

Research Article

Generalized Parabolic Cylinder Function Distribution

Ilir F. Proгри¹

¹*Giftet Inc., 5 Euclid Ave. #3, Worcester, MA 01610, USA*

ORCID: 0000-0001-5197-1278

Correspondence should be addressed to Ilir Proгри; iproгри@giftet.com

Received December 8, 2015; Revised December 19, 21, 2015-January 7, 2016, June 1-July 1, 2016 Partially Presented January 28, 2016, Accepted July 16, 2016; Published November 1, 2016.

Scientific Editor-in-Chief/Editor: Ilir F. Proгри

Copyright © 2016 Giftet Inc. All rights reserved. This work may not be translated or copied in whole or in part without written permission to the publisher (Giftet Inc., 5 Euclid Ave. #3, Worcester, MA 01610, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software or by similar or dissimilar methodology now known or hereafter developed is forbidden. The use of the publication of trade names, trademarks, service marks, or similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

This paper discussed generalized parabolic cylinder function (or GPCF) distributionⁱ (or GPCFD) probability density function (pdf) in a manner that is original and never presented before in the literature. For GPCF the closed form expression of the cumulative distribution function (cdf) is given by means of series expansion of the four Kampé de Fériet functions and six confluent hypergeometric series of two variables. Numerical results are derived for each case to validate the theoretical models presented in the paper.ⁱⁱ

Index Terms—Bessel functions, modified Bessel functions, cumulative distribution function, Kampé de Fériet function, series expansion, closed form expression, signal analysis, generalized functions, pdf cdf analysis, hypergeometric series, confluent hypergeometric series of two variables.

1 Introduction

The parabolic cylinder function (PCF) [1] distributions (PCFDs) are distributions that involve the computation of PCF.

The application of PCFDs is limited to gamma like properties as described in (Mathai and Saxena, 1968) [2]. In (Skjong and Madsen 1987) [3] a PCFD is known as the Pierson and Holmes distribution.

Even in Electrical and Computer Engineering the discussion of PCF applications is limited to a small number of publications [4]-[7].

Moreover, the numerical evaluation of the PCF is also

limited to special software such as (Wolfram 1999-2016) [8] or a special MATLAB toolbox (Cojocaru, 2009) [9].

Although the pdf of GPCFD was known, I found no references that discuss the cdf of GPCFD. These facts alone demonstrate that the cdf of GPCFD in closed form expression perhaps did not exist prior to this publication.

The real application that generated a very serious interest researching GPCFDs is because of adaptive GPS signal detection based on DCAC and more specifically under the assumption with GPS signal present, the statistics are GPCFDs (Proгри et al. 2016, [14], [15]).

This paper presents the first attempt to produce a closed form expression of the cdf of GPCFD. Although I have produced a closed form expression of the cdf of GPCFD, which contains four Kampé de Fériet functions (Progr 2016, [13]) and six confluent hypergeometric series of two variables [10], I do not believe that this closed form expression is simplified enough to enable a fast numerical computation of the cdf of GPCFD in MATLAB.

More theorems are needed to prove that indeed the closed form expression of the cdf of the GPCFD is indeed a valid cdf.

This paper is organized as follows: in Sect. 2 GPCFD pdf is presented. The series expansion of the GPCFD is discussed in Sect. 3. Section IV contains numerical results; Conclusion is provided in Section V along with a list of references.

2 GPCFD PDF

The GPCFD has its pdf given by

$$f_{PCD}(x) = \frac{d_0(x) \sum_{k=0}^{N-1} \frac{(N-k-1)! [b_0(x) D_{k-N}[c_0(x)] + b_1(x) D_{k-N}[c_1(x)]]}{H_k(N)^{-1} a^{k-N}}}{c} \quad (1)$$

where $D_k(z)$ is called the PCF (see [10] pg. 1028 9.24-9.25) and $H_k(N)$ is the Hermite polynomials (see [10] pg. 996 8.95)

$$a_0(x) = \frac{x-b}{2a^2}; \quad a_0^-(x) = \frac{1}{c^2} - a_0(x); \quad a_0^+(x) = \frac{1}{c^2} + a_0(x) \quad (2)$$

$$b_0(x) = e^{\frac{a^2}{4} a_0^2(x)}; \quad b_1(x) = e^{\frac{a^2}{4} a_0^2(x)} \quad (3)$$

$$c_0(x) = a a_0^-(x); \quad c_1(x) = a a_0^+(x) \quad (4)$$

$$d_0(x) = e^{-a_0(x)(x-b)} \quad (5)$$

$$b = \mu_k \mu_{k+1} \quad (6)$$

$$a^2 = c^2 \sum_{k=1}^N \mu_k^2; \quad c^2 = \sigma_k^2 \quad (7)$$

$$D_k(z) = 2^{\frac{1}{4} + \frac{k}{2}} W_{\frac{1}{4} + \frac{k}{2}, \frac{1}{4}} \left(\frac{z^2}{2} \right) z^{-\frac{1}{2}}; \quad z = \{c_0(x), c_1(x)\} \quad (8)$$

$$\frac{1}{c} = \frac{2^N \sigma_w \sigma^{2N}}{\sqrt{2\pi}(N-1)!} \quad (9)$$

Theorem 1: Prove that the GPCFD $f_{PCD}(x; a, b, c, N)$ given by (1) is a valid pdf; i.e., prove that two conditions must be met:

$$f_{PCD,N}(x; a, b, c) \geq 0 \quad (10)$$

and

$$\int_{-\infty}^{\infty} f_{PCD,N}(x, a, b, c) dx = 1 \quad (11)$$

where $-\infty < a, b, c < \infty$ and $N \geq 1$ positive integer.

Proof of Theorem 1: Let us prove them one at a time. First in order to prove (10) we must show that for all values of $-\infty < x < \infty$

$$\frac{d_0(x) \sum_{k=0}^{N-1} \frac{(N-k-1)! [b_0(x) D_{k-N}[c_0(x)] + b_1(x) D_{k-N}[c_1(x)]]}{H_k(N)^{-1} a^{k-N}}}{c} \geq 0 \quad (12)$$

Let us consider the parameters of (12). First, since N is a positive integer then

$$N \geq 1 \Rightarrow N-1 \geq 0 \Rightarrow (N-1)! \geq 0! = 1 > 0 \quad (13)$$

Hence, substituting (13) into and given that $\sigma_w > 0$ and $\sigma > 0$ we have

$$\frac{1}{c} = \frac{2^N \sigma_w \sigma^{2N}}{\sqrt{2\pi}(N-1)!} > 0 \quad (14)$$

Next, from (5) we have

$$d_0(x) = e^{-a_0(x-b)} > 0 \quad (15)$$

Next, in order to show that (12) is true we must show that for $-\infty < x < \infty$

$$H_k(N) \frac{n! [b_0(x) D_{k-N}[c_0(x)] + b_1(x) D_{k-N}[c_1(x)]]}{a^{k-N}} \geq 0 \quad (16)$$

where

$$n = N-1-k > 0, \quad k = \{0, 1, \dots, N-1\} \quad (17)$$

Let us consider the elements in (17) one at a time. First, since $a > 0$ then

$$a^{N-k} > 0 \quad (18)$$

regardless of N and k . Next, from (17) we have

$$n! = (N-1-k)! > 0, \quad k = \{0, 1, \dots, N-1\} \quad (19)$$

Also, from (see [10] pg. 996 8.95 2.) we have

$$H_k(N) = (2N)^k \left[1 - 2^{-1} \binom{k}{2} N^{-2} + 2^{-2} \cdot 1 \cdot 3 \cdot \binom{k}{4} N^{-4} - 2^{-3} \cdot 1 \cdot 3 \cdot 5 \cdot \binom{k}{6} N^{-6} + \dots \right] \quad (20)$$

In order to show that $H_k(N) > 0$ we must show that from (20) the following holds

$$2^{-1} \binom{k}{2} N^{-2} < 1 \quad (21)$$

and

$$2^{-3} \cdot 1 \cdot 3 \cdot 5 \binom{k}{6} N^{-6} < 2^{-2} \cdot 1 \cdot 3 \cdot \binom{k}{4} N^{-4} \quad (22)$$

or we must show that

$$\binom{k}{2} < 2N^2; \text{ for } k \geq 2 \text{ and } N \geq 3 \quad (23)$$

and

$$5 \binom{k}{6} < 2 \cdot \binom{k}{4} N^2; \text{ for } k \geq 4 \text{ and } N \geq 5 \quad (24)$$

Or we must show that

$$\frac{k(k-1)}{2} < 2N^2; \quad k \geq 2 \text{ and } N \geq 3 \quad (25)$$

and

$$5 \frac{k(k-1)(k-2)(k-3)(k-4)(k-5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} < 2 \cdot \frac{k(k-1)(k-2)(k-3)}{1 \cdot 2 \cdot 3 \cdot 4} N^2 \quad (26)$$

Or we must show that

$$k^2 - k < 4N^2, \quad k \geq 2, \quad N \geq 3 \quad (27)$$

and

$$\frac{(k-4)(k-5)}{6} < 2N^2 \quad (28)$$

Or we must show that

$$4N^2 - k^2 + k > 0, \quad k \geq 2, \quad N \geq 3 \quad (29)$$

And

$$(k-4)(k-5) < (k-1)(k-1) < N < 12N^2 \quad (30)$$

Hence, $H_k(N) > 0$ for $k < N$. Hence, in order to prove (16) it remains to show that for $-\infty < x < \infty$

$$b_0(x)D_{k-N}[c_0(x)] + b_1(x)D_{k-N}[c_1(x)] \geq 0 \quad (31)$$

From the definition of b_0 and b_1 (3) we have

$$b_0(x) = e^{\frac{a^2}{4}a_0^2(x)} > 0; \quad b_1(x) = e^{\frac{a^2}{4}a_0^2(x)} > 0 \quad (32)$$

From the definition of the PCF (see [10] pg. 1028 9.24-9.25) we have

$$D_{k-N}[c_i(x)] > 0 \text{ for } i = \{0,1\} \text{ if } k < N \quad (33)$$

which is true anyway. So, we have proven that (1) is true.

