{13p-42}{p} - \frac{3(4p-13)}{p} = \frac{p-3}{p} = \left\{ \frac{2}{p=5}, \frac{3}{p=6}, \frac{4}{p=7}, \frac{5}{p=8} \right\} \quad (94)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}(\tau = \tau_{+9}) = \frac{13p-42}{p} - \frac{3(4p-13)}{p} = \frac{p-3}{p} = R_{VBOC1(p,n,\alpha)}(\tau_{-9}) = \left\{ \frac{2}{p=5}, \frac{3}{p=6}, \frac{4}{p=7}, \frac{5}{p=8} \right\} \quad (95)$$

Tenth, we substitute values of $\tau = \tau_{+10}(\alpha)$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), we get

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+10}(\alpha)] = \frac{13p-42}{p} - \frac{(7-\alpha)(4p-13)}{2p} = \frac{-2p+7+(4p-13)\alpha}{2p} = \left\{ \frac{-3+7\alpha}{p=5}, \frac{-5+11\alpha}{p=6}, \frac{-7+15\alpha}{p=7}, \frac{-9+19\alpha}{p=8} \right\} \quad (96)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+10}(\alpha)] = \frac{-(p-7)+(2p-7)\alpha}{p} - \frac{7-\alpha}{2p} = \frac{7-2p+(4p-13)\alpha}{2p} = R_{VBOC1(p,n,\alpha)}[\tau_{-10}(\alpha)] = \left\{ \frac{-3+7\alpha}{p=5}, \frac{-5+11\alpha}{p=6}, \frac{-7+15\alpha}{p=7}, \frac{-9+19\alpha}{p=8} \right\} \quad (97)$$

Eleventh, we substitute values of $\tau = \tau_{+11}(\alpha)$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), we get

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+11}(\alpha)] = \frac{-(p-7)+(2p-7)\alpha}{p} - \frac{7+\alpha}{2p} = \frac{7-2p+(4p-15)\alpha}{2p} = \left\{ \frac{-3+5\alpha}{p=5}, \frac{-5+9\alpha}{p=6}, \frac{-7+13\alpha}{p=7}, \frac{-9+17\alpha}{p=8} \right\} \quad (98)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+11}(\alpha)] = -\frac{(15p-56)}{p} + \frac{(4p-15)(7+\alpha)}{2p} = \frac{7-2p+(4p-15)\alpha}{2p} = R_{VBOC1(p,n,\alpha)}[\tau_{-11}(\alpha)] = \left\{ \frac{-3+5\alpha}{\underbrace{10}_{p=5}}, \frac{-5+9\alpha}{\underbrace{12}_{p=6}}, \frac{-7+13\alpha}{\underbrace{14}_{p=7}}, \frac{-9+17\alpha}{\underbrace{16}_{p=8}} \right\} \quad (99)$$

Twelfth, we substitute values of $\tau = \tau_{\mp 12}$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), we get

$$R_{VBOC1(p,n,\alpha)}(\tau = \tau_{-12}) = -\frac{(15p-56)}{p} + \frac{4(4p-15)}{p} = \frac{p-4}{p} = \left\{ \frac{1}{\underbrace{5}_{p=5}}, \frac{2}{\underbrace{6}_{p=6}}, \frac{3}{\underbrace{7}_{p=7}}, \frac{4}{\underbrace{8}_{p=8}} \right\} \quad (100)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}(\tau = \tau_{+12}) = \frac{17p-72}{p} - \frac{4(4p-17)}{p} = \frac{p-4}{p} = R_{VBOC1(p,n,\alpha)}(\tau_{-12}) = \left\{ \frac{1}{\underbrace{5}_{p=5}}, \frac{2}{\underbrace{6}_{p=6}}, \frac{3}{\underbrace{7}_{p=7}}, \frac{4}{\underbrace{8}_{p=8}} \right\} \quad (101)$$

Equations (76) through (101) complete the proof of *theorem 4* for the first twelve values of $\tau_{+1,2,\dots,12}(\alpha)$ and $p = \{5,6,7,8\}$.

The next content is entirely new because now we will test the continuity theorem for next twelve values of $\tau_{+13,14,\dots,24}(\alpha)$ and

$p = \{5,6,7,8\}$. Thirteen, we substitute values of $\tau = \tau_{\mp 13}(\alpha)$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), and we obtain

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{-13}(\alpha)] = \frac{17p-72}{p} - \frac{(4p-17)(9-\alpha)}{2p} = \frac{9-2p+(4p-17)\alpha}{2p} = \left\{ \frac{-1+3\alpha}{\underbrace{10}_{p=5}}, \frac{-3+7\alpha}{\underbrace{12}_{p=6}}, \frac{-5+11\alpha}{\underbrace{14}_{p=7}}, \frac{-7+15\alpha}{\underbrace{16}_{p=8}} \right\} \quad (102)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+13}(\alpha)] = \frac{-(p-9)+(2p-9)\alpha}{p} - \frac{9-\alpha}{2p} = \frac{9-2p+(4p-17)\alpha}{2p} = R_{VBOC1(p,n,\alpha)}[\tau_{-13}(\alpha)] = \left\{ \frac{-1+3\alpha}{\underbrace{10}_{p=5}}, \frac{-3+7\alpha}{\underbrace{12}_{p=6}}, \frac{-5+11\alpha}{\underbrace{14}_{p=7}}, \frac{-7+15\alpha}{\underbrace{16}_{p=8}} \right\} \quad (103)$$

Fourteenth, we substitute values of $\tau = \tau_{\mp 14}(\alpha)$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17) we obtain

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{-14}(\alpha)] = \frac{-(p-9)+(2p-9)\alpha}{p} - \frac{9+\alpha}{2p} = \frac{9-2p+(4p-19)\alpha}{2p} = \left\{ \frac{-1+\alpha}{\underbrace{10}_{p=5}}, \frac{-3+5\alpha}{\underbrace{12}_{p=6}}, \frac{-5+9\alpha}{\underbrace{14}_{p=7}}, \frac{-7+13\alpha}{\underbrace{16}_{p=8}} \right\} \quad (104)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+14}(\alpha)] = -\frac{19p-90}{p} + \frac{(4p-19)(9+\alpha)}{2p} = \frac{9-2p+(4p-19)\alpha}{2p} = \left\{ \frac{-1+\alpha}{\underbrace{10}_{p=5}}, \frac{-3+5\alpha}{\underbrace{12}_{p=6}}, \frac{-5+9\alpha}{\underbrace{14}_{p=7}}, \frac{-7+13\alpha}{\underbrace{16}_{p=8}} \right\} = R_{VBOC1(p,n,\alpha)}[\tau_{-14}(\alpha)] \quad (105)$$

Fifteenth, we substitute values of $\tau = \tau_{\mp 15}$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), and we get

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{-15}] = -\frac{19p-90}{p} + \frac{5(4p-19)}{p} = \frac{p-5}{p} = \left\{ \frac{0}{\underbrace{5}_{p=5}}, \frac{1}{\underbrace{6}_{p=6}}, \frac{2}{\underbrace{7}_{p=7}}, \frac{3}{\underbrace{8}_{p=8}} \right\} \quad (106)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+15}] = \frac{21p-110}{p} - \frac{5(4p-21)}{p} = \frac{p-5}{p} = \left\{ \frac{0}{\underbrace{5}_{p=5}}, \frac{1}{\underbrace{6}_{p=6}}, \frac{2}{\underbrace{7}_{p=7}}, \frac{3}{\underbrace{8}_{p=8}} \right\} = R_{VBOC1(p,n,\alpha)}[\tau = \tau_{-15}] \quad (107)$$

Sixteenth, we substitute values of $\tau = \tau_{\mp 16}(\alpha)$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), and we obtain the following

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{-16}(\alpha)] = \frac{21p-110}{p} - \frac{(4p-21)(11-\alpha)}{2p} = \frac{11-2p+(4p-21)\alpha}{2p} = \left\{ \frac{-1+3\alpha}{\underbrace{12}_{p=6}}, \frac{-3+7\alpha}{\underbrace{14}_{p=7}}, \frac{-5+11\alpha}{\underbrace{16}_{p=8}} \right\} \quad (108)$$

On the other hand we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+16}(\alpha)] = \frac{-(p-11)+(2p-11)\alpha}{p} - \frac{11-\alpha}{2p} = \frac{11-2p+(4p-21)\alpha}{2p} = \left\{ \frac{-1+3\alpha}{\frac{12}{p=6}}, \frac{-3+7\alpha}{\frac{14}{p=7}}, \frac{-5+11\alpha}{\frac{16}{p=8}} \right\} = R_{VBOC1(p,n,\alpha)}[\tau_{-16}(\alpha)] \quad (109)$$

Seventeenth, we substitute values of $\tau = \tau_{+17}(\alpha)$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), and we get

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{-17}(\alpha)] = \frac{-(p-11)+(2p-11)\alpha}{p} - \frac{11+\alpha}{2p} = \frac{11-2p+(4p-23)\alpha}{2p} = \left\{ \frac{-1+\alpha}{\frac{12}{p=6}}, \frac{-3+5\alpha}{\frac{14}{p=7}}, \frac{-5+9\alpha}{\frac{16}{p=8}} \right\} \quad (110)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+17}(\alpha)] = -\frac{23p-132}{p} + \frac{(4p-23)(11+\alpha)}{2p} = \frac{11-2p+(4p-23)\alpha}{2p} = \left\{ \frac{-1+\alpha}{\frac{12}{p=6}}, \frac{-3+5\alpha}{\frac{14}{p=7}}, \frac{-5+9\alpha}{\frac{16}{p=8}} \right\} = R_{VBOC1(p,n,\alpha)}[\tau_{-17}(\alpha)] \quad (111)$$