Next, we have to show that (2) is true. In order to show that (2) is true, we must show that

$$\int_{-\infty}^{\infty} \frac{d_0(x) \sum_{k=0}^{N-1} \frac{H_k(N)n! [b_0(x)D_{k-N}[c_0(x)] + b_1(x)D_{k-N}[c_1(x)]]}{a^{k-N}}}{C} dx = 1 \quad (34)$$

Or by changing the order of summation and integration we have

$$\frac{\sum_{k=0}^{N-1} \frac{H_k(N)n! \int_{-\infty}^{\infty} d_0(x) [b_0(x)D_{k-N}[c_0(x)] + b_1(x)D_{k-N}[c_1(x)]] dx}{a^{k-N}}}{C} = 1 \quad (35)$$

Or

$$\frac{\sum_{k=0}^{N-1} \frac{n! \left[\int_{-\infty}^{\infty} d_0(x) b_0 D_{k-N}[c_0(x)] dx + \int_{-\infty}^{\infty} d_0(x) b_1 D_{k-N}[c_1(x)] dx \right]}{H_k(N)^{-1} a^{k-N}}}{C} = 1 \quad (36)$$

Let us evaluate these integrals separately

$$g_{0k,N}(a, b, c) = \int_{-\infty}^{\infty} d_0(x) b_0(x) D_{k-N}[c_0(x)] dx \quad (37)$$

and

$$g_{1k,N}(a, b, c) = \int_{-\infty}^{\infty} d_1(x) b_1(x) D_{k-N}[c_1(x)] dx \quad (38)$$

First, from (2) through (5) and (37) we have

$$g_{0k,N}(a, b, c) = \int_{-\infty}^{\infty} \frac{D_{k-N} \left[a \left(\frac{1}{c^2} - \frac{x-b}{2a^2} \right) \right]}{e^{\frac{(x-b)^2}{2a^2}} e^{-\frac{a^2}{4} \left(\frac{1}{c^2} - \frac{x-b}{2a^2} \right)^2}} dx \quad (39)$$

Let us make the substitution

$$y = a \left(\frac{1}{c^2} - \frac{x-b}{2a^2} \right) \quad (40)$$

We have the following

$$dy = -\frac{dx}{2a}; \quad x \rightarrow |_{-\infty}^{\infty} \Rightarrow y \rightarrow |_{\infty}^{-\infty} \quad (41)$$

Substituting (40) and (41) into (39) we have

$$g_{0k,N}(a, b, c) = -2a \int_{\infty}^{-\infty} e^{-2a^2 \left(\frac{y}{a} - \frac{1}{c^2} \right)^2} e^{\frac{y^2}{4}} D_{k-N}(y) dy \quad (42)$$

which is equal to

$$g_{0k,N}(a, b, c) = 2a \int_{-\infty}^{\infty} e^{-\frac{\left(y - \frac{a}{c^2} \right)^2}{2}} e^{\frac{y^2}{4}} D_{k-N}(y) dy \quad (43)$$

Substituting the result in (see [10] pg. 842 7.724) we have

$$g_{0k,N}(a, b, c) = \frac{\sqrt{2\pi} \frac{1}{4} \left(1 - \frac{1}{4} \right)^{\frac{k-N}{2}} e^{-\frac{a^2}{4} \frac{1}{c^4}} D_{k-N} \left[\frac{a}{c^2} \left(1 - \frac{1}{4} \right)^{-\frac{1}{2}} \right]}{(2a)^{-1}} \quad (44)$$

which can be simplified as

$$g_{0k,N}(a, b, c) = 2a \sqrt{\frac{\pi}{2}} \left(\frac{3}{4} \right)^{\frac{k-N}{2}} e^{-\frac{a^2}{3c^4}} D_{k-N} \left(\frac{2a}{\sqrt{3}c^2} \right) \quad (45)$$

Similarly, let us solve the second integral.

First, from (2) through (5) and (38) we have

$$g_{1k,N}(a, b, c) = \int_{-\infty}^{\infty} \frac{D_{k-N} \left[a \left(\frac{1}{c^2} + \frac{x-b}{2a^2} \right) \right]}{e^{\frac{(x-b)^2}{2a^2}} e^{-\frac{a^2}{4} \left(\frac{1}{c^2} + \frac{x-b}{2a^2} \right)^2}} dx \quad (46)$$

Let us make the substitution

$$y = a \left(\frac{1}{c^2} + \frac{x-b}{2a^2} \right) \quad (47)$$

We have the following

$$dy = \frac{dx}{2a}; \quad x \rightarrow |_{-\infty}^{\infty} \Rightarrow y \rightarrow |_{-\infty}^{\infty} \quad (48)$$

Substituting (47) and (48) into (46) we have

$$g_{1k,N}(a, b, c) = 2a \int_{-\infty}^{\infty} e^{-2a^2 \left(\frac{y}{a} + \frac{1}{c^2} \right)^2} e^{\frac{y^2}{4}} D_{k-N}(y) dy \quad (49)$$

which is equal to

$$g_{1k,N}(a, b, c) = 2a \int_{-\infty}^{\infty} e^{-\frac{\left(y + \frac{a}{c^2} \right)^2}{2}} e^{\frac{y^2}{4}} D_{k-N}(y) dy \quad (50)$$

Substituting the result in (see [10] pg. 842 7.724) we have

$$g_{1k,N}(a, b, c) = \frac{\sqrt{2\pi} \frac{1}{4} \left(1 - \frac{1}{4}\right)^{\frac{k-N}{2}} e^{\frac{a^2}{4c^4}} D_{k-N} \left[\frac{a}{c^2} \left(1 - \frac{1}{4}\right)^{\frac{1}{2}} \right]}{(2a)^{-1}} \quad (51)$$

which can be simplified as

$$g_{1k,N}(a, b, c) = 2a \sqrt{\frac{\pi}{2}} \left(\frac{3}{4}\right)^{\frac{k-N}{2}} e^{\frac{a^2}{3c^4}} D_{k-N} \left(\frac{2a}{\sqrt{3}c^2} \right) \quad (52)$$

It turns out that

$$g_{0k,N}(a, b, c) \equiv g_{1k,N}(a, b, c) = \frac{\sqrt{\frac{\pi}{2}} \left(\frac{3}{4}\right)^{\frac{k-N}{2}} e^{\frac{a^2}{3c^4}} D_{k-N} \left(\frac{2a}{\sqrt{3}c^2} \right)}{(2a)^{-1}} \quad (53)$$

Finally, substituting (53) into (35) we have

$$\frac{4}{c} \sum_{k=0}^{N-1} \frac{H_k(N) n! \sqrt{\frac{\pi}{2}} \left(\frac{3}{4}\right)^{\frac{k-N}{2}} e^{\frac{a^2}{3c^4}} D_{k-N} \left(\frac{2a}{\sqrt{3}c^2} \right)}{a^{k-N-1}} = 1 \quad (54)$$

Or we need to set

$$\frac{4}{c} \left(\frac{3}{4}\right)^{\frac{-N}{2}} e^{\frac{a^2}{3c^4}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{N-1} \frac{H_k(N) n! \left(\frac{3}{4}\right)^{\frac{k}{2}} D_{k-N} \left(\frac{2a}{\sqrt{3}c^2} \right)}{a^k} = 1 \quad (55)$$

Substituting the value of (9) into (55) we obtain

$$C = \frac{\sqrt{2\pi}(N-1)!}{2^N \sigma_w \sigma^2 N} \equiv \frac{4\sqrt{\pi} e^{\frac{a^2}{3c^4}} \left(\frac{2a}{\sqrt{3}}\right)^N a \sum_{k=0}^{N-1} H_k(N) n! \left(\frac{\sqrt{3}}{2a}\right)^k D_{k-N} \left(\frac{2a}{\sqrt{3}c^2} \right)}{\sqrt{2}} \quad (56)$$

Corollary 1: Let us prove that in fact (see [10] pg. 842 7.724) is correct.

Proof of Corollary 1. Let us assume that $a = 0$, $k = 0$, and $N = 1$. From (42) we obtain

$$\int_{-\infty}^{\infty} e^{-\frac{(y-a')^2}{2\frac{1}{4}}} e^{\frac{y^2}{4}} D_{k-N}(y) dy = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\frac{1}{4}}} e^{\frac{y^2}{4}} D_{-1}(y) dy \quad (57)$$

From (see [10] pg. 1030 9.254 1.) we have

$$D_{-1}(y) = e^{\frac{y^2}{4}} \sqrt{\frac{\pi}{2}} \left[1 - \Phi \left(\frac{y}{\sqrt{2}} \right) \right] \quad (58)$$

Substituting (44) into (43) we obtain

$$\sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} \frac{[1 - \Phi(\frac{y}{\sqrt{2}})]}{e^{\frac{3y^2}{2}}} dy = \frac{\int_{-\infty}^{\infty} e^{-\frac{3y^2}{2}} dy - \int_{-\infty}^{\infty} e^{-\frac{3y^2}{2}} \Phi(\frac{y}{\sqrt{2}}) dy}{\frac{1}{2\sqrt{2}} \pi^{\frac{1}{2}}} \quad (59)$$

Let us solve these two integrals one at a time. The first integral is given by

$$\int_{-\infty}^{\infty} e^{-\frac{3y^2}{2}} dy = \sqrt{\frac{2}{3}} \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\frac{2\pi}{3}} \quad (60)$$

The second integral is given by

$$\int_{-\infty}^{\infty} e^{-\frac{3y^2}{2}} \Phi \left(\frac{y}{\sqrt{2}} \right) dy = 0 \quad (61)$$

Using the series expansion of the error function $\Phi(y)$ (see [10] pg. 889 8.253 1.) we have

$$\Phi(y) = \frac{2}{\sqrt{\pi}} e^{-y^2} \sum_{k=0}^{\infty} \frac{2^k y^{2k+1}}{(2k+1)!!} \quad (62)$$

Or the series expansion of error function $\Phi \left(\frac{y}{\sqrt{2}} \right)$ is given by

$$\Phi \left(\frac{y}{\sqrt{2}} \right) = \frac{2e^{-\frac{y^2}{2}} \sum_{k=0}^{\infty} \frac{2^k y^{2k+1}}{2^{k+\frac{1}{2}} (2k+1)!!}}{\sqrt{\pi}} = \frac{2e^{-\frac{y^2}{2}} \sum_{k=0}^{\infty} \frac{y^{2k+1}}{2^{\frac{1}{2}} (2k+1)!!}}{\sqrt{\pi}} \quad (63)$$

Substituting (63) into (61) we obtain

$$\frac{2 \int_{-\infty}^{\infty} e^{-2y^2} \sum_{k=0}^{\infty} \frac{y^{2k+1}}{2^{\frac{1}{2}} (2k+1)!!} dy}{\sqrt{\pi}} = \frac{2 \sum_{k=0}^{\infty} \frac{\int_{-\infty}^{\infty} e^{-2y^2} y^{2k+1} dy}{2^{\frac{1}{2}} (2k+1)!!}}{\sqrt{\pi}} = 0 \quad (64)$$

Finally, the answer to (61) becomes

$$\sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} \frac{[1 - \Phi(\frac{y}{\sqrt{2}})]}{e^{\frac{3y^2}{2}}} dy = \sqrt{\frac{\pi}{2}} \sqrt{\frac{2\pi}{3}} = \frac{\pi}{\sqrt{3}} \quad (65)$$

Instead, if we employ the result from (see [10] pg. 842 7.724) we have

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\frac{1}{4}}} e^{\frac{y^2}{4}} D_{-1}(y) dy = \sqrt{\frac{\pi}{2}} \frac{2}{\sqrt{3}} D_{-1}(0) = \sqrt{\frac{\pi}{2}} \frac{2}{\sqrt{3}} \sqrt{\frac{\pi}{2}} = \frac{\pi}{\sqrt{3}} \quad (66)$$

Hence, (65) is identical to (66).

Let us assume that $a \neq 0$, $a' \equiv \frac{a}{c^2}$, $k = 0$, and $N = 1$. From

(46) we obtain

$$\int_{-\infty}^{\infty} e^{-\frac{(y-a')^2}{2\frac{1}{4}}} e^{\frac{y^2}{4}} D_{k-N}(y) dy = \int_{-\infty}^{\infty} e^{\frac{y^2}{4}} \frac{(y-a')^2}{2\frac{1}{4}} D_{-1}(y) dy \quad (67)$$

which is equivalent with

$$\int_{-\infty}^{\infty} e^{\frac{y^2}{4}} \frac{(y-a')^2}{2\frac{1}{4}} D_{-1}(y) dy = \frac{\int_{-\infty}^{\infty} e^{\frac{y^2}{4}} \frac{(\sqrt{3}y - \frac{4a'}{\sqrt{3}})^2}{2} [1 - \Phi(\frac{y}{\sqrt{2}})] dy}{\sqrt{\frac{2}{\pi}} e^{-\frac{2a'^2}{3}}} \quad (68)$$

which can be decomposed into two integrals the first of which is

$$\frac{e^{\frac{2a'^2}{3}} \int_{-\infty}^{\infty} e^{\frac{y^2}{4}} \frac{(\sqrt{3}y - \frac{4a'}{\sqrt{3}})^2}{2} dy}{\sqrt{\frac{2}{\pi}}} = \sqrt{\frac{2}{3}} e^{\frac{2a'^2}{3}} \frac{\int_{-\infty}^{\infty} e^{-x^2} dx}{\sqrt{\frac{2}{\pi}}} = \frac{\pi e^{\frac{2a'^2}{3}}}{\sqrt{3}} \quad (69)$$

And the second of which is given by (if we make the substitution $x = \sqrt{3}y - \frac{4a'}{\sqrt{3}}$)

$$\frac{e^{\frac{2a'^2}{3}} \int_{-\infty}^{\infty} e^{-\frac{(\sqrt{3}y - \frac{4a'}{\sqrt{3}})^2}{2}} \Phi\left(\frac{y}{\sqrt{2}}\right) dy}{\sqrt{\frac{2}{\pi}}} = \frac{e^{\frac{2a'^2}{3}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \Phi\left(\frac{x + \frac{4a'}{\sqrt{3}}}{\sqrt{2}}\right) dx}{\sqrt{\frac{2}{\pi}} \sqrt{3}} \quad (70)$$

which can be written as (see [10] pg. 891 ex. 8.259 1.)

$$\frac{e^{\frac{2a'^2}{3}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \Phi\left(\frac{x + \frac{4a'}{\sqrt{3}}}{\sqrt{2}}\right) dx}{\sqrt{\frac{6}{\pi}}} = \frac{e^{\frac{2a'^2}{3}} \sqrt{2\pi} \Phi\left(\frac{4a'}{3 \times 2 \sqrt{\frac{1}{6} + \frac{1}{2}}}\right)}{\sqrt{\frac{6}{\pi}}} \quad (71)$$

which can be simplified as

$$\frac{e^{\frac{2a'^2}{3}} \int_{-\infty}^{\infty} e^{-\frac{(\sqrt{3}y - \frac{4a'}{\sqrt{3}})^2}{2}} \Phi\left(\frac{y}{\sqrt{2}}\right) dy}{\sqrt{\frac{2}{\pi}}} = \frac{e^{\frac{2a'^2}{3}} \pi \Phi\left(\sqrt{\frac{2}{3}} a'\right)}{\sqrt{3}} \quad (72)$$

Substituting (72) and (69) into (68) we obtain

$$\int_{-\infty}^{\infty} e^{\frac{y^2}{4} - \frac{(y-a')^2}{2\frac{1}{4}}} D_{-1}(y) dy = \frac{\pi e^{\frac{2a'^2}{3}} \left[1 - \Phi\left(\sqrt{\frac{2}{3}} a'\right)\right]}{\sqrt{3}} \quad (73)$$

Finally, substituting the relations of $D_{-1}(y)$ in (43) we obtain

$$\int_{-\infty}^{\infty} e^{\frac{y^2}{4} - \frac{(y-a')^2}{2\frac{1}{4}}} D_{-1}(y) dy = \sqrt{\frac{2\pi}{3}} e^{\frac{a'^2}{3}} D_{-1}\left(\frac{2}{\sqrt{3}} a'\right) \quad (74)$$

If we had to use (44) we would have obtained

$$\int_{-\infty}^{\infty} e^{\frac{y^2}{4} - \frac{(y-a')^2}{2\frac{1}{4}}} D_{-1}(y) dy = \sqrt{\frac{2\pi}{3}} e^{\frac{a'^2}{3c^4}} D_{-1}\left(\frac{2}{\sqrt{3}} \frac{a'}{c^2}\right) \quad (75)$$

Again, we see that (75) is identical to (74).

This concluded the discussion on the parabolic cylinder function distribution.

3 Series Expansion of GPCFD CDF

In this section the series expansion of GPCFD cdf is given. The work that is needed to produce the cdf of GPCFD is not yet completed. Although I have been successful to produce the closed form expression of the cdf of GPCFD its simplification still remains an issue.

3.1 GPCFD CDF

From the definition of the pdf of GPCFD, we define its corresponding cdf as

$$F_{PCD,N}(x; a, b, c) = \int_{-\infty}^x f_{PCD,N}(t; a, b, c) dt \quad (76)$$

$-\infty < a, b, c < \infty$ and $N \geq 1$, which can also be written as

$$F_{PCD,N}(x) = \frac{\int_{-\infty}^x d_0(t) \sum_{k=0}^{N-1} \frac{n! [b_0(t) D_{k-N}[c_0(t)] + b_1(t) D_{k-N}[c_1(t)]] dt}{H_k(N)^{-1} a^{k-N}}}{C} \quad (77)$$

where C is given by (56).

Interchanging the order of summation and integration we obtain

$$F_{PCD,N}(x) = \frac{\sum_{k=0}^{N-1} \frac{n! \int_{-\infty}^x d_0(t) [b_0(t) D_{k-N}[c_0(t)] + b_1(t) D_{k-N}[c_1(t)]] dt}{H_k(N)^{-1} a^{k-N}}}{C} \quad (78)$$

If we define next

$$g_{0k,N}(x; a, b, c) = \int_{-\infty}^x d_0(t) b_0(t) D_{k-N}[c_0(t)] dt \quad (79)$$

and

$$g_{1k,N}(x; a, b, c) = \int_{-\infty}^x d_0(t) b_1(t) D_{k-N}[c_1(t)] dt \quad (80)$$

then if we substitute (79) and (80) into (78) we obtain

$$F_{PCD,N}(x; a, b, c) = \frac{\sum_{k=0}^{N-1} \frac{n! [g_{0k,N}(x; a, b, c) + g_{1k,N}(x; a, b, c)]}{H_k(N)^{-1} a^{k-N}}}{C} \quad (81)$$

It remains to compute the functions $g_{ik,N}(x; a, b, c)$ for $i = \{0, 1\}$.

First, from (2) through (5) and (79) we have

$$g_{0k,N}(x; a, b, c) = \int_{-\infty}^x \frac{D_{k-N}\left[a\left(\frac{1}{c^2} - \frac{t-b}{2a^2}\right)\right]}{e^{\frac{(t-b)^2}{2a^2}} e^{-\frac{a^2}{4}\left(\frac{1}{c^2} - \frac{t-b}{2a^2}\right)^2}} dt \quad (82)$$

Let us make the substitution,

$$y = a\left(\frac{1}{c^2} - \frac{t-b}{2a^2}\right) \quad (83)$$

We have the following

$$dy = -\frac{dt}{2a}; t \rightarrow \left|_{-\infty}^x \right. y \rightarrow \left|_{\infty}^{a\left(\frac{1}{c^2} - \frac{x-b}{2a^2}\right)} \right. \quad (84)$$

Substituting (83) and (84) into (82) yields

$$g_{0k,N}(a, b, c) = -2a \int_{\infty}^{x_0} \frac{D_{k-N}(y)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2} e^{-\frac{y^2}{4}}} dy \quad (85)$$

where

$$D_{k-N}(y) = \frac{\frac{\sqrt{\pi}}{\Gamma(\mu)} \Phi\left(\lambda, \frac{1}{2}, \frac{y^2}{2}\right) - \frac{\sqrt{2\pi} y}{\Gamma(\lambda)} \Phi\left(\mu, \frac{3}{2}, \frac{y^2}{2}\right)}{2\lambda e^{\frac{y^2}{4}}} \quad (86)$$

where

$$x_0 = a\left(\frac{1}{c^2} - \frac{x-b}{2a^2}\right) \quad (87)$$

$$\lambda = \frac{N-k}{2} \quad (88)$$

$$\mu = \frac{1}{2} - \frac{N-k}{2} = \frac{1}{2} - \lambda \quad (89)$$

Substituting (86) into (85) yields

$$g_{0k,N}(a, b, c) = \frac{\frac{\sqrt{\pi}}{2\lambda} \int_{x_0}^{\infty} \frac{\Phi\left(\lambda \frac{1}{2}, \frac{y^2}{2}\right) \sqrt{2} y \Phi\left(\mu \frac{3}{2}, \frac{y^2}{2}\right)}{\Gamma(\mu) \Gamma(\lambda)} \frac{e^{-2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}}{e^{0.5a^{-1}}} dy}{0.5a^{-1}} \quad (90)$$

As we can see the integral (90) can be split into two integrals

$$g_{0k1,N}(a, b, c) = \frac{a\sqrt{\pi}}{2^{\lambda-1}\Gamma(\mu)} \int_{x_0}^{\infty} \frac{\Phi\left(\lambda \frac{1}{2}, \frac{y^2}{2}\right)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy \quad (91)$$