Eighteenth, we substitute values of $\tau = \tau_{+18}$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), yields

$$R_{VBOC1(p,n,\alpha)}(\tau = \tau_{-18}) = -\frac{23p-132}{p} + \frac{(4p-23)6}{p} = \frac{p-6}{p} = \left\{ \frac{0}{\frac{12}{p=6}}, \frac{\frac{1}{7}}{\frac{14}{p=7}}, \frac{\frac{2}{8}}{\frac{16}{p=8}} \right\} \quad (112)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}(\tau = \tau_{+18}) = \frac{25p-156}{p} - \frac{6(4p-25)}{p} = \frac{p-6}{p} = \left\{ \frac{0}{\frac{12}{p=6}}, \frac{\frac{1}{7}}{\frac{14}{p=7}}, \frac{\frac{2}{8}}{\frac{16}{p=8}} \right\} = R_{VBOC1(p,n,\alpha)}(\tau_{-18}) \quad (113)$$

Nineteenth, we substitute values of $\tau = \tau_{+19}(\alpha)$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), we obtain

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{-19}(\alpha)] = \frac{25p-156}{p} - \frac{(4p-25)(13-\alpha)}{2p} = \frac{13-2p+(4p-25)\alpha}{2p} = \left\{ \frac{-1+3\alpha}{\frac{14}{p=7}}, \frac{-3+7\alpha}{\frac{16}{p=8}} \right\} \quad (114)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+19}(\alpha)] = \frac{-(p-13)+(2p-13)\alpha}{p} - \frac{13-\alpha}{2p} = \frac{13-2p+(4p-25)\alpha}{2p} = R_{VBOC1(p,n,\alpha)}[\tau_{-19}(\alpha)] = \left\{ \frac{-1+3\alpha}{\frac{14}{p=7}}, \frac{-3+7\alpha}{\frac{16}{p=8}} \right\} \quad (115)$$

Twentieth, we substitute values of $\tau = \tau_{+20}(\alpha)$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), and we get

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{-20}(\alpha)] = \frac{-(p-13)+(2p-13)\alpha}{p} - \frac{13+\alpha}{2p} = \frac{13-2p+(4p-27)\alpha}{2p} = \left\{ \frac{-1+\alpha}{\frac{14}{p=7}}, \frac{-3+5\alpha}{\frac{16}{p=8}} \right\} \quad (116)$$

On the other hand, we have

$$R_{VBOC1(p,n,\alpha)}[\tau = \tau_{+20}(\alpha)] = -\frac{27p-182}{p} + \frac{(4p-27)(13+\alpha)}{2p} = \frac{13-2p+(4p-27)\alpha}{2p} = R_{VBOC1(p,n,\alpha)}[\tau_{-20}(\alpha)] = \left\{ \frac{-1+\alpha}{\frac{14}{p=7}}, \frac{-3+5\alpha}{\frac{16}{p=8}} \right\} \quad (117)$$

Twenty-first, we substitute values of $\tau = \tau_{+21}$ in $R_{VBOC1(p,n,\alpha)}(\tau)$, given by (17), we get

$$R_{VBOC1(p,n,\alpha)}(\tau = \tau_{-21}) = -\frac{27p-182}{p} + \frac{7(4p-27)}{p} = \frac{p-7}{p} = \left\{ \frac{0}{\frac{14}{p=7}}, \frac{\frac{1}{8}}{\frac{16}{p=8}} \right\} \quad (118)$$

On the other hand we have

$$R_{VBOC1(pn,n,\alpha)}(\tau = \tau_{+21}) = \frac{29p-210}{p} - \frac{7(4p-29)}{p} = \frac{p-7}{p} = R_{VBOC1(pn,n,\alpha)}(\tau_{-9}) = \left\{ \frac{0}{\underbrace{7}_{p=7}}, \frac{1}{\underbrace{8}_{p=8}} \right\} \quad (119)$$

Twenty-second, we substitute values of $\tau = \tau_{\mp 22}(\alpha)$ in $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), we get

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{-22}(\alpha)] = \frac{29p-210}{p} - \frac{(4p-29)(15-\alpha)}{2p} = \frac{15-2p+(4p-29)\alpha}{2p} = \left\{ \frac{-1+3\alpha}{\underbrace{16}_{p=8}} \right\} \quad (120)$$

On the other hand, we have

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{+22}(\alpha)] = \frac{-(p-15)+(2p-15)\alpha}{p} - \frac{15-\alpha}{2p} = \frac{15-2p+(4p-29)\alpha}{2p} = R_{VBOC1(pn,n,\alpha)}[\tau_{-22}(\alpha)] = \left\{ \frac{-1+3\alpha}{\underbrace{16}_{p=8}} \right\} \quad (121)$$

Twenty-third, we substitute values of $\tau = \tau_{\mp 23}(\alpha)$ in $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), we get

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{-23}(\alpha)] = \frac{-(p-15)+(2p-15)\alpha}{p} - \frac{15+\alpha}{2p} = \frac{15-2p+(4p-31)\alpha}{2p} = \left\{ \frac{-1+\alpha}{\underbrace{16}_{p=8}} \right\} \quad (122)$$

On the other hand, we have

$$R_{VBOC1(pn,n,\alpha)}[\tau = \tau_{+23}(\alpha)] = -\frac{31p-240}{p} + \frac{(4p-31)(15+\alpha)}{2p} = \frac{15-2p+(4p-31)\alpha}{2p} = R_{VBOC1(pn,n,\alpha)}[\tau_{-23}(\alpha)] = \left\{ \frac{-1+\alpha}{\underbrace{16}_{p=8}} \right\} \quad (123)$$

Twenty-fourth and finally, we substitute values of $\tau = \tau_{\mp 24}$ in $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), we get

$$R_{VBOC1(pn,n,\alpha)}(\tau = \tau_{-24}) = -\frac{31p-240}{p} + \frac{8(4p-31)}{p} = \frac{p-8}{p} = \left\{ \frac{0}{\underbrace{8}_{p=8}} \right\} \quad (124)$$

On the other hand, we have

$$R_{VBOC1(pn,n,\alpha)}(\tau = \tau_{+24}) = 0 = R_{VBOC1(pn,n,\alpha)}(\tau_{-24}) = \left\{ \frac{0}{\underbrace{8}_{p=8}} \right\} \quad (125)$$

Equations (102) through (125) complete the proof of *theorem 4* for the second twelve values of $\tau_{+13,14,\dots,24}(\alpha)$ and $p = \{5,6,7,8\}$.

Equations (76) through (126) complete the proof of *theorem 4* for all twenty-four values of $\tau_{+1,2,\dots,24}(\alpha)$ and $p = \{5,6,7,8\}$.

Theorem 5: Prove that the ACF $R_{VBOC1(m=pn,n,\alpha)}(\tau)$, given by (15), obeys the continuity theorem of the second kind; i.e., it is a continuous function for every value of τ and $p = \{5,6,7,8\}$ and $\alpha = \{0,1\}$.

Proof of theorem 5: The proof of theorem 5 is straightforward. In order to prove that the ACF, $R_{VBOC1(m=pn,n,\alpha)}(\tau)$, given by (17), obeys the continuity theorem of the second kind; i.e., it is

a continuous function for every values of τ , $p = \{5,6,7,8\}$ and $\alpha = \{0,1\}$ it suffices to prove $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), obeys the continuity theorem of the second kind; i.e., it is a continuous function for every values of τ .

Since, we already proved in *theorem 5*; i.e., continuity theorem of the first kind that $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), for or every value of τ , $p = \{5,6,7,8\}$ and for $\alpha = [0,1]$ we can employ the results of *theorem 4* to prove *theorem 5*.

First, we prove that $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), obeys the continuity theorem of the second kind for $\{\tau_1(\alpha), \tau_2(\alpha)\}$.

From (78) and (79), $\alpha = 0$, and $p = \{5,6,7,8\}$ we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_1(0)] = \frac{1-2p+(4p-1)\alpha}{2p} = -1 + \frac{1}{2p} = \left\{ \frac{-9}{10}, \frac{-11}{12}, \frac{-13}{14}, \frac{-15}{16} \right\} \quad (126)$$

On the other hand, from (80) and (81), $\alpha = 0$, and $p = \{5,6,7,8\}$ we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_2(0)] = \frac{1-2p+(4p-3)\alpha}{2p} = -1 + \frac{1}{2p} = \left\{\frac{-9}{10}, \frac{-11}{12}, \frac{-13}{14}, \frac{-15}{16}\right\} = R_{VBOC1(pn,n,\alpha=0)}[\tau_1(0)] \quad (127)$$

Next, from (78) and (79), $\alpha = 1$, and $p = \{5,6,7,8\}$ we have

Equation (128) results from *definition 2* (Progri 2015) [1].