And

$$g_{0k2,N}(a, b, c) = -\frac{a\sqrt{2\pi}}{2^{\lambda-1}\Gamma(\lambda)} \int_{x_0}^{\infty} \frac{y \Phi\left(\mu \frac{3}{2}, \frac{y^2}{2}\right)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy \quad (92)$$

Let us try to solve integral (91) based on one definition of the confluent hypergeometric function [11] as follows

$$\int_{x_0}^{\infty} \frac{\Phi\left(\lambda \frac{1}{2}, \frac{y^2}{2}\right)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy = \frac{\Gamma\left(\frac{1}{2}\right) \int_{x_0}^{\infty} \int_0^1 \frac{e^{\frac{uy^2}{2}} u^{\lambda-1}(1-u)^{\mu-1} du}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy}{\Gamma(\lambda)\Gamma(\mu)} \quad (93)$$

Changing the order of integration in (93) produces

$$\int_{x_0}^{\infty} \frac{\Phi\left(\lambda \frac{1}{2}, \frac{y^2}{2}\right)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy = \frac{\int_0^1 u^{\lambda-1}(1-u)^{\mu-1} \int_{x_0}^{\infty} \frac{e^{\frac{uy^2}{2}}}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy du}{\Gamma^{-1}\left(\frac{1}{2}\right)\Gamma(\lambda)\Gamma(\mu)} \quad (94)$$

In order to solve (94) first we have to solve the following integral

$$\int_{x_0}^{\infty} \frac{e^{\frac{uy^2}{2}}}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy = \int_{x_0}^{\infty} e^{\frac{uy^2}{2} - 2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2} dy \quad (95)$$

From (Gradshteyn, Ryzhik 2007, [10] pg. 108 2.33 ex. 1) we have

$$\int_{x_0}^{\infty} \frac{e^{\frac{uy^2}{2}}}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy = \frac{1}{2} \frac{e^{-\frac{2a^2}{c^4}} \sqrt{\frac{\pi}{2-\frac{u}{c^2}}} \operatorname{erf}\left(\sqrt{2-\frac{u}{c^2}} y - \frac{2a}{c^2 \sqrt{2-\frac{u}{c^2}}}\right)}{\frac{2a^2}{c^4 \left(1-\frac{u}{c^2}\right)}} \Bigg|_{x_0}^{\infty} \quad (96)$$

Equation (94) can be simplified as

$$\int_{x_0}^{\infty} \frac{e^{\frac{uy^2}{2}}}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy = \sqrt{\frac{\pi}{2}} \frac{1}{2\tilde{u}} \frac{e^{-2a^2c^{-4}} [1 - \operatorname{erf}(\sqrt{2\tilde{u}}u')]}{e^{-2a^2c^{-4}\tilde{u}^{-2}}} \quad (97)$$

where

$$u' = \tilde{u}x_0 - \frac{a}{\tilde{u}c^2} \quad (98)$$

And

$$\tilde{u} = \sqrt{1 - \frac{u}{c^2}} \quad (99)$$

Next, we have to solve the following integral

$$\sqrt{\frac{\pi}{2}} \int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1}}{2\tilde{u}e^{2a^2c^{-4}(1-\tilde{u}^{-2})}} du = \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{\int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1} \tilde{u}}{e^{-2a^2c^{-4}\tilde{u}^{-2}}} du}{e^{2a^2c^{-4}}} \quad (100)$$

The expression in (100) cannot be solved by means of normal integration. Instead, we will use the series expansion of the exponential function (Gradshteyn, Ryzhik 2007, [10] pg. 26 1.211 ex. 1)

$$e^{2a^2c^{-4}\tilde{u}^{-2}} = \sum_{m=0}^{\infty} \frac{(2a^2c^{-4}\tilde{u}^{-2})^m}{m!} \quad (101)$$

Substituting (101) into (100) produces

$$\sqrt{\frac{\pi}{2}} \int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1}}{2\tilde{u}e^{2a^2c^{-4}(1-\tilde{u}^{-2})}} du = \frac{\int_0^1 \sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\tilde{u}^2\right]^m}{m!} \frac{u^{\lambda-1}(1-u)^{\mu-1}}{\tilde{u}^{-1}} du}{\frac{3}{2^2\pi} \frac{1}{2e} \frac{2a^2}{c^4}} \quad (102)$$

Changing the order of summation and integration in (102) yields

$$\sqrt{\frac{\pi}{2}} \int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1}}{2\tilde{u}e^{2a^2c^{-4}(1-\tilde{u}^{-2})}} du = \frac{\sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m \int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1}}{\tilde{u}^{2m-1}} du}{\pi \frac{1}{2^2\pi} \frac{3}{2e} \frac{2a^2}{c^4}}}{\pi \frac{1}{2^2\pi} \frac{3}{2e} \frac{2a^2}{c^4}} \quad (103)$$

Finally, it remains to solve the integral in (103) as follows (see Gradshteyn, Ryzhik 2007, [10] pg. 317 3.197 ex. 3)

$$\int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1}}{(1-\beta u)^v} du = B(\lambda, \mu) F(v, \lambda; \lambda + \mu; \beta) \quad (104)$$

where

$$v = m - \frac{1}{2} \quad (105)$$

$$\beta = \frac{1}{4} \quad (106)$$

Finally, substituting (104) into (103) produces

$$\sqrt{\frac{\pi}{2}} \int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1}}{2\tilde{u}e^{2a^2c^{-4}(1-\tilde{u}^{-2})}} du = \frac{\sqrt{\pi} \sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m B(\lambda, \mu) F(v, \lambda; \lambda + \mu; \frac{1}{4})}{m!}}{2\sqrt{2}e \frac{2a^2}{c^4}} \quad (107)$$

In order to solve the following integral

$$\int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1} \sqrt{\frac{\pi}{2}} \operatorname{erf}(\sqrt{2\tilde{u}}u')}{2\tilde{u}e^{2a^2c^{-4}(1-\tilde{u}^{-2})}} du = \frac{\int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1} \operatorname{erf}(\sqrt{2\tilde{u}}u')}{\tilde{u}e^{-\frac{2a^2}{c^4}\tilde{u}^{-2}}}}{\pi \frac{1}{2^2\pi} \frac{3}{2e} \frac{2a^2}{c^4}} du \quad (108)$$

We need to the series expansion of the $\operatorname{erf}(\sqrt{2\tilde{u}}u')$ (see Gradshteyn, Ryzhik 2007, [10] pg. 889 8.253 ex. 1) as follows

$$\operatorname{erf}(\sqrt{2\tilde{u}}u') = \frac{3}{\sqrt{\pi}} e^{-2u'^2} \sum_{m=0}^{\infty} \frac{2^{2m} u'^{2m+1}}{(2m+1)!!} \quad (109)$$

Substituting (109) into (108) produces

$$\int_0^1 \frac{u^{1-\lambda}(1-u)^{\mu-1} \sqrt{\pi} \operatorname{erf}(\sqrt{2\tilde{u}}u')}{2^{\frac{3}{2}} \tilde{u} e^{-\frac{2a^2}{c^4}\tilde{u}^{-2}}} du = \frac{\int_0^1 \sum_{m=0}^{\infty} \frac{2^{2m} u'^{2m+1}}{(2m+1)!!} du}{e^{-\frac{4x_0a}{c^2} + \frac{2a^2}{c^4}}} \quad (110)$$

where

$$2u'^2 - \frac{2a^2}{c^4\tilde{u}^2} = 2\tilde{u}^2 x_0^2 - \frac{4x_0a}{c^2} \quad (111)$$

In (110) we change the order of summation and integration and we obtain

$$\int_0^1 \frac{u^{1-\lambda}(1-u)^{\mu-1} \sqrt{\pi} \operatorname{erf}(\sqrt{2}u') du}{\frac{3}{2^2} \tilde{u} e^{\frac{2a^2}{c^4}} e^{-\frac{2a^2}{c^4} \tilde{u}^2}} = \frac{\sum_{m=0}^{\infty} \frac{\int_0^1 \frac{u^{1-\lambda}(1-u)^{\mu-1} \sqrt{\pi} \operatorname{erf}(\sqrt{2}u') du}{2^{-2m} (2m+1)!!}}{\frac{2a^2}{e} \frac{4x_0 a}{c^2}} \quad (112)$$

It remains to simplify and solve the integral in (112). But first we must simplify the following

$$u'^{2m+1} = (\tilde{u}x_0 - a\tilde{u}^{-1}c^{-2})^{2m+1} \quad (113)$$

We expand (113) using the binomial expansion (see Arfken and Weber 1995, [12], pg. 317 binomial theorem) as follows

$$u'^{m'} = x_0^{m'} \tilde{u}^{2m+1} (1 - a\tilde{u}^{-1}c^{-2})^{m'} \quad (114)$$

where

$$m' = 2m + 1 \quad (115)$$

Or

$$u'^{m'} = x_0^{m'} \tilde{u}^{2m+1} \sum_{q=0}^{\infty} \binom{m'}{q} (-a\tilde{u}^{-1}c^{-2})^q \quad (116)$$

Substituting (116) into (112) yields,

$$\int_0^1 \frac{u^{1-\lambda} \sqrt{\pi} \operatorname{erf}(\sqrt{2}u') du}{(1-u)^{\mu-1} \frac{3}{2^2} \tilde{u} e^{\frac{2a^2}{c^4}} e^{-\frac{2a^2}{c^4} \tilde{u}^2}} = \frac{\sum_{m=0}^{\infty} \frac{\int_0^1 \frac{\tilde{u}^{2m} \sum_{q=0}^{\infty} \binom{m'}{q} \left(\frac{-a}{\tilde{u}c^2}\right)^q du}{x_0^{-m'} 2^{-2m} (2m+1)!!}}{\frac{2a^2}{e} \frac{4x_0 a}{c^2}} \quad (117)$$

Changing again the order of summation and integration in (117) yields

$$\int_0^1 \frac{u^{1-\lambda} \sqrt{\pi} \operatorname{erf}(\sqrt{2}u') du}{[1-u]^{1-\mu} \frac{3}{2^2} \tilde{u} e^{\frac{2a^2}{c^4}} e^{-\frac{2a^2}{c^4} \tilde{u}^2}} = \frac{\sum_{m=0}^{\infty} \frac{\sum_{q=0}^{\infty} \frac{\binom{m'}{q} x_0^{m'} \int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1} du}{\tilde{u}^{(q-2m)} e^{\frac{x_0^2}{2} u}}}{\frac{2^{-2m} (2m+1)!!}}}{e^{-2x_0^2} \frac{4x_0 a}{c^2} + \frac{2a^2}{c^4}} \quad (118)$$

Finally, it remains to solve the integral in (118) as follows (see Gradshteyn, Ryzhik 2007, [10] pg. 349 3.285)

$$\int_0^1 \frac{u^{\lambda-1}(1-u)^{\mu-1}}{\left(1-\frac{u}{4}\right)^{(q-2m)} e^{\frac{x_0^2}{2} u}} du = \frac{\Phi_1\left(\lambda, q-2m; \lambda+\mu; -\frac{x_0^2}{2} \frac{1}{4}\right)}{B^{-1}(\lambda, \mu)} \quad (119)$$

Next, substituting (119) into (118) we obtain

$$\frac{\int_0^1 \frac{u^{1-\lambda} \sqrt{\pi} \operatorname{erf}(\sqrt{2}u') du}{[1-u]^{1-\mu} \tilde{u} e^{\frac{2a^2}{c^4}} e^{-\frac{2a^2}{c^4} \tilde{u}^2}}}{\frac{3}{2^2}} = \frac{\sum_{m=0}^{\infty} \frac{\sum_{q=0}^{\infty} \frac{\binom{m'}{q} x_0^{m'} \left(\frac{\sqrt{2}a}{c^2}\right)^{-q} \Phi_1\left(\lambda, q-2m; \lambda+\mu; -\frac{x_0^2}{2} \frac{1}{4}\right)}{2^{-2m} (2m+1)!!}}{B^{-1}(\lambda, \mu) e^{-2x_0^2} \frac{4x_0 a}{c^2} + \frac{2a^2}{c^4}} \quad (120)$$

Next, substituting (107) and (120) into (93) yields

$$\int_{x_0}^{\infty} \frac{\Phi\left(\lambda, \frac{1}{2}, \frac{y^2}{2}\right) dy}{e^{2a^2 \left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} = \frac{\frac{\sqrt{\pi} \sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m F\left(v, \lambda; \lambda+\mu; \frac{1}{4}\right)}{m!}}{\frac{3}{2^2} e^{\frac{2a^2}{c^4}} B^{-1}(\lambda, \mu) \Gamma^{-1}\left(\frac{1}{2}\right) \Gamma(\lambda) \Gamma(\mu)} - \frac{\sum_{q=0}^{\infty} \frac{\binom{m'}{q} x_0^{m'} \left(\frac{-a}{c^2}\right)^{-q} \Phi_1\left(\lambda, q-2m; \lambda+\mu; -\frac{x_0^2}{2} \frac{1}{4}\right)}{\frac{2^{-2m} (2m+1)!!}}}{\frac{2a^2}{e} \frac{4x_0 a}{c^2} B^{-1}(\lambda, \mu) \Gamma^{-1}\left(\frac{1}{2}\right) \Gamma(\lambda) \Gamma(\mu)} \quad (121)$$

Next, substituting (121) into (91) yields,

$$g_{0k1,N}(a, b, c) = \frac{a \sqrt{\pi} \left[\frac{\sqrt{\pi} \sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m F\left(v, \lambda; \lambda+\mu; \frac{1}{4}\right)}{m!}}{\sum_{q=0}^{\infty} \frac{\binom{m'}{q} \Phi_1\left(\lambda, q-2m; \lambda+\mu; -\frac{x_0^2}{2} \frac{1}{4}\right)}{\left(\frac{-a}{c^2}\right)^{-q}}} \right]}{B^{-1}(\lambda, \mu) \Gamma(\lambda) \Gamma^2(\mu) 2^{\lambda-1} e^{\frac{2a^2}{c^4}}} \quad (122)$$

Next, let us try to solve integral (92) based on one definition of the confluent hypergeometric function [11] as follows

$$\int_{x_0}^{\infty} \frac{y \Phi\left(\mu, \frac{3}{2}, \frac{y^2}{2}\right) dy}{e^{2a^2 \left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} = \frac{\Gamma\left(\frac{3}{2}\right) \int_{x_0}^{\infty} y \int_0^1 \frac{u^{\mu-1} (1-u)^{\lambda} du}{e^{2a^2 \left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy}{\Gamma(\mu) \Gamma(1+\lambda)} \quad (123)$$

Changing the order of integration in (123) produces

$$\int_{x_0}^{\infty} \frac{y \Phi\left(\mu, \frac{3}{2}, \frac{y^2}{2}\right) dy}{e^{2a^2 \left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} = \frac{\int_0^1 u^{\mu-1} (1-u)^{\lambda} \int_{x_0}^{\infty} \frac{y e^{-\frac{y^2}{2}}}{e^{2a^2 \left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy du}{\Gamma^{-1}\left(\frac{3}{2}\right) \Gamma(\mu) \Gamma(1+\lambda)} \quad (124)$$

In order to solve (124) first we have to solve the following integral

$$\int_{x_0}^{\infty} \frac{y e^{-\frac{y^2}{2}}}{e^{2a^2 \left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy = \int_{x_0}^{\infty} y e^{\frac{y^2}{2} - 2a^2 \left(\frac{y}{a} - \frac{1}{c^2}\right)^2} dy \quad (125)$$

Equation (125) can be written as follows

$$\int_{x_0}^{\infty} y e^{\frac{y^2}{2} - 2a^2 \left(\frac{y}{a} - \frac{1}{c^2}\right)^2} dy = \int_{x_0}^{\infty} y e^{-2\left(y^2 \tilde{u}^2 - 2y \frac{a}{c^2} + \frac{a^2}{c^4}\right)} dy \quad (126)$$

Next, we recognize that exponent of the exponential function inside the integral in (125) can be written as

$$y^2 \tilde{u}^2 - \frac{2ya}{c^2} + \frac{a^2}{c^4} = \left(\tilde{u}y - \frac{a}{\tilde{u}c^2}\right)^2 - \frac{a^2}{c^4} \left(\frac{u}{4-u}\right) \quad (127)$$

Next, substituting (127) into (125) yields

$$\int_{x_0}^{\infty} y e^{\frac{y^2}{2} - 2a^2 \left(\frac{y}{a} - \frac{1}{c^2}\right)^2} dy = \frac{\int_{x_0}^{\infty} y e^{-2\left(\tilde{u}y - \frac{a}{\tilde{u}c^2}\right)^2} dy}{e^{-2\frac{a^2}{c^4} \left(\frac{u}{4-u}\right)}} \quad (128)$$

Next, in (128) we make the substitution

$$y' = \tilde{u}y - \frac{a}{\tilde{u}c^2} \Rightarrow y = \frac{y'}{\tilde{u}} + \frac{a}{\tilde{u}^2 c^2} \quad (129)$$

Hence, we obtain the following

$$dy' = \tilde{u} dy \Rightarrow dy = \frac{dy'}{\tilde{u}} \quad (130)$$

and

$$y \rightarrow |_{x_0}^{\infty} \Rightarrow y' \rightarrow |_w^{\infty} \quad (131)$$

Next, substituting (129), (130), and (131) into (128) produces

$$\int_{x_0}^{\infty} y e^{\frac{uy^2}{2} - 2a^2(\frac{y}{a} - \frac{1}{c^2})^2} dy = \frac{\int_w^{\infty} (y' + \frac{a}{\tilde{u}c^2}) e^{-2y'^2} dy'}{\tilde{u}^2 e^{-2\frac{a^2}{c^4}(\frac{u}{4-u})}} \quad (132)$$

Equation (132) can be broken into two integrals:

$$\int_{x_0}^{\infty} y e^{\frac{uy^2}{2} - 2a^2(\frac{y}{a} - \frac{1}{c^2})^2} dy = \frac{\int_w^{\infty} y' e^{-2y'^2} dy' + a \int_w^{\infty} \frac{e^{-2y'^2} dy'}{\tilde{u}c^2}}{\tilde{u}^2 e^{-2\frac{a^2}{c^4}(\frac{u}{4-u})}} \quad (133)$$

The first integral in (133) is solved very easily as follows:

$$\int_w^{\infty} y' e^{-2y'^2} dy' = \frac{\int_w^{\infty} e^{-2y'^2} d2y'^2}{4} = -\frac{e^{-2y'^2}}{4} \Big|_w^{\infty} = \frac{e^{-2uw^2}}{4} \quad (134)$$

The second integral in (133) is also solved very easily as follows (Gradshteyn, Ryzhik 2007, [10] pg. 108 2.33 ex. 1):

$$\int_w^{\infty} e^{-2y'^2} dy' = \sqrt{\frac{\pi}{2}} \frac{\text{erf}(\sqrt{2}y')}{2} \Big|_w^{\infty} = \sqrt{\frac{\pi}{2}} \frac{1 - \text{erf}(\sqrt{2}w)}{2} \quad (135)$$

Substituting (134) and (135) into (133) we obtain

$$\int_{x_0}^{\infty} y e^{\frac{uy^2}{2} - 2a^2(\frac{y}{a} - \frac{1}{c^2})^2} dy = \frac{\frac{e^{-2uw^2}}{4} + a \sqrt{\frac{\pi}{2}} \frac{1 - \text{erf}(\sqrt{2}w)}{2}}{\tilde{u}^2 e^{-2\frac{a^2}{c^4}(\frac{u}{4-u})}} \quad (136)$$

Before computing the integral we recognize the following simplification

$$2 \left[\frac{a^2}{c^4} \left(\frac{u}{4-u} \right) - u'^2 \right] = \left[\frac{2a(2x_0 - \frac{a}{c^2})}{c^2} - 2\tilde{u}^2 x_0^2 \right] = \left[\frac{2a(2x_0 - \frac{a}{c^2})}{c^2} - 2x_0^2 + \frac{x_0^2 u}{4} \right] \quad (137)$$

Next, we have to solve the following integral (Gradshteyn, Ryzhik 2007, [10] pg. 349 3.385)

$$\frac{\int_0^1 \frac{u^{v-1}(1-u)^{w-1} e^{-\left(\frac{x_0^2}{4}\right)u}}{\tilde{u}^2} du = B(v, w) \Phi_1 \left(v, 1; w + v; \frac{x_0^2}{4} \frac{1}{4} \right)}{4e^{2x_0^2 - \frac{2a}{c^2}(2x_0 - \frac{a}{c^2})}} \quad (138)$$

where

$$v = \mu = \frac{1}{2} - \frac{N-k}{2} = \frac{1}{2} - \lambda \quad (139)$$

$$w = \lambda + 1 = \frac{N-k}{2} + 1 = \frac{3}{2} - \mu \quad (140)$$

Next, we have to solve the following integral by substituting (101) as follows

$$\left[\frac{\sqrt{\frac{\pi}{2}} \int_0^1 \frac{u^{v-1}(1-u)^{w-1}}{\left(1 - \frac{u}{4}\right)^{\frac{3}{2}} e^{-2\frac{a^2}{c^4}\left(1 - \frac{u}{4}\right)}} du = \int_0^1 \frac{u^{v-1}(1-u)^{w-1} \sum_{m=0}^{\infty} \frac{(2a^2 c^{-4} \tilde{u}^{-2})^m}{m!}}{\left(1 - \frac{u}{4}\right)^{\frac{3}{2}}} du}{2a^{-1} c^2 e^{2\frac{a^2}{c^4}}} \right] \quad (141)$$