Finally, from (80)-(83), $\alpha = 1$, for $p = \{5,6,7,8\}$ we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_1(1)] = \frac{1-2p+4p-1}{2p} = 1 = R_{BPSK_0}[0] \quad (128)$$

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_2(1)] = \frac{1-2p+4p-3}{2p} = 1 - \frac{1}{p} = \left\{\frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}\right\} = R_{VBOC1(pn,n,\alpha=1)}[\tau_3(1)] \quad (129)$$

Second, we prove that $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), $\{\tau_4(\alpha), \tau_5(\alpha)\}$.

obeys the *continuity theorem of the second kind* for From (84) and (85), $\alpha = 0$, and for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_4(0)] = \frac{3-2p+(4p-5)\alpha}{2p} = -1 + \frac{3}{2p} = \left\{\frac{-7}{10}, \frac{-9}{12}, \frac{-11}{14}, \frac{-13}{16}\right\} \quad (130)$$

On the other hand, from (84)-(87), $\alpha = 0$, and for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_5(0)] = \frac{3-2p+(4p-5)\alpha}{2p} = -1 + \frac{3}{2p} = \left\{\frac{-7}{10}, \frac{-9}{12}, \frac{-11}{14}, \frac{-13}{16}\right\} = R_{VBOC1(pn,n,\alpha=0)}[\tau_4(0)] \quad (131)$$

Next, from (82)-(85), $\alpha = 1$, for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_4(1)] = \frac{3-2p+4p-5}{2p} = 1 - \frac{1}{p} = \left\{\frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}\right\} = R_{VBOC1(pn,n,\alpha=1)}[\tau_3(1)] \quad (132)$$

On the other hand, from (86)-(89), $\alpha = 1$, and for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_5(1)] = \frac{3-2p+4p-7}{2p} = 1 - \frac{2}{p} = \left\{\frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}\right\} = R_{VBOC1(pn,n,\alpha=1)}[\tau_6(1)] \quad (133)$$

Third, we prove that $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), obeys the *continuity theorem of the second kind* for $\{\tau_7(\alpha), \tau_8(\alpha)\}$.

From (90) and (91), $\alpha = 0$, and for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_7(0)] = \frac{5-2p+(4p-9)\alpha}{2p} = -1 + \frac{5}{2p} = \left\{\frac{-5}{10}, \frac{-7}{12}, \frac{-9}{14}, \frac{-11}{16}\right\} \quad (134)$$

On the other hand, from (90)-(93), $\alpha = 0$, and for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_8(0)] = \frac{5-2p+(4p-11)\alpha}{2p} = -1 + \frac{5}{2p} = \left\{\frac{-5}{10}, \frac{-7}{12}, \frac{-9}{14}, \frac{-11}{16}\right\} = R_{VBOC1(pn,n,\alpha=0)}[\tau_7(0)] \quad (135)$$

Next, from (88)-(91), $\alpha = 1$, and for $p = \{5,6,7,8\}$ we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_7(1)] = \frac{5-2p+(4p-9)\alpha}{2p} = 1 - \frac{2}{p} = \left\{\frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}\right\} = R_{VBOC1(pn,n,\alpha)}[\tau_6(1)] \quad (136)$$

On the other hand, from (92)-(95), $\alpha = 1$, for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_8(1)] = \frac{5-2p+(4p-11)\alpha}{2p} = 1 - \frac{3}{p} = \left\{\frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}\right\} = R_{VBOC1(pn,n,\alpha)}[\tau_9(1)] \quad (137)$$

Fourth, we prove that $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), obeys the *continuity theorem of the second kind* for $\{\tau_{10}(\alpha), \tau_{11}(\alpha)\}$.

From (96), (97), $\alpha = 0$, and for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_{10}(0)] = \frac{7-2p+(4p-13)\alpha}{2p} = -1 + \frac{7}{2p} = \left\{\frac{-3}{10}, \frac{-5}{12}, \frac{-7}{14}, \frac{-9}{16}\right\} \quad (138)$$

On the other hand, from and (96)-(99), $\alpha = 0$, for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_{11}(0)] = \frac{7-2p+(4p-15)\alpha}{2p} = -1 + \frac{7}{2p} = \left\{\frac{-3}{10}, \frac{-5}{12}, \frac{-7}{14}, \frac{-9}{16}\right\} = R_{VBOC1(pn,n,\alpha=0)}[\tau_{10}(0)] \quad (139)$$

Next, from (94)-(97), $\alpha = 1$, for $p = \{5,6,7,8\}$ we have,

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_{10}(1)] = \frac{7-2p+(4p-13)\alpha}{2p} = 1 - \frac{3}{p} = \left\{\frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}\right\} = R_{VBOC1(pn,n,\alpha)}[\tau_9(1)] \quad (140)$$

On the other hand, from (98)-(101), $\alpha = 1$, and for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_{11}(1)] = \frac{7-2p+(4p-15)\alpha}{2p} = 1 - \frac{4}{p} = \left\{\frac{1}{5}, \frac{2}{6}, \frac{3}{7}, \frac{4}{8}\right\} = R_{VBOC1(pn,n,\alpha)}[\tau_{12}(1)] \quad (141)$$

Equations (126) through (141) complete the proof of *theorem 2* or *continuity theorem of the second kind* for the first twelve values of $\tau_{1,2,\dots,12}(\alpha = \{0,1\})$ and $p = \{5,6,7,8\}$.

Fifth, we prove that $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), obeys the *continuity theorem of the second kind* for $\{\tau_{13}(\alpha), \tau_{14}(\alpha)\}$.

From (102), (103), $\alpha = 0$, and $p = \{5,6,7,8\}$ we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_{13}(0)] = \frac{9-2p+(4p-17)\alpha}{2p} = -1 + \frac{9}{2p} = \left\{\frac{-1}{10}, \frac{-3}{12}, \frac{-5}{14}, \frac{-7}{16}\right\} \quad (142)$$

On the other hand, from (104), (105), $\alpha = 0$, and $p = \{5,6,7,8\}$ we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_{14}(0)] = \frac{9-2p+(4p-19)\alpha}{2p} = -1 + \frac{9}{2p} = \left\{\frac{-1}{10}, \frac{-3}{12}, \frac{-5}{14}, \frac{-7}{16}\right\} = R_{VBOC1(pn,n,\alpha=0)}[\tau_{13}(0)] \quad (143)$$

Next, from (102), (103), $\alpha = 1$, and $p = \{5,6,7,8\}$ we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_{13}(1)] = \frac{9-2p+4p-17}{2p} = 1 - \frac{4}{p} = \left\{\frac{1}{5}, \frac{2}{6}, \frac{3}{7}, \frac{8}{8}\right\} \quad (144)$$

Finally, from (104)-(107), $\alpha = 1$, and for $p = \{5,6,7,8\}$ we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_{14}(1)] = \frac{9-2p+(4p-19)\alpha}{2p} = 1 - \frac{5}{p} = \left\{0, \frac{1}{6}, \frac{2}{7}, \frac{3}{8}\right\} = R_{VBOC1(pn,n,\alpha=1)}[\tau_{15}(1)] \quad (145)$$

Sixth, we prove that $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), obeys the *continuity theorem of the second kind* for $\{\tau_{16}(\alpha), \tau_{17}(\alpha)\}$.

From (108), (109), $\alpha = 0$, and for $p = \{6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_{16}(0)] = \frac{11-2p+(4p-21)\alpha}{2p} = -1 + \frac{11}{2p} = \left\{\frac{-1}{12}, \frac{-3}{14}, \frac{-5}{16}\right\} \quad (146)$$

On the other hand, from (108)-(111), for $p = \{6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_{17}(0)] = \frac{11-2p+(4p-23)\alpha}{2p} = -1 + \frac{11}{2p} = \left\{\frac{-1}{12}, \frac{-3}{14}, \frac{-5}{16}\right\} = R_{VBOC1(pn,n,\alpha=0)}[\tau_{16}(0)] \quad (147)$$

Next, from (106)-(109), $\alpha = 1$, and for $p = \{5,6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_{16}(1)] = \frac{11-2p+4p-21}{2p} = 1 - \frac{5}{p} = \left\{0, \frac{1}{6}, \frac{2}{7}, \frac{3}{8}\right\} = R_{VBOC1(pn,n,\alpha=1)}[\tau_{15}(1)] \quad (148)$$

On the other hand, from (110)-(113) $\alpha = 1$, for $p = \{6,7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_{17}(1)] = \frac{11-2p+4p-23}{2p} = 1 - \frac{6}{p} = \left\{0, \frac{1}{7}, \frac{2}{8}\right\} = R_{VBOC1(pn,n,\alpha=1)}[\tau_{18}(1)] \quad (149)$$

Seventh, we prove that $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), obeys the *continuity theorem of the second kind* for $\{\tau_{19}(\alpha), \tau_{20}(\alpha)\}$.

From (114), (115), $\alpha = 0$, and for $p = \{7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_{19}(0)] = \frac{13-2p+(4p-25)\alpha}{2p} = -1 + \frac{13}{2p} = \left\{\frac{-1}{14}, \frac{-3}{16}\right\} \quad (150)$$

On the other hand, from (114)-(117), $\alpha = 0$, and for $p = \{7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_{20}(0)] = \frac{13-2p+(4p-27)\alpha}{2p} = -1 + \frac{13}{2p} = \left\{\frac{-1}{14}, \frac{-3}{16}\right\} = R_{VBOC1(pn,n,\alpha=0)}[\tau_{19}(0)] \quad (151)$$

Next, from (112)-(115), $\alpha = 1$, and for $p = \{6,7,8\}$ we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_{19}(1)] = \frac{13-2p+(4p-25)\alpha}{2p} = 1 - \frac{6}{p} = \left\{0, \frac{1}{7}, \frac{2}{8}\right\} = R_{VBOC1(pn,n,\alpha)}[\tau_{18}(1)] \quad (152)$$

On the other hand, from (116)-(119), $\alpha = 1$, and for $p = \{7,8\}$, we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_{20}(1)] = \frac{13-2p+(4p-27)\alpha}{2p} = 1 - \frac{7}{p} = \left\{0, \frac{1}{8}\right\} = R_{VBOC1(pn,n,\alpha)}[\tau_{21}(1)] \quad (153)$$

Eighth, we prove that $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), obeys the continuity theorem of the second kind for $\{\tau_{22}(\alpha), \tau_{23}(\alpha)\}$.