Changing the order of summation and integration into (141) yields

$$\frac{\sqrt{\frac{\pi}{2}} \left[\int_0^1 \frac{u^{v-1}(1-u)^{w-1}}{\left(1 - \frac{u}{4}\right)^{\frac{3}{2}} e^{-2\frac{a^2}{c^4}\left(1 - \frac{u}{4}\right)}} du = \sum_{m=0}^{\infty} \frac{(2a^2 c^{-4})^m \int_0^1 \frac{u^{v-1}(1-u)^{w-1}}{\left(1 - \frac{u}{4}\right)^{\frac{3}{2}}} du}{m!} \right]}{2a^{-1} c^2 e^{2\frac{a^2}{c^4}}} \quad (142)$$

Substituting (104) into (142) produces

$$\frac{\sqrt{\frac{\pi}{2}} \left[\int_0^1 \frac{u^{v-1}(1-u)^{w-1}}{\left(1 - \frac{u}{4}\right)^{\frac{3}{2}} e^{-2\frac{a^2}{c^4}\left(1 - \frac{u}{4}\right)}} du = \sum_{m=0}^{\infty} \frac{(2a^2 c^{-4})^m F\left(m + \frac{3}{2}; v, w; \frac{1}{4}\right)}{B^{-1}(v, w)} \right]}{2a^{-1} c^2 e^{2\frac{a^2}{c^4}}} \quad (143)$$

Next, we have to solve the following integral using (109)

$$\frac{\int_0^1 \frac{u^{v-1}(1-u)^{w-1} \text{erf}(\sqrt{2}u)}{\tilde{u}^3 e^{-2\frac{a^2}{c^4}\tilde{u}^2}} du = \int_0^1 \frac{u^{v-1}(1-u)^{w-1} \sum_{m=0}^{\infty} \frac{2^{2m} u'^{2m+1}}{(2m+1)!!}}{\sqrt{\pi} 2^{-\frac{3}{2}} \tilde{u}^3 e^{2u'^2} e^{-2\frac{a^2}{c^4}\tilde{u}^2}} du}{a^{-1} \pi^{-1} 2 \sqrt{2} c^2 e^{2\frac{a^2}{c^4}}} \quad (144)$$

In (144) changing the order of summation and integration and substituting (111) yields

$$\frac{\int_0^1 \frac{u^{v-1}(1-u)^{w-1} \text{erf}(\sqrt{2}u)}{\tilde{u}^3 e^{-2\frac{a^2}{c^4}\tilde{u}^2}} du}{a^{-1} \pi^{-1} 2 \sqrt{2} c^2 e^{2\frac{a^2}{c^4}}} = \frac{\sum_{m=0}^{\infty} \frac{2^{2m} \int_0^1 \frac{u'^{2m+1}}{\tilde{u}^3 e^{2\tilde{u}^2 x_0^2}} du}{(2m+1)!!}}{a^{-1} c^2 e^{2\frac{a^2}{c^4}} \frac{4x_0 a}{c^2}} \quad (145)$$

Next, we substitute (116) into (145) and we obtain

$$\frac{\int_0^1 \frac{u^{v-1}(1-u)^{w-1} \text{erf}(\sqrt{2}u)}{u^{1-v} \tilde{u}^3 e^{-2\frac{a^2}{c^4}\tilde{u}^2}} du}{a^{-1} \pi^{-1} 2 \sqrt{2} c^2 e^{2\frac{a^2}{c^4}}} = \frac{\sum_{m=0}^{\infty} \frac{\int_0^1 \frac{u^{v-1}(1-u)^{w-1}}{u^{1-v} \tilde{u}^{2-2m} e^{2\tilde{u}^2 x_0^2}} du}{2^{-2m} x_0^{-m} (2m+1)!!}}{a^{-1} c^2 e^{2\frac{a^2}{c^4}} \frac{4x_0 a}{c^2}} \quad (146)$$

Changing again the order of summation and integration in and after simplifications (146) yields

$$\frac{\int_0^1 \frac{u^{v-1}(1-u)^{w-1} \text{erf}(\sqrt{2}u)}{u^{1-v} \tilde{u}^3 e^{-2\frac{a^2}{c^4}\tilde{u}^2}} du}{a^{-1} \pi^{-1} 2 \sqrt{2} c^2 e^{2\frac{a^2}{c^4}}} = \frac{\sum_{q=0}^{\infty} \left(\frac{m'}{q} \right) \left[\frac{-a}{c^2} \right]^q \int_0^1 \frac{u^{v-1}(1-u)^{w-1}}{u^{1-v} \tilde{u}^{q-2-2m} e^{2\tilde{u}^2 x_0^2}} du}{\sum_{m=0}^{\infty} \frac{2^{-2m} x_0^{-m} (2m+1)!!}}{a^{-1} c^2 e^{-2x_0^2 - \frac{4x_0 a}{c^2} + 2\frac{a^2}{c^4}}} \quad (147)$$

Finally, it remains to solve the integral in (147) as follows (see Gradshteyn, Ryzhik 2007, [10] pg. 349 3.385)

$$\int_0^1 \frac{u^{v-1}(1-u)^{w-1}}{\left(1 - \frac{u}{4}\right)^{(q-2-2m)} e^{\frac{x_0^2}{2}u}} du = \frac{\Phi_1 \left(v, q-2-2m; v+w; -\frac{x_0^2}{2} \frac{1}{4} \right)}{B^{-1}(v, w)} \quad (148)$$

Next, substituting (147) into (148) we obtain

$$\frac{\int_0^1 \frac{u^{v-1}(1-u)^{w-1} \text{erf}(\sqrt{2}u)}{u^{1-v} \tilde{u}^3 e^{-2\frac{a^2}{c^4}\tilde{u}^2}} du}{a^{-1} \pi^{-1} c^2 e^{2\frac{a^2}{c^4}}} = \frac{\sum_{q=0}^{\infty} \left(\frac{m'}{q} \right) \left[\frac{-a}{c^2} \right]^q \Phi_1 \left(v, q-2-2m; v+w; -\frac{x_0^2}{2} \frac{1}{4} \right)}{\sum_{m=0}^{\infty} \frac{2^{-2m} x_0^{-m} (2m+1)!!}}{B^{-1}(v, w) a^{-1} c^2 e^{-2x_0^2 - \frac{4x_0 a}{c^2} + 2\frac{a^2}{c^4}}} \quad (149)$$

Next, substituting (138), (143), and (149) into (123) yields

$$\int_{x_0}^{\infty} \frac{y \Phi\left(\mu, \frac{3}{2}, \frac{y^2}{2}\right) dy}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} = \frac{\frac{\Phi_1\left(v, 1; w+v; \frac{x_0^2}{4}, \frac{1}{4}\right)}{4e^{2x_0^2 - \frac{2a}{c^2}\left(2x_0 - \frac{a}{c^2}\right)}} + \frac{\sqrt{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{(2a^2 c^{-4})^m F\left(m+\frac{3}{2}, v; w+\frac{1}{4}\right)}{m!}}{2a^{-1} c^2 e^{\frac{2a^2}{c^4}}} - \frac{\sum_{m=0}^{\infty} \frac{\binom{m'}{q} \left[\frac{-a}{c^2}\right]^q \Phi_1\left(v, q-2-2m; v+w; -\frac{x_0^2}{2}, \frac{1}{4}\right)}{2^{-2m} x_0^{-m'} (2m+1)!!}}{a^{-1} c^2 e^{-2x_0^2 - \frac{4x_0 a}{c^2} + \frac{2a^2}{c^4}}} \quad (150)$$

Next, substituting (150) into (92) yields,

$$g_{0k2,N}(a, b, c) = \frac{\frac{\Phi_1\left(v, 1; w+v; \frac{x_0^2}{4}, \frac{1}{4}\right)}{4e^{2x_0^2 - \frac{2a}{c^2}\left(2x_0 - \frac{a}{c^2}\right)}} + \frac{\sqrt{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{(2a^2 c^{-4})^m F\left(m+\frac{3}{2}, v; w+\frac{1}{4}\right)}{m!}}{2a^{-1} c^2 e^{\frac{2a^2}{c^4}}} - \frac{\sum_{m=0}^{\infty} \frac{\binom{m'}{q} \left[\frac{-a}{c^2}\right]^q \Phi_1\left(v, q-2-2m; v+w; -\frac{x_0^2}{2}, \frac{1}{4}\right)}{2^{-2m} x_0^{-m'} (2m+1)!!}}{a^{-1} c^2 e^{-2x_0^2 - \frac{4x_0 a}{c^2} + \frac{2a^2}{c^4}}} \quad (151)$$

Finally, if we substituting (122) and (151) into (90) yields

$$g_{0k,N}(a, b, c) = \frac{\frac{\sqrt{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m F\left(v, \lambda; \lambda+\mu; \frac{1}{4}\right)}{m!}}{2^{\frac{3}{2}}} - \frac{\sum_{q=0}^{\infty} \frac{\binom{m'}{q} \Phi_1\left(\lambda, q-2m; \lambda+\mu; -\frac{x_0^2}{2}, \frac{1}{4}\right)}{\left[\frac{-a}{c^2}\right]^{-q}}}{x_0^{-m'} 2^{-2m} (2m+1)!!}}{a^{-1} \pi^{-\frac{1}{2}} B^{-1}(\lambda, \mu) \Gamma(\lambda) \Gamma^2(\mu) 2^{\lambda-1} e^{\frac{2a^2}{c^4}}} - \frac{\frac{\Phi_1\left(v, 1; w+v; \frac{x_0^2}{4}, \frac{1}{4}\right)}{4e^{2x_0^2 - \frac{2a}{c^2}\left(2x_0 - \frac{a}{c^2}\right)}} + \frac{\sqrt{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{(2a^2 c^{-4})^m F\left(m+\frac{3}{2}, v; w+\frac{1}{4}\right)}{m!}}{2a^{-1} c^2 e^{\frac{2a^2}{c^4}}} - \frac{\sum_{m=0}^{\infty} \frac{\binom{m'}{q} \left[\frac{-a}{c^2}\right]^q \Phi_1\left(v, q-2-2m; v+w; -\frac{x_0^2}{2}, \frac{1}{4}\right)}{2^{-2m} x_0^{-m'} (2m+1)!!}}{a^{-1} c^2 e^{-2x_0^2 - \frac{4x_0 a}{c^2} + \frac{2a^2}{c^4}}} \quad (152)$$