From (120), (121), $\alpha = 0$, and for $p = 8$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_{22}(0)] = \frac{15-2p+(4p-29)\alpha}{2p} = -1 + \frac{15}{2p} = \left\{\frac{-1}{16}\right\} \quad (154)$$

On the other hand, from and (120)-(123), $\alpha = 0$, for $p = 8$, we have

$$R_{VBOC1(pn,n,\alpha=0)}[\tau_{23}(0)] = \frac{15-2p+(4p-31)\alpha}{2p} = -1 + \frac{15}{2p} = \left\{\frac{-1}{16}\right\} = R_{VBOC1(pn,n,\alpha=0)}[\tau_{22}(0)] \quad (155)$$

Next, from (118)-(121), $\alpha = 1$, and for $p = 8$ we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_{22}(1)] = \frac{15-2p+(4p-29)\alpha}{2p} = 1 - \frac{7}{p} = \left\{\frac{1}{8}\right\} = R_{VBOC1(pn,n,\alpha)}[\tau_{21}(1)] \quad (156)$$

On the other hand, from (122)-(125), for $p = 8$, we have

$$R_{VBOC1(pn,n,\alpha=1)}[\tau_{23}(1)] = \frac{15-2p+(4p-31)\alpha}{2p} = 0 = \{0\} = R_{VBOC1(pn,n,\alpha)}[\tau_{24}(1)] \quad (157)$$

Equations (142) through (157) complete the proof of *theorem 5* or *continuity theorem of the second kind* for the second twelve values of $\tau_{13,14,\dots,24}(\alpha = \{0,1\})$ and $p = \{5,6,7,8\}$.

Equations (126) through (157) complete the proof of *theorem 5* or *continuity theorem of the second kind* for the second twelve values of $\tau_{1,2,\dots,24}(\alpha = \{0,1\})$ and $p = \{5,6,7,8\}$.

Corollary 1 in [2] holds; i.e., $R_{VBOC1(pn,n,\alpha)}[0] = 1 = \max$ regardless of p or α ; i.e., $R_{VBOC1(pn,n,\alpha)}[\tau_i(\alpha)] \leq 1$ regardless of p or α .

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(pn,n,\alpha)}[\tau^1(\alpha)] & \dots & \\ \dots & \ddots & \vdots \\ \dots & R_{VBOC1(pn,n,\alpha)}[\tau^{2p}(\alpha)] \end{bmatrix} \rightarrow \arg \min_{\alpha} \sum_{i=1}^{2p} R_{VBOC1(pn,n,\alpha)}[\tau^i(\alpha)] \quad (158)$$

where

$$\tau^i(\alpha) = \left[\frac{T_c(1+2j-\alpha)}{2p} \equiv \tau^{[1+3j(\alpha)]} \quad \frac{T_c(1+2j+\alpha)}{2p} \equiv \tau^{[2+3j(\alpha)]} \right] \quad (159)$$

And

$$i = \{1,2\} + 3j = [\{1,2\}, \dots \{2p-1, 2p\}], \quad p = j+1 = \{1, \dots 8\}, \text{ and } j = \{0,1, \dots 7\}. \quad (160)$$

Proof of theorem 6: The proof of theorem 6 is also straightforward. Because the min-max of the ACF of $VBOC1(pn,n,\alpha)(t)$, or $R_{VBOC1(pn,n,\alpha)}(\tau)$, that depends α on for values of $p = \{5,6,7,8\}$ and $\tau^i(\alpha)$ for $i = \{1,2, \dots 2p\}$ given

9.2 VBOC1(α) ACF Optimization Theorems

At this point we are certain that closed form expression of the ACF, $R_{VBOC1(pn,n,\alpha)}(\tau)$, given by (17), is the correct one and we are ready to find the optimum $0 \leq \alpha \leq 1$ that minimizes the out of phase ACF based on certain criteria known as the $VBOC1(\alpha)$ ACF Optimization Theorems.

Theorem 6: Prove that the ACF $R_{VBOC1(m=pn,n,\alpha)}(\tau)$, given by (17), obeys the optimization theorem of the first kind; i.e., find the values of α that minimizes the out of phase ACF,

by (158); hence, it suffices to show that show that (158) holds.

First, for $p = 5$ find the value of α that minimizes $\tau^1(\alpha)$, ..., $\tau^{10}(\alpha)$

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(p,n,\alpha)}[\tau^1(\alpha)] \cdots R_{VBOC1(p,n,\alpha)}[\tau^4(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^5(\alpha)] \cdots R_{VBOC1(p,n,\alpha)}[\tau^8(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^9(\alpha)] \quad R_{VBOC1(p,n,\alpha)}[\tau^{10}(\alpha)] \end{bmatrix} \rightarrow \arg \min_{\alpha} \begin{bmatrix} 1-2p-(1-4p)\alpha & 1-2p-(3-4p)\alpha & 3-2p-(5-4p)\alpha & 3-2p-(7-4p)\alpha \\ 5-2p-(9-4p)\alpha & 5-2p-(11-4p)\alpha & 7-2p-(13-4p)\alpha & 7-2p-(15-4p)\alpha \\ 9-2p-(17-4p)\alpha & 9-2p-(19-4p)\alpha & & \end{bmatrix} \quad (161)$$

Optimization theorem of the first kind says that proving (161) is equivalent with proving the following

$$\arg \min_{\alpha} \sum_{i=1}^{10} R_{VBOC1(p,n,\alpha)}[\tau^i(\alpha)] = \arg \min_{\alpha} \left[\frac{50-20p-(100-40p)\alpha}{2p} \right]; \rightarrow -1 + 2\alpha = 0; 0 \leq \alpha \leq 1 \quad (162)$$

Hence, we find minimum $\alpha = 0.5$ and it is independent of p . Substituting, $\alpha = 0.5$ in (161) we obtain p . This completes the *proof of theorem 6* for $p = 5$.

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c0.5}}{2p}\right] \cdots R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c4.5}}{2p}\right] \\ R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c5.5}}{2p}\right] \cdots R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c9.5}}{2p}\right] \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & & \end{bmatrix} \quad (163)$$

Second, we find the value of α that minimizes $\tau^1(\alpha)$, ..., $\tau^{12}(\alpha)$ for $p = 6$

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(p,n,\alpha)}[\tau^1(\alpha)] \cdots R_{VBOC1(p,n,\alpha)}[\tau^4(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^5(\alpha)] \cdots R_{VBOC1(p,n,\alpha)}[\tau^8(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^9(\alpha)] \cdots R_{VBOC1(p,n,\alpha)}[\tau^{12}(\alpha)] \end{bmatrix} = \arg \min_{\alpha} \begin{bmatrix} 1-2p-(1-4p)\alpha & 1-2p-(3-4p)\alpha & 3-2p-(5-4p)\alpha & 3-2p-(7-4p)\alpha \\ 5-2p-(9-4p)\alpha & 5-2p-(11-4p)\alpha & 7-2p-(13-4p)\alpha & 7-2p-(15-4p)\alpha \\ 9-2p-(17-4p)\alpha & 9-2p-(19-4p)\alpha & 11-2p-(21-4p)\alpha & 11-2p-(23-4p)\alpha \end{bmatrix} \quad (164)$$

Optimization theorem of the first kind says that proving (164) is equivalent with proving the following

$$\arg \min_{\alpha} \sum_{i=1}^{12} R_{VBOC1(p,n,\alpha)}[\tau^i(\alpha)] = \arg \min_{\alpha} \left[\frac{72-24p-(144-48p)\alpha}{2p} \right] \quad (165)$$

Hence, we find minimum $\alpha = 0.5$ and it is independent of p . Substituting, $\alpha = 0.5$ in (158) we obtain p . This completes the *proof of theorem 6* for $p = 6$.

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c0.5}}{2p}\right] \cdots R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c3.5}}{2p}\right] \\ R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c4.5}}{2p}\right] \cdots R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c7.5}}{2p}\right] \\ R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c8.5}}{2p}\right] \cdots R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c11.5}}{2p}\right] \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \end{bmatrix} \quad (166)$$

Third, we find the value of α that minimizes $\tau^1(\alpha)$, ..., $\tau^{14}(\alpha)$ for $p = 7$

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(p,n,\alpha)}[\tau^1(\alpha)] \cdots R_{VBOC1(p,n,\alpha)}[\tau^4(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^5(\alpha)] \cdots R_{VBOC1(p,n,\alpha)}[\tau^8(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^9(\alpha)] \cdots R_{VBOC1(p,n,\alpha)}[\tau^{12}(\alpha)] \\ R_{VBOC1(p,n,\alpha)}[\tau^{13}(\alpha)] \quad R_{VBOC1(p,n,\alpha)}[\tau^{14}(\alpha)] \end{bmatrix} = \arg \min_{\alpha} \begin{bmatrix} 1-2p-(1-4p)\alpha & 1-2p-(3-4p)\alpha & 3-2p-(5-4p)\alpha & 3-2p-(7-4p)\alpha \\ 5-2p-(9-4p)\alpha & 5-2p-(11-4p)\alpha & 7-2p-(13-4p)\alpha & 7-2p-(15-4p)\alpha \\ 9-2p-(17-4p)\alpha & 9-2p-(19-4p)\alpha & 11-2p-(21-4p)\alpha & 11-2p-(23-4p)\alpha \\ 13-2p-(25-4p)\alpha & 13-2p-(27-4p)\alpha & & \end{bmatrix} \quad (167)$$

Optimization theorem of the first kind says that proving (167) is equivalent with proving the following

$$\arg \min_{\alpha} \sum_{i=1}^{14} R_{VBOC1(p,n,\alpha)}[\tau^i(\alpha)] = \arg \min_{\alpha} \left[\frac{98-28p-(196-56p)\alpha}{2p} \right] \quad (168)$$

Hence, we find minimum $\alpha = 0.5$ and it is independent of p . Substituting, $\alpha = 0.5$ in (167) we obtain p . This completes the *proof of theorem 3* for $p = 7$.

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c0.5}}{2p}\right] \cdots R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c3.5}}{2p}\right] \\ R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c4.5}}{2p}\right] \cdots R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c7.5}}{2p}\right] \\ R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c8.5}}{2p}\right] \cdots R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c11.5}}{2p}\right] \\ R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c12.5}}{2p}\right] \quad R_{VBOC1(p,n,\alpha)}\left[\frac{T_{c13.5}}{2p}\right] \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & & \end{bmatrix} \quad (169)$$

Fourth, we find the value of α that minimizes $\tau^1(\alpha), \dots, \tau^{16}(\alpha)$ for $p = 8$

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(pn,n,\alpha)}[\tau^1(\alpha)] \cdots R_{VBOC1(pn,n,\alpha)}[\tau^4(\alpha)] \\ R_{VBOC1(pn,n,\alpha)}[\tau^5(\alpha)] \cdots R_{VBOC1(pn,n,\alpha)}[\tau^8(\alpha)] \\ R_{VBOC1(pn,n,\alpha)}[\tau^9(\alpha)] \cdots R_{VBOC1(pn,n,\alpha)}[\tau^{12}(\alpha)] \\ R_{VBOC1(pn,n,\alpha)}[\tau^{13}(\alpha)] \cdots R_{VBOC1(pn,n,\alpha)}[\tau^{16}(\alpha)] \end{bmatrix} = \arg \min_{\alpha} \frac{\begin{bmatrix} 1-2p-(1-4p)\alpha & 1-2p-(3-4p)\alpha & 3-2p-(5-4p)\alpha & 3-2p-(7-4p)\alpha \\ 5-2p-(9-4p)\alpha & 5-2p-(11-4p)\alpha & 7-2p-(13-4p)\alpha & 7-2p-(15-4p)\alpha \\ 9-2p-(17-4p)\alpha & 9-2p-(19-4p)\alpha & 11-2p-(21-4p)\alpha & 11-2p-(23-4p)\alpha \\ 13-2p-(25-4p)\alpha & 13-2p-(27-4p)\alpha & 15-2p-(29-4p)\alpha & 15-2p-(31-4p)\alpha \end{bmatrix}}{2p} \quad (170)$$

Optimization theorem of the first kind says that proving (170) is equivalent with proving the following

$$\arg \min_{\alpha} \sum_{i=1}^{16} R_{VBOC1(pn,n,\alpha)}[\tau^i(\alpha)] = \arg \min_{\alpha} \frac{\min[128-32p-(256-56p)\alpha]}{2p} \quad (171)$$

Hence, we find minimum $\alpha = 0.5$ and it is independent of p . This completes the proof of theorem 3. Substituting, $\alpha = 0.5$ in (170) we obtain

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 0.5}{2p}\right] \cdots R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 3.5}{2p}\right] \\ R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 4.5}{2p}\right] \cdots R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 7.5}{2p}\right] \\ R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 8.5}{2p}\right] \cdots R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 11.5}{2p}\right] \\ R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 12.5}{2p}\right] \cdots R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 15.5}{2p}\right] \end{bmatrix} = \frac{\begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \end{bmatrix}}{2p} \quad (172)$$

Equations (161) through (172) complete the proof of theorem 6; i.e., the optimization theorem of the first kind proves that $\alpha = 0.5$ minimizes the out-of-phase autocorrelation peaks of

the ACF of $VBOC1_{(pn,n,\alpha)}(t)$, or $R_{VBOC1(pn,n,\alpha)}(\tau)$, for all values of p .

Corollary 10: It follows immediately from (172) that

$$\lim_{p \rightarrow \infty} \begin{bmatrix} R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 0.5}{2p}\right] \cdots R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 3.5}{2p}\right] \\ R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 4.5}{2p}\right] \cdots R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 7.5}{2p}\right] \\ R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 8.5}{2p}\right] \cdots R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 11.5}{2p}\right] \\ R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 12.5}{2p}\right] \cdots R_{VBOC1(pn,n,\alpha)}\left[\frac{T_c 15.5}{2p}\right] \end{bmatrix} = \lim_{p \rightarrow \infty} \frac{\begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \end{bmatrix}}{2p} = 0 \quad (173)$$

which is another reason why $\alpha = 0.5$ is the optimum delay for the out-of-phase autocorrelation peaks of the ACF of $VBOC1_{(pn,n,\alpha)}(t)$, or $R_{VBOC1(pn,n,\alpha)}(\tau)$, for all values of p .

We will leave as an exercise to the reader to prove Theorem

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(pn,n,\alpha)}[\tau^1(\alpha)] & \cdots & \\ \cdots & \ddots & \vdots \\ \cdots & R_{VBOC1(pn,n,\alpha)}[\tau^{2p}(\alpha)] \end{bmatrix} \rightarrow \arg \min_{\alpha} \prod_{i=1}^{2p} R_{VBOC1(pn,n,\alpha)}[\tau_i(\alpha)] = 0 \quad (174)$$

$\tau^i(\alpha)$ for $i = \{1, 2, \dots, 2p\}$ given by (159) in theorem 4 and $p = \{5, 6, 7, 8\}$.

Proof of theorem 7: The proof of theorem 7 is also straightforward. Because the min-max of the ACF of $VBOC1_{(pn,n,\alpha)}(t)$, or $R_{VBOC1(pn,n,\alpha)}(\tau)$, that depends α on for values of $p = \{5, 6, 7, 8\}$, $i = \{1, 2, \dots, 2p\}$, and $\tau^i(\alpha)$ given by

4 in Progi (2015, [2]) for $p = \{5, 6, 7, 8\}$.

Theorem 7: Prove that the ACF $R_{VBOC1(m=pn,n,\alpha)}(\tau)$, given by (17), obeys the optimization theorem of the third kind; i.e., find the values of α that minimizes the out-of-phase ACF

(159); hence, it suffices to show that (174) holds.

First, we find the value of α that minimizes $\tau^1(\alpha) \cdots \tau^{10}(\alpha)$ in (159) for $p = 5$. Optimization theorem of the third kind says that proving (159) is equivalent with proving the following

$$\arg \min_{\alpha} \left\{ \prod_{i=1}^{10} R_{VBOC1(pn,n,\alpha)}[\tau_i(\alpha)] \right\} \rightarrow \arg \min_{\alpha} \prod_{i=1}^5 \left[\frac{2i-1-2p-[4(i-1)+1-4p]\alpha}{2p} \right] \left[\frac{2i-1-2p-[4(i-1)+3-4p]\alpha}{2p} \right] = 0 \quad (175)$$

$$\begin{cases} 2i-1-2p-[4(i-1)+1-4p]\alpha = 0 \\ 2i-1-2p-[4(i-1)+3-4p]\alpha = 0 \end{cases} \rightarrow \begin{cases} \alpha = \frac{2i-1-2p}{4(i-1)+1-4p} \Big|_{p=5, i=\{1,2,\dots,5\}} = \left\{ \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\} \\ \alpha = \frac{2i-1-2p}{4(i-1)+3-4p} \Big|_{p=5, i=\{1,2,\dots,5\}} = \left\{ \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \end{cases} \quad (176)$$

There are three observations from (176). The first is that α depends on p . And the second is that minimum $\alpha =$

$\left\{ \left\{ \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\}, \left\{ \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \right\}$ for $p = 5$. Substituting,

minimum $\alpha = \left\{ \left\{ \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\}, \left\{ \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \right\}$ in (161) we obtain

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(p,n,\alpha)} \left[\frac{9T_c}{19p} \right] \cdots R_{VBOC1(p,n,\alpha)} \left[\frac{7T_c}{13p} \right] \\ R_{VBOC1(p,n,\alpha)} \left[\frac{5T_c}{11p} \right] \cdots R_{VBOC1(p,n,\alpha)} \left[\frac{3T_c}{5p} \right] \\ R_{VBOC1(p,n,\alpha)} \left[\frac{T_c}{3p} \right] \quad R_{VBOC1(p,n,\alpha)} \left[\frac{T_c}{p} \right] \end{bmatrix} \rightarrow \arg \min_{\alpha} \frac{\begin{bmatrix} \frac{5-p}{19} & \frac{p-5}{17} & \frac{5-p}{15} & \frac{p-5}{13} \\ \frac{5-p}{11} & \frac{p-5}{9} & \frac{5-p}{7} & \frac{p-5}{5} \\ \frac{5-p}{3} & p-5 & & \end{bmatrix}}{p} \Big|_{p=5} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (177)$$

From where (175) is obtained. The third observation is that minimum value of (177) according to *the optimization theorem*

of third kind is 0 because we set it to be equal to 0.