Second, from (2) through (5) and (80) we have

$$g_{1k,N}(x; a, b, c) = \int_{-\infty}^x \frac{D_{k-N}\left[a\left(\frac{1}{c^2} + \frac{t-b}{2a^2}\right)\right]}{e^{2a^2\left(\frac{t-b}{a} - \frac{1}{c^2}\right)^2}} dt \quad (153)$$

Let us make the substitution,

$$y = a\left(\frac{1}{c^2} + \frac{t-b}{2a^2}\right) \quad (154)$$

We have the following

$$dy = \frac{dt}{2a}; \quad t \rightarrow \left|_{-\infty}^x \Rightarrow y \rightarrow \left|_{-\infty}^{a\left(\frac{1}{c^2} + \frac{x-b}{2a^2}\right)} \right. \quad (155)$$

Substituting (155) and (154) into (153) yields

$$g_{1k,N}(a, b, c) = 2a \int_{-\infty}^{x_1} \frac{D_{k-N}(y)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2} e^{-\frac{y^2}{4}}} dy \quad (156)$$

where

$$x_1 = a\left(\frac{1}{c^2} + \frac{x-b}{2a^2}\right) \quad (157)$$

Substituting (86) into (156) yields

$$g_{1k,N}(a, b, c) = \frac{\frac{\Phi\left(\lambda, \frac{1}{2}, \frac{y^2}{2}\right)}{\Gamma(\mu)} \frac{\sqrt{2} y \Phi\left(\mu, \frac{3}{2}, \frac{y^2}{2}\right)}{\Gamma(\lambda)}}{\frac{\sqrt{\pi}}{2^{\lambda}} \int_{-\infty}^{x_1} \frac{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}}{0.5a^{-1}} dy} \quad (158)$$

As we can see the integral (158) can be split into two integrals

$$g_{1k1,N}(a, b, c) = \frac{a\sqrt{\pi}}{2^{\lambda-1}\Gamma(\mu)} \int_{-\infty}^{x_1} \frac{\Phi\left(\lambda, \frac{1}{2}, \frac{y^2}{2}\right)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy \quad (159)$$

and

$$g_{1k2,N}(a, b, c) = -\frac{a\sqrt{2\pi}}{2^{\lambda-1}\Gamma(\lambda)} \int_{-\infty}^{x_1} \frac{y \Phi\left(\mu, \frac{3}{2}, \frac{y^2}{2}\right)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy \quad (160)$$

There are two ways to solve the integral in (159). The first way would be as follows:

$$\int_{-\infty}^{x_1} \frac{\Phi\left(\lambda, \frac{1}{2}, \frac{y^2}{2}\right)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy = 1 - \int_{x_1}^{\infty} \frac{\Phi\left(\lambda, \frac{1}{2}, \frac{y^2}{2}\right)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy \quad (161)$$

Hence, (159) can be obtained from (122) as follows

$$g_{1k1,N}(a, b, c) = 1 - g_{0k1,N}(a, b, c, x_{0 \rightarrow 1}) \quad (162)$$

Similarly, (160) can be obtained from (151) as follows

$$g_{1k2,N}(a, b, c) = 1 - g_{0k2,N}(a, b, c, x_{0 \rightarrow 1}) \quad (163)$$

The second way is by means of direct integration.

Let us try to solve integral (159) based on one definition of the confluent hypergeometric function [11] as follows

$$\int_{-\infty}^{x_1} \frac{\Phi\left(\lambda, \frac{1}{2}, \frac{y^2}{2}\right)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy = \frac{\Gamma\left(\frac{1}{2}\right) \int_{-\infty}^{x_1} \int_0^1 e^{\frac{uy^2}{2}} u^{\lambda-1} (1-u)^{\mu-1} du dy}{\Gamma(\lambda) \Gamma(1-\mu)} \quad (164)$$

Changing the order of integration in (164) produces

$$\int_{-\infty}^{x_1} \frac{\Phi\left(\lambda, \frac{1}{2}, \frac{y^2}{2}\right)}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy = \frac{\int_0^1 u^{\lambda-1} (1-u)^{\mu-1} \int_{-\infty}^{x_1} \frac{e^{\frac{uy^2}{2}}}{e^{2a^2\left(\frac{y}{a} - \frac{1}{c^2}\right)^2}} dy du}{\Gamma^{-1}\left(\frac{1}{2}\right) \Gamma(\lambda) \Gamma(\mu-1)} \quad (165)$$

In order to solve (165) first we have to solve the following integral

$$\int_{-\infty}^{x_1} \frac{e^{\frac{uy^2}{2}}}{e^{2a^2\left(\frac{y}{a}-\frac{1}{c^2}\right)^2}} dy = \int_{-\infty}^{x_1} e^{\frac{uy^2}{2}-2a^2\left(\frac{y}{a}-\frac{1}{c^2}\right)^2} dy \quad (166)$$

From (Gradshteyn, Ryzhik 2007, [10] pg. 108 2.33 ex. 1) we have

$$\int_{-\infty}^{x_1} \frac{e^{\frac{uy^2}{2}}}{e^{2a^2\left(\frac{y}{a}-\frac{1}{c^2}\right)^2}} dy = \frac{1}{2} \frac{e^{-\frac{2a^2}{c^4}} \sqrt{\frac{\pi}{(2-\frac{u}{c^2})}} \operatorname{erf}\left(\sqrt{\frac{2-\frac{u}{c^2}}{2}} y - \frac{2a}{c^2 \sqrt{2-\frac{u}{c^2}}}\right)}{e^{-\frac{2a^2}{c^4(1-\frac{u}{c^2})}}} \Bigg|_{-\infty}^{x_1} \quad (167)$$

Equation (167) can be simplified as

$$\int_{-\infty}^{x_1} \frac{e^{\frac{uy^2}{2}}}{e^{2a^2\left(\frac{y}{a}-\frac{1}{c^2}\right)^2}} dy = \sqrt{\frac{\pi}{2}} \frac{1}{2\tilde{u}} \frac{e^{-2a^2c^{-4}[1+\operatorname{erf}(\sqrt{2}u)]}}{e^{-2a^2c^{-4}\tilde{u}^{-2}}} \quad (168)$$

Without any loss of generality following the derivations from (100) to (121) and keeping in mind (167) we can arrive at the following result

$$\int_{-\infty}^{x_1} \frac{\Phi\left(\lambda, \frac{1}{2}, \frac{y^2}{2}\right) dy}{e^{2a^2\left(\frac{y}{a}-\frac{1}{c^2}\right)^2}} = \left| \begin{array}{l} \frac{\sqrt{\pi} \sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m}{m!} F(v, \lambda; \lambda + \mu; \frac{1}{4})}{\frac{3}{22} e^{\frac{2a^2}{c^4}} B^{-1}(\lambda, \mu) \Gamma^{-1}\left(\frac{1}{2}\right) \Gamma(\lambda) \Gamma(\mu)} + \\ \frac{\sum_{q=0}^{\infty} \binom{m'}{q} x_1^{m'} \Phi_1\left(\lambda, q-2m; \lambda + \mu; -\frac{x_1^2}{2}, \frac{1}{4}\right)}{\frac{\sum_{m=0}^{\infty} \left[\frac{-a}{c^2}\right]^m}{2^{-2m(2m+1)}!!}} \\ \frac{2a^2}{e c^4} - 2x_1^2 - \frac{4x_1 a}{c^2} B^{-1}(\lambda, \mu) \Gamma^{-1}\left(\frac{1}{2}\right) \Gamma(\lambda) \Gamma(\mu) \end{array} \right| \quad (169)$$

Substituting (169) into (159) produces

$$g_{1k1,N}(a, b, c) = \left| \begin{array}{l} \frac{\sqrt{\pi} \sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m}{m!} F(v, \lambda; \lambda + \mu; \frac{1}{4})}{\frac{3}{22} e^{\frac{2a^2}{c^4}} B^{-1}(\lambda, \mu) \Gamma^{-1}\left(\frac{1}{2}\right) \Gamma(\lambda) \Gamma(\mu)} + \\ \frac{\sum_{q=0}^{\infty} \binom{m'}{q} x_1^{m'} \Phi_1\left(\lambda, q-2m; \lambda + \mu; -\frac{x_1^2}{2}, \frac{1}{4}\right)}{\frac{\sum_{m=0}^{\infty} \left[\frac{-a}{c^2}\right]^m}{2^{-2m(2m+1)}!!}} \\ \frac{2a^2}{e c^4} - 2x_1^2 - \frac{4x_1 a}{c^2} B^{-1}(\lambda, \mu) \Gamma^{-1}\left(\frac{1}{2}\right) \Gamma(\lambda) \Gamma(\mu) \end{array} \right| \quad (170)$$

Without any loss of generality, we can obtain

$$g_{1k2,N}(a, b, c) = \left| \begin{array}{l} \frac{\Phi_1\left(v, 1; w + v; \frac{x_1^2}{4}, \frac{1}{4}\right)}{4e^{\frac{2x_1^2}{c^2} - \frac{2a}{c^2}\left(2x_1 - \frac{a}{c^2}\right)}} + \\ \frac{\sqrt{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{(2a^2 c^{-4})^m F\left(m + \frac{3}{2}; v; w + v; \frac{1}{4}\right)}{m!}}{2a^{-1} c^2 e^{\frac{2a^2}{c^4}}} + \\ \frac{\sum_{q=0}^{\infty} \binom{m'}{q} \left[\frac{-a}{c^2}\right]^q \Phi_1\left(v, q-2-2m; v + w; -\frac{x_1^2}{2}, \frac{1}{4}\right)}{2^{-2m} x_1^{-m'} (2m+1)!!} \\ \frac{a^{-1} c^2 e^{-2x_1^2 - \frac{4x_1 a}{c^2} + 2\frac{a^2}{c^4}}}{2^{\lambda-\frac{3}{2}} a^{-1} \Gamma(\lambda) \pi^{-\frac{1}{2}} B^{-1}(v, w) \Gamma^{-1}\left(\frac{3}{2}\right) \Gamma(v) \Gamma(w)} \end{array} \right| \quad (171)$$

Finally, if we substitute (170) and (171) into (158) yields

$$g_{1k,N}(a, b, c) = \left| \begin{array}{l} \frac{\sqrt{\pi} \sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m}{m!} F(v, \lambda; \lambda + \mu; \beta)}{\frac{3}{22} e^{\frac{2a^2}{c^4}} B^{-1}(\lambda, \mu) \Gamma^{-1}\left(\frac{1}{2}\right) \Gamma(\lambda) \Gamma(\mu)} + \\ \frac{\sum_{q=0}^{\infty} \binom{m'}{q} x_1^{m'} \Phi_1\left(\lambda, q-2m; \lambda + \mu; -\frac{x_1^2}{2}, \frac{1}{4}\right)}{\frac{\sum_{m=0}^{\infty} \left[\frac{-a}{c^2}\right]^m}{2^{-2m(2m+1)}!!}} \\ \frac{2a^2}{e c^4} - 2x_1^2 - \frac{4x_1 a}{c^2} B^{-1}(\lambda, \mu) \Gamma^{-1}\left(\frac{1}{2}\right) \Gamma(\lambda) \Gamma(\mu) \end{array} \right| \quad (172)$$

Next, one can substitute (152) and (172) into (81) which yields for the first time the closed form series expression of the cdf of the generalized parabolic cylinder function.