Second, it is also straight forward to show that for $p = 6$

$$\arg \min_{\alpha} \begin{cases} 2i - 1 - 2p - [4(i - 1) + 1 - 4p]\alpha = 0 \\ 2i - 1 - 2p - [4(i - 1) + 3 - 4p]\alpha = 0 \end{cases} \rightarrow \begin{cases} \alpha = \frac{2i-1-2p}{4(i-1)+1-4p} \Big|_{p=6, i=\{1,2,\dots,6\}} = \left\{ \frac{11}{23}, \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\} \\ \alpha = \frac{2i-1-2p}{4(i-1)+3-4p} \Big|_{p=6, i=\{1,2,\dots,6\}} = \left\{ \frac{11}{21}, \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \end{cases} \quad (178)$$

There are three observations from (178). The first is that α depends on p . And the second is that minimum $\alpha =$

minimum $\alpha = \left\{ \left\{ \frac{11}{23}, \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\}, \left\{ \frac{11}{21}, \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \right\}$ in (174)

we obtain

$\left\{ \left\{ \frac{11}{23}, \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\}, \left\{ \frac{11}{21}, \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \right\}$ for $p = 6$. Substituting,

$$\arg \min_{\alpha} \begin{cases} 2i - 1 - 2p - [4(i - 1) + 1 - 4p]\alpha = 0 \\ 2i - 1 - 2p - [4(i - 1) + 3 - 4p]\alpha = 0 \end{cases} \rightarrow \begin{cases} \alpha = \frac{2i-1-2p}{4(i-1)+1-4p} \Big|_{p=6, i=\{1,2,\dots,6\}} = \left\{ \frac{11}{23}, \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\} \\ \alpha = \frac{2i-1-2p}{4(i-1)+3-4p} \Big|_{p=6, i=\{1,2,\dots,6\}} = \left\{ \frac{11}{21}, \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \end{cases} \quad (179)$$

The third observation is that minimum value of (179) according to optimization theorem of third kind is 0 because

we set it to be equal to 0.

Third, it is also straight forward to show that for $p = 7$

$$\arg \min_{\alpha} \begin{cases} 2i - 1 - 2p - [4(i - 1) + 1 - 4p]\alpha = 0 \\ 2i - 1 - 2p - [4(i - 1) + 3 - 4p]\alpha = 0 \end{cases} \rightarrow \begin{cases} \alpha = \frac{2i-1-2p}{4(i-1)+1-4p} \Big|_{p=7, i=\{1,2,\dots,7\}} = \left\{ \frac{13}{27}, \frac{11}{23}, \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\} \\ \alpha = \frac{2i-1-2p}{4(i-1)+3-4p} \Big|_{p=7, i=\{1,2,\dots,7\}} = \left\{ \frac{13}{25}, \frac{11}{21}, \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \end{cases} \quad (180)$$

There are three observations from (180). The first is that α depends on p . The second is that minimum $\alpha =$

Substituting,

minimum

$\alpha =$

$\left\{ \left\{ \frac{13}{27}, \frac{11}{23}, \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\}, \left\{ \frac{13}{25}, \frac{11}{21}, \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \right\}$ for $p = 7$.

$\left\{ \left\{ \frac{13}{27}, \frac{11}{23}, \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\}, \left\{ \frac{13}{25}, \frac{11}{21}, \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \right\}$ in (174) we

obtain

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(p,n,\alpha)} \left[\frac{13T_c}{27p} \right] \cdots R_{VBOC1(p,n,\alpha)} \left[\frac{11T_c}{21p} \right] \\ R_{VBOC1(p,n,\alpha)} \left[\frac{9T_c}{19p} \right] \cdots R_{VBOC1(p,n,\alpha)} \left[\frac{7T_c}{13p} \right] \\ R_{VBOC1(p,n,\alpha)} \left[\frac{5T_c}{11p} \right] \cdots R_{VBOC1(p,n,\alpha)} \left[\frac{3T_c}{5p} \right] \\ R_{VBOC1(p,n,\alpha)} \left[\frac{T_c}{3p} \right] \quad R_{VBOC1(p,n,\alpha)} \left[\frac{T_c}{p} \right] \end{bmatrix} \rightarrow \arg \min_{\alpha} \frac{\begin{bmatrix} \frac{7-p}{27} & \frac{p-7}{25} & \frac{7-p}{23} & \frac{p-7}{21} \\ \frac{7-p}{19} & \frac{p-7}{17} & \frac{7-p}{15} & \frac{p-7}{13} \\ \frac{7-p}{11} & \frac{p-7}{9} & \frac{7-p}{7} & \frac{p-7}{5} \\ \frac{7-p}{3} & p-7 & & \end{bmatrix}}{p} \Big|_{p=7} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (181)$$

The third observation is that minimum value of (181) according to *the optimization theorem of third kind* is 0

because we set it to be equal to 0.

Fourth, it is also straight forward to show that for $p = 8$

$$\arg \min_{\alpha} \begin{cases} 2i - 1 - 2p - [4(i - 1) + 1 - 4p]\alpha = 0 \\ 2i - 1 - 2p - [4(i - 1) + 3 - 4p]\alpha = 0 \end{cases} \rightarrow \begin{cases} \alpha = \frac{2i-1-2p}{4(i-1)+1-4p} \Big|_{p=8, i=\{1,2,\dots,8\}} = \left\{ \frac{15}{31}, \frac{13}{27}, \frac{11}{23}, \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\} \\ \alpha = \frac{2i-1-2p}{4(i-1)+3-4p} \Big|_{p=8, i=\{1,2,\dots,8\}} = \left\{ \frac{15}{29}, \frac{13}{25}, \frac{11}{21}, \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \end{cases} \quad (182)$$

There are three observations from (182). The first is that α depends on p . And the second is that minimum $\alpha =$

Substituting

the values

of $\alpha =$

$\left\{ \left\{ \frac{15}{31}, \frac{13}{27}, \frac{11}{23}, \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\}, \left\{ \frac{15}{29}, \frac{13}{25}, \frac{11}{21}, \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \right\}$ for $p = 8$.

$\left\{ \left\{ \frac{15}{31}, \frac{13}{27}, \frac{11}{23}, \frac{9}{19}, \frac{7}{15}, \frac{5}{11}, \frac{3}{7}, \frac{1}{3} \right\}, \left\{ \frac{15}{29}, \frac{13}{25}, \frac{11}{21}, \frac{9}{17}, \frac{7}{13}, \frac{5}{9}, \frac{3}{5}, 1 \right\} \right\}$ in (174)

yields

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(pn,n,\alpha)} \left[\frac{15T_c}{31p} \right] \cdots R_{VBOC1(pn,n,\alpha)} \left[\frac{13T_c}{25p} \right] \\ R_{VBOC1(pn,n,\alpha)} \left[\frac{11T_c}{23p} \right] \cdots R_{VBOC1(pn,n,\alpha)} \left[\frac{9T_c}{17p} \right] \\ R_{VBOC1(pn,n,\alpha)} \left[\frac{7T_c}{15p} \right] \cdots R_{VBOC1(pn,n,\alpha)} \left[\frac{5T_c}{9p} \right] \\ R_{VBOC1(pn,n,\alpha)} \left[\frac{3T_c}{7p} \right] \cdots R_{VBOC1(pn,n,\alpha)} \left[\frac{T_c}{p} \right] \end{bmatrix} \rightarrow \arg \min_{\alpha} \begin{matrix} \begin{matrix} \frac{8-p}{31} & \frac{p-8}{29} & \frac{8-p}{27} & \frac{p-8}{25} \\ \frac{8-p}{23} & \frac{p-8}{21} & \frac{8-p}{19} & \frac{p-8}{17} \\ \frac{8-p}{15} & \frac{p-8}{13} & \frac{8-p}{11} & \frac{p-8}{9} \\ \frac{8-p}{7} & \frac{p-8}{5} & \frac{8-p}{3} & \frac{p-8}{1} \end{matrix} \\ p \end{matrix}_{p=8} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (183)$$

The third observation is that minimum value of (183) according to optimization theorem of third kind is 0 because we set it to be equal to 0.

Equations (175) through (183) complete the proof of *theorem 7*.

We leave as an exercise to the reader to prove that *Corollary 3* in [2] holds; i.e., $\forall p$ half of the α s (or odd ones) are smaller than 0.5 and the remaining half (or even ones) are greater than 0.5 for $p = \{5, 6, 7, 8\}$.

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(pn,n,\alpha)} [\tau^1(\alpha)] & \cdots & \\ \cdots & \ddots & \\ & & \vdots \\ & & R_{VBOC1(pn,n,\alpha)} [\tau^{2p}(\alpha)] \end{bmatrix} \rightarrow \arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(pn,n,\alpha)} [\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1(pn,n,\alpha)} [\tau^{2p}(\alpha)] \end{bmatrix} \rightarrow 0 \quad (184)$$

which is equivalent with

$$\alpha_{\min} = \frac{\sum_{i=1}^{2p} a_i(p) b_i(p)}{\sum_{i=1}^{2p} [a_i(p)]^2} = \frac{\mathbf{a}(p)^T \mathbf{b}(p)}{\mathbf{a}(p)^T \mathbf{a}(p)} = \frac{\mathbf{a}(p)^T \mathbf{b}(p)}{\|\mathbf{a}(p)\|^2} \quad \text{vii} \quad (185)$$

where $\tau^i(\alpha)$ for $i = \{1, 2, \dots, 2p\}$ given by (159) in *theorem 3* and $p = \{5, 6, 7, 8\}$ and vectors $\mathbf{a}(p)$ and $\mathbf{b}(p)$ are given by

$$\begin{bmatrix} R_{VBOC1(pn,n,\alpha)} [\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1(pn,n,\alpha)} [\tau^{16}(\alpha)] \end{bmatrix} = \frac{\begin{matrix} \begin{matrix} 1-2p \\ 1-2p \\ \vdots \\ 1-2p \end{matrix} \\ p=\{1,\dots,8\} \\ \begin{matrix} 3-2p \\ 3-2p \\ \vdots \\ 3-2p \end{matrix} \\ p=\{2,\dots,8\} \\ \begin{matrix} 5-2p \\ 5-2p \\ \vdots \\ 5-2p \end{matrix} \\ p=\{3,\dots,8\} \\ \begin{matrix} 7-2p \\ 7-2p \\ \vdots \\ 7-2p \end{matrix} \\ p=\{4,\dots,8\} \\ \begin{matrix} 9-2p \\ 9-2p \\ \vdots \\ 9-2p \end{matrix} \\ p=\{5,\dots,8\} \\ \begin{matrix} 11-2p \\ 11-2p \\ \vdots \\ 11-2p \end{matrix} \\ p=\{6,\dots,8\} \\ \begin{matrix} 13-2p \\ 13-2p \\ \vdots \\ 13-2p \end{matrix} \\ p=\{7,8\} \\ \begin{matrix} 15-2p \\ 15-2p \\ \vdots \\ 15-2p \end{matrix} \\ p=\{8\} \end{matrix}}{\mathbf{b}(p)} - \frac{\begin{matrix} \begin{matrix} 1-4p \\ 3-4p \\ \vdots \\ 1-4p \end{matrix} \\ p=\{1,\dots,8\} \\ \begin{matrix} 5-4p \\ 7-4p \\ \vdots \\ 5-4p \end{matrix} \\ p=\{2,\dots,8\} \\ \begin{matrix} 9-4p \\ 11-4p \\ \vdots \\ 9-4p \end{matrix} \\ p=\{3,\dots,8\} \\ \begin{matrix} 13-4p \\ 15-4p \\ \vdots \\ 13-4p \end{matrix} \\ p=\{4,\dots,8\} \\ \begin{matrix} 17-4p \\ 19-4p \\ \vdots \\ 17-4p \end{matrix} \\ p=\{5,\dots,8\} \\ \begin{matrix} 21-4p \\ 23-4p \\ \vdots \\ 21-4p \end{matrix} \\ p=\{6,\dots,8\} \\ \begin{matrix} 25-4p \\ 27-4p \\ \vdots \\ 25-4p \end{matrix} \\ p=\{7,8\} \\ \begin{matrix} 29-4p \\ 31-4p \\ \vdots \\ 29-4p \end{matrix} \\ p=\{8\} \end{matrix}}{\mathbf{a}(p)} \alpha \equiv \mathbf{r}(p, \alpha) \quad (186)$$

Proof of theorem 8: The proof of *theorem 8* is also straight forward.

First for $p = 5$ from the *optimization theorem of the fourth kind (or in the mean square sense)*; i.e., find the values of α that minimizes the out-of-phase ACF,

$$\arg \min_{\alpha} \begin{bmatrix} R_{VBOC1(pn,n,\alpha)} [\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1(pn,n,\alpha)} [\tau^{2p}(\alpha)] \end{bmatrix} \rightarrow 0 \quad (187)$$

It is also straightforward to show that *Corollary 4* in [2] holds; i.e., $\alpha = 0.5$ is the asymptotic value for which the product optimum α s from the *optimization theorem of the third kind* approaches the sum optimum α from the *optimization theorem of the first and second kinds*.

Theorem 8: Prove that the ACF $R_{VBOC1(m=np,n,\alpha)}(\tau)$, given by (17), obeys the *optimization theorem of the fourth kind (or in the mean square sense)*; i.e., find the values of α that minimizes the out-of-phase ACF,

which is equivalent with

$$\alpha_{\min} = \frac{\sum_{i=1}^{10} a_i(p) b_i(p)}{\sum_{i=1}^{10} [a_i(p)]^2} = \frac{\mathbf{a}(p)^T \mathbf{b}(p)}{\mathbf{a}(p)^T \mathbf{a}(p)} = \frac{\mathbf{a}(p)^T \mathbf{b}(p)}{\|\mathbf{a}(p)\|^2} = 2 \frac{\|\mathbf{b}(p)\|^2}{\|\mathbf{a}(p)\|^2} \quad (188)$$

where

$$\begin{bmatrix} R_{VBOC1(pn,n,\alpha)} [\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1(pn,n,\alpha)} [\tau^{10}(\alpha)] \end{bmatrix} = \frac{\begin{matrix} \begin{matrix} 1-2p \\ 1-2p \\ \vdots \\ 1-2p \end{matrix} \\ \begin{matrix} 3-2p \\ 3-2p \\ \vdots \\ 3-2p \end{matrix} \\ \begin{matrix} 5-2p \\ 5-2p \\ \vdots \\ 5-2p \end{matrix} \\ \begin{matrix} 7-2p \\ 7-2p \\ \vdots \\ 7-2p \end{matrix} \\ \begin{matrix} 9-2p \\ 9-2p \\ \vdots \\ 9-2p \end{matrix} \\ \begin{matrix} 11-2p \\ 11-2p \\ \vdots \\ 11-2p \end{matrix} \\ \begin{matrix} 13-2p \\ 13-2p \\ \vdots \\ 13-2p \end{matrix} \\ \begin{matrix} 15-2p \\ 15-2p \\ \vdots \\ 15-2p \end{matrix} \\ \begin{matrix} 17-2p \\ 17-2p \\ \vdots \\ 17-2p \end{matrix} \\ \begin{matrix} 19-2p \\ 19-2p \\ \vdots \\ 19-2p \end{matrix} \end{matrix}}{\mathbf{b}(p)} - \frac{\begin{matrix} \begin{matrix} 1-4p \\ 3-4p \\ \vdots \\ 1-4p \end{matrix} \\ \begin{matrix} 5-4p \\ 7-4p \\ \vdots \\ 5-4p \end{matrix} \\ \begin{matrix} 9-4p \\ 11-4p \\ \vdots \\ 9-4p \end{matrix} \\ \begin{matrix} 13-4p \\ 15-4p \\ \vdots \\ 13-4p \end{matrix} \\ \begin{matrix} 17-4p \\ 19-4p \\ \vdots \\ 17-4p \end{matrix} \\ \begin{matrix} 21-4p \\ 23-4p \\ \vdots \\ 21-4p \end{matrix} \\ \begin{matrix} 25-4p \\ 27-4p \\ \vdots \\ 25-4p \end{matrix} \\ \begin{matrix} 29-4p \\ 31-4p \\ \vdots \\ 29-4p \end{matrix} \\ \begin{matrix} 33-4p \\ 35-4p \\ \vdots \\ 33-4p \end{matrix} \\ \begin{matrix} 37-4p \\ 39-4p \\ \vdots \\ 37-4p \end{matrix} \end{matrix}}{\mathbf{a}(p)} \alpha \equiv \mathbf{r}(p, \alpha) \quad (189)$$

Hence, substituting (189) into (188) we obtain

$$\alpha_{\min} = \frac{4[(2p-1)^2 + (2p-3)^2 + (2p-5)^2 + (2p-7)^2 + (2p-9)^2]}{(4p-1)^2 + (4p-3)^2 + (4p-5)^2 + (4p-7)^2 + (4p-9)^2 + (4p-11)^2 + (4p-13)^2 + (4p-15)^2 + (4p-17)^2 + (4p-19)^2} = \frac{4 \times 165}{1330} = 0.4962 \quad (190)$$

Second, we leave as an exercise to the reader to show that for $p = 6$ from the *optimization theorem of the fourth kind (or in the minimum mean square error (MMSE) sense)*; i.e., one finds the values of α that minimizes the out-of-phase ACF, based on the solution that we found for $p = 5$

$$\alpha_{\min} = 2 \frac{\frac{\|\mathbf{b}(p)\|^2}{p=6}}{\frac{\|\mathbf{a}(p)\|^2}{p=6}} = 2 \frac{\frac{\|\mathbf{b}(p)\|^2}{p=5} + 2(2p-11)^2}{\frac{\|\mathbf{a}(p)\|^2}{p=5} + (4p-21)^2 + (4p-23)^2}$$

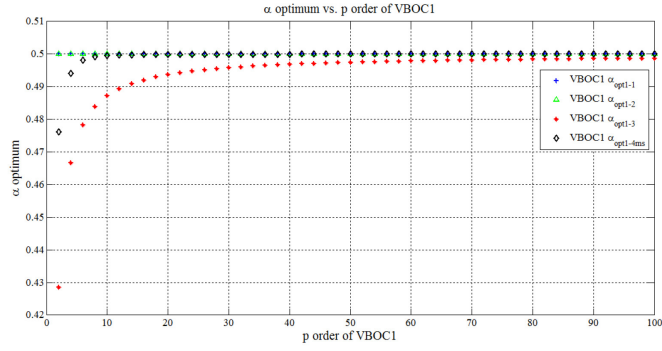
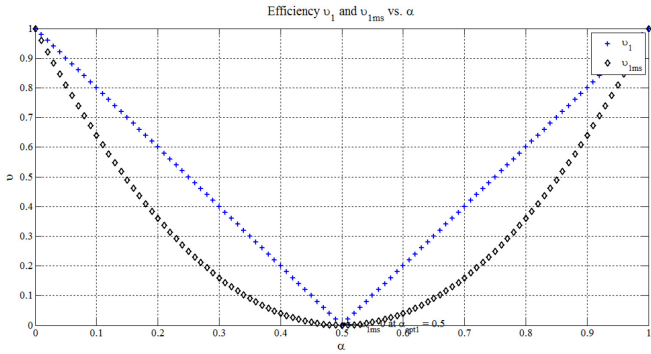
(a) α optimum vs. order p of $VBOC1_{(pn,n,\alpha)}(t)$.(b) Efficiency $\Upsilon_{1(\alpha)}$ (+) and $\Upsilon_{1MS(\alpha)}$ (◇) vs α .

FIGURE 2: Illustration of ACF optimization theorems.

$$= \frac{4 \times 286}{2300} = 0.4974 \quad (191)$$

Third, we leave as an exercise to the reader to show that for $p = 7$ from the *optimization theorem of the fourth kind* (or in the *minimum mean square error (MMSE) sense*); i.e., one finds the values of α that minimizes the out-of-phase ACF, based on the solution that we found for $p = 6$

$$\alpha_{\min} = 2 \frac{\frac{\|\mathbf{b}(p)\|^2}{p=7}}{\frac{\|\mathbf{a}(p)\|^2}{p=7}} = 2 \frac{\frac{\|\mathbf{b}(p)\|^2}{p=6} + 2(2p-13)^2}{\frac{\|\mathbf{a}(p)\|^2}{p=6} + (4p-25)^2 + (4p-27)^2} \quad (192)$$

$$= \frac{4 \times 455}{3654} = 0.4981$$

Fourth, we leave as an exercise to the reader to show that for $p = 8$ from the *optimization theorem of the fourth kind* (or in the *minimum mean square error (MMSE) sense*); i.e., one finds the values of α that minimizes the out-of-phase ACF, based on the solution that we found for $p = 7$

$$\alpha_{\min} = 2 \frac{\frac{\|\mathbf{b}(p)\|^2}{p=8}}{\frac{\|\mathbf{a}(p)\|^2}{p=8}} = 2 \frac{\frac{\|\mathbf{b}(p)\|^2}{p=7} + 2(2p-15)^2}{\frac{\|\mathbf{a}(p)\|^2}{p=7} + (4p-29)^2 + (4p-31)^2}$$

$$= \frac{4 \times 680}{5456} = 0.4985 \quad (193)$$

Equations (187) through (193) complete the *proof of theorem 8*.

We leave as an exercise to the reader to show that *Corollary 5* in [2] holds; i.e., $\alpha = 0.5$ is the asymptotic value for which the mean square optimum α s from the *optimization theorem of the fourth kind* approach the optimum α .

Corollary 11: or *performance enhancement or efficiency* both in terms of optimization theorems of the *first kind* and *fourth kinds*.

Performance enhancement or efficiency based on the *optimization theorem of the first kind* or $\Upsilon_{1(p,\alpha)}$ is defined as absolute value of the ratio or the sum of all out-of-phase autocorrelation peaks that depend on α by their number as follows:

$$\Upsilon_{1(p,\alpha)} = \frac{|\sum_{i=1}^{2p} R_{VBOC1(pn,n,\alpha)}[\tau_i(\alpha)]|}{2p} = \left|1 - \frac{4}{p}\right| |2\alpha - 1| \quad (194)$$

As we take the limit as $p \rightarrow \infty$ we find efficiency only as function of α as follows:

$$\Upsilon_{1(\alpha)} = \lim_{p \rightarrow \infty} \Upsilon_{1(p,\alpha)} = |2\alpha - 1| \quad (195)$$

or $|2\alpha - 1| = 0$ or $\alpha = 0.5$ gives efficiency equal to zero or infinite improvements of the *performance enhancement*.

Performance enhancement or efficiency based on the *optimization theorem of the fourth kind* is defined as the ratio of the sum of all out-of-phase autocorrelation peaks squared by their number and denoted by $\Upsilon_{1MS(p,\alpha)}$ as follows:

$$\begin{aligned} \Upsilon_{1MS(p,\alpha)} &= \frac{\sum_{i=1}^{2p} [R_{VBOC1(pn,n,\alpha)}[\tau_i(\alpha)]]^2}{2p} = \frac{\mathbf{r}(p,\alpha)^\dagger \mathbf{r}(p,\alpha)}{2p} \\ &= \frac{\mathbf{b}(p)^\dagger \mathbf{b}(p) - [\mathbf{b}(p)^\dagger \mathbf{a}(p) + \mathbf{a}(p)^\dagger \mathbf{b}(p)] \alpha + \mathbf{a}(p)^\dagger \mathbf{a}(p) \alpha^2}{2p} \\ &= \frac{\|\mathbf{b}(p)\|^2 - 4\|\mathbf{b}(p)\|^2 \alpha + \|\mathbf{a}(p)\|^2 \alpha^2}{2p} \\ &= \frac{\|\mathbf{b}(p)\|^2 - 4\|\mathbf{b}(p)\|^2 \alpha + 2\frac{\|\mathbf{b}(p)\|^2}{\alpha_{\min}(p)} \alpha^2}{2p} \\ &= \frac{\|\mathbf{b}(p)\|^2}{2p} \left[1 - 4\alpha + \frac{2}{\alpha_{\min}(p)} \alpha^2\right] \end{aligned} \quad (196)$$

As we take the limit as $p \rightarrow \infty \Rightarrow \alpha_{\min}(p) \rightarrow 0.5$ we find efficiency only as function of α as follows:

$$\Upsilon_{1MS(\alpha)} = \lim_{p \rightarrow \infty} \Upsilon_{1MS(p,\alpha)} = 1 - 4\alpha + 4\alpha^2$$

$$\equiv (1 - 2\alpha)^2 \equiv 4(0.5 - \alpha)^2 \quad (197)$$

For $\alpha = \{0,1\}$ we have $Y_{1MS(\alpha=\{0,1\})} = 1$ as opposed to for $\alpha = 0.5$ we have $Y_{1MS(\alpha=0.5)} = 0$; hence, $\alpha = 0.5$ offers 100% efficiency or for $\alpha = 0.5$ $VBOC1_{(pn,n,\alpha)}$ is 100% more efficient than either $BOC_{(pn,n)}$ or $BPSK_0$ in mean square sense in terms of out-of-phase ACF.

This concludes the detailed discussion on $VBOC1(\alpha)$ ACF pure signal optimization: (1) $VBOC1(\alpha)$ generalized ACF continuity theorems and (2) generalized ACF optimization theorems. Next, we discuss some example on numerical theoretical results.

9.3 Numerical, Theoretical Results

Very accurate simulation and interference results on $VBOC1(\alpha)$ and $VBOC2(\alpha, 1 - \alpha)$ ACFs and PSDs are given in Chap. 7 of (Progri 2015) [6] and (Progri 2012) [7].

However, detailed discussion on optimum α ; i.e., the simulation results of the optimization theorems 3 through 6; corollaries 5 and 6; simulation results with optimum $\alpha = 0.5$ $VBOC1_{(n,n,\alpha)}$ and $VBOC1_{(2n,n,\alpha)}$ are unique and original to this journal paper that the reader cannot find in (Progri 2015) [6] and (Progri 2012) [7].

Figure 2(a) presents the α optimum vs. order p of $VBOC1_{(pn,n,\alpha)}(t)$. There are four curves shown in Figure 2(a): (1)-(2) first through third correspond to $VBOC1_{(pn,n,\alpha)}(t)$ optimization theorems 3 and 4; (3) correspond to $VBOC1_{(pn,n,\alpha)}(t)$ optimization theorem 5; and (4) correspond to $VBOC1_{(pn,n,\alpha)}(t)$ optimization theorem 6 or in the mean square sense. The first curve which corresponds to

$VBOC1_{(pn,n,\alpha)}(t)$ shows that as p changes from two to one hundred α_{opt} is either equal to or converges to 0.5 regardless of generalized optimization parameter p (or of sub-carrier frequency) and regardless of the optimization theorem(s).

The argument can be made that the choice we make for α will equally affect the standardization of the waveforms regardless of sub-carrier frequency; i.e., even though the choices on $BOC_{(1,1)}(t) = VBOC1_{(1,1,\alpha=0)}(t)$ on GPS L1 data signal and $BOC_{(2,1)}(t) = BOC_{(10,5)}(t) = VBOC1_{(2,1,\alpha=0)}(t)$ or the GPS military M-code (Progri 2015) [1], (Progri 2012) [7], (Progri et. al. 2007) [8], (Betz 2001-2002) [12], and (Sousa and Nunes, 2013) [24] on both GPS L1 and L2 frequencies were arbitrary it only happened by accident that they have the same efficiency. Next, we consider how efficient are these choices?

Figure 2(b) depicts the efficiency $Y_{1(\alpha)}$ and $Y_{1MS(\alpha)}$ corresponding to $VBOC1_{(pn,n,\alpha)}(t)$ respectively vs. signal design and optimization parameter α . As shown in Figure 2(b) efficiency for either $Y_{1(\alpha)}$ or $Y_{1MS(\alpha)}$ at $\alpha = 0.5$ equals to $Y_{1(\alpha=0.5)} = Y_{1MS(\alpha=0.5)} = 0$ in corollary 6 of (Progri 2015, [2]).

The argument can be made that the choices on $BOC_{(1,1)}(t) = VBOC1_{(1,1,\alpha=0)}(t)$ on GPS L1 data signal and $BOC_{(2,1)}(t) = BOC_{(10,5)}(t) = VBOC1_{(2,1,\alpha=0)}(t)$ or the GPS military M-code (Progri 2015) [6], (Progri 2012) [7], (Progri et. al. 2007) [8], (Betz 2001-2002) [12], and (Sousa and Nunes, 2013) [24] on both GPS L1 and L2 frequencies are arbitrary and 100% inefficient in terms of out of phase ACF; i.e., BOC modulation is just as inefficient as the BPSK modulation in terms of out of phase ACF.

ⁱ The definition given here avoids that two possible signals are defined: one where the $+1/w_f$ part is longer than the $-1/w_f$ and one where the $-1/w_f$ part is longer than the positive $+1/w_f$; hence, avoiding allowing the parameter α to take on values in the range $-1 \leq \alpha \leq 1$; hence, reducing the complexity of continuity and optimization theorems by a factor of two.

ⁱⁱ One can observe a repeatability pattern in (53). It is this exact repeatability pattern that can be employed to visually check the accuracy of (53). This is not a guaranty but it provides some form of trust into the accuracy of (53). The

exact check is given by means of the continuity theorem in the generalized form in Appendix A.

ⁱⁱⁱ Ditto (53).

^{iv} Ditto (53).

^v Ditto comment on (69) can be applied in (70).

^{vi} There was an error in (72) in [7]. Equation (72) is the correct expression.

^{vii} '†' denotes the transpose of a vector $\mathbf{a}$.