3.2 A Simplified Series Expansion of GPCFD CDF

Now that we have produced GPCFD cdf we will simplify it to enable its calculation via MATLAB.

The first simplification comes from the definition of the Beta function as follows

$$B(\lambda, \mu) = \frac{\Gamma(\lambda) \Gamma(\mu)}{\Gamma(\lambda + \mu)} \Rightarrow \frac{B(\lambda, \mu)}{\Gamma(\lambda) \Gamma(\mu)} = \frac{1}{\Gamma(\lambda + \mu)} \quad (173)$$

The second simplification is due to the numerical values of $\lambda + \mu$ in (89); hence,

$$\frac{B(\lambda, \mu)}{\Gamma(\lambda) \Gamma(\mu)} = \frac{1}{\Gamma(\lambda + \mu)} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi}} \Rightarrow \frac{\sqrt{\pi} B(\lambda, \mu)}{\Gamma(\lambda) \Gamma(\mu)} = 1 \quad (174)$$

The third simplification is because of recursion formula of the gamma function

$$\Gamma(\lambda + 1) = \lambda \Gamma(\lambda) \Rightarrow \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \quad (175)$$

The fourth simplification results from the following

$$\sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m}{m!} F(v, \lambda; \frac{1}{2}, \frac{1}{4}) = \sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m}{m!} \sum_{l=0}^{\infty} \frac{\left(\frac{m-\frac{1}{2}}{2}\right)_l \left(\frac{1}{4}\right)_l}{\left(\frac{1}{2}\right)_l l!} \quad (176)$$

which can be written as

$$\sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m}{m!} F(v, \lambda; \frac{1}{2}, \frac{1}{4}) = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{l+m} \left(\lambda\right)_l \left[\frac{2a^2}{c^4}\right]^m}{\left(\frac{1}{2}\right)_l m! l!} \left(\frac{1}{4}\right)_l \quad (177)$$

Or this can be written with the help of the Kampé de Fériet function [13] as follows

$$\sum_{m=0}^{\infty} \frac{\left[\frac{2a^2}{c^4}\right]^m F\left(v, \lambda; \frac{1}{2}, \frac{1}{4}\right)}{m!} = F_{0:0;1}^{1:0;1} \left[\begin{matrix} -\frac{1}{2}; -; \lambda; \frac{2a^2}{c^4}, \frac{1}{4} \\ -; -; \frac{1}{2}; \end{matrix} \right] \quad (178)$$

Similarly,

$$\sum_{m=0}^{\infty} \frac{(2a^2 c^{-4})^m F\left(m+\frac{3}{2}, v; \frac{3}{2}, \frac{1}{4}\right)}{m!} = F_{0:0;1}^{1:0;1} \left[\begin{matrix} \frac{3}{2}; -; v; \frac{2a^2}{c^4}, \frac{1}{4} \\ -; -; \frac{3}{2}; \end{matrix} \right] \quad (179)$$

Substituting (174) through (179) into (152) and (172) yields

$$g_{0k,N}(a, b, c) = \frac{\sqrt{\pi} F_{0:0;1}^{1:0;1} \left[\begin{matrix} -\frac{1}{2}; -; \lambda; \frac{2a^2}{c^4}, \frac{1}{4} \\ -; -; \frac{1}{2}; \end{matrix} \right]}{\frac{3}{2^2} e^{\frac{2a^2}{c^4}}} + \frac{\sum_{q=0}^{\infty} \binom{m'}{q} \left[\frac{-a}{c^2} \right]^q \Phi_1 \left(\lambda, q-2m; \frac{1}{2}; -\frac{x_0^2}{2}, \frac{1}{4} \right)}{2^{-2m} x_0^{-m} (2m+1)!!} \quad (180)$$

$$- \frac{\sqrt{\frac{\pi}{2}} F_{0:0;1}^{1:0;1} \left[\begin{matrix} \frac{3}{2}; -; v; \frac{2a^2}{c^4}, \frac{1}{4} \\ -; -; \frac{3}{2}; \end{matrix} \right]}{a^{-1} c^2 e^{\frac{2a^2}{c^4}}} + \frac{\sum_{q=0}^{\infty} \binom{m'}{q} \left[\frac{-a}{c^2} \right]^q \Phi_1 \left(v, q-2-2m; \frac{3}{2}; -\frac{x_0^2}{2}, \frac{1}{4} \right)}{2^{-2m} x_0^{-m} (2m+1)!!}$$

$$- \frac{e^{-2x_0^2} - \frac{4x_0 a}{c^2} + \frac{2a^2}{c^4}}{a^{-1} c^2 e^{-2x_0^2} - \frac{4x_0 a}{c^2} + \frac{2a^2}{c^4}} \frac{2^{\lambda-\frac{1}{2}} a^{-1} \Gamma(\lambda) \Gamma^{-1}\left(\frac{1}{2}\right)}{2^{\lambda-\frac{1}{2}} a^{-1} \Gamma(\lambda) \Gamma^{-1}\left(\frac{1}{2}\right)}$$

And

$$g_{1k,N}(a, b, c) = \frac{\sqrt{\pi} F_{0:0;1}^{1:0;1} \left[\begin{matrix} -\frac{1}{2}; -; \lambda; \frac{2a^2}{c^4}, \frac{1}{4} \\ -; -; \frac{1}{2}; \end{matrix} \right]}{\frac{3}{2^2} e^{\frac{2a^2}{c^4}}} + \frac{\sum_{q=0}^{\infty} \binom{m'}{q} x_1^{m'} \left[\frac{-a}{c^2} \right]^q \Phi_1 \left(\lambda, q-2m; \frac{1}{2}; -\frac{x_1^2}{2}, \frac{1}{4} \right)}{2^{-2m} (2m+1)!!} \quad (181)$$

$$- \frac{2a^2}{e^{\frac{2a^2}{c^4}} - 2x_1^2 - \frac{4x_1 a}{c^2} + \frac{2a^2}{c^4}} + \frac{\sum_{q=0}^{\infty} \binom{m'}{q} x_1^{m'} \left[\frac{-a}{c^2} \right]^q \Phi_1 \left(v, q-2-2m; \frac{3}{2}; -\frac{x_1^2}{2}, \frac{1}{4} \right)}{2^{-2m} x_1^{-m} (2m+1)!!}$$

$$- \frac{a^{-1} c^2 e^{-2x_1^2} - \frac{4x_1 a}{c^2} + \frac{2a^2}{c^4}}{a^{-1} c^2 e^{-2x_1^2} - \frac{4x_1 a}{c^2} + \frac{2a^2}{c^4}} \frac{2^{\lambda-\frac{1}{2}} a^{-1} \Gamma(\lambda) \Gamma^{-1}\left(\frac{1}{2}\right)}{2^{\lambda-\frac{1}{2}} a^{-1} \Gamma(\lambda) \Gamma^{-1}\left(\frac{1}{2}\right)}$$

Next, the simplification of the following term is needed:

$$\sum_{m=0}^{\infty} \frac{\sum_{q=0}^{\infty} \binom{m'}{q} x_1^{m'} \left[\frac{-a}{c^2} \right]^q \Phi_1 \left(\lambda, q-2m; \frac{1}{2}; -\frac{x_1^2}{2}, \frac{1}{4} \right)}{2^{-2m} (2m+1)!!} = ? \quad (182)$$

The simplification of the (182) is not easy obtainable. After the simplification of (182) is completed then I believe the next step would be the computation of the closed form expression of

the cdf of GPCFD via MATLAB.

At that time I will also construct a series of theorems that are needed to prove that indeed the GPCFD cdf given by (81) is a valid cdf.

4 Numerical, Theoretical Results

The numerical, theoretical results section contains a few simplified examples.

4.1 Examples

Figure 1 presents three scenarios of the PCFDs pdf and cdf. In Fig. 1 (a) we have assumed that $a = 1$, $p = 1$; and (top) $b = 1$; (bottom) $b = 0$. The results of Fig. 1 (b) are similar to those of Fig. 1 (a) for values of $a = 2$, $p = 2$; and (top) $b = 2$; (bottom) $b = 0$. It appears that the mean of the parabolic cylinder random variable is equal to b . The shape of the parabolic cylinder pdf and cdf is very close to that of a normal or Gaussian. We should also say that GPCFD pdf and cdf exists only for integer values of p . In Fig. 1 (c) we have employed the results of Fig. 1 (b) but we have produced half GPCFD pdf and cdf.

This concluded the discussion on examples and numerical, theoretical results.

Once, the analytical work for the computation of the cdf of the GPCFD is completed and verified then its computation is possible via MATLAB.

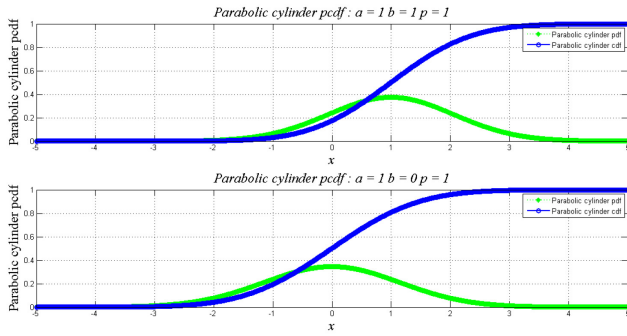
I am going to publish it as soon as it becomes available.

5 Conclusions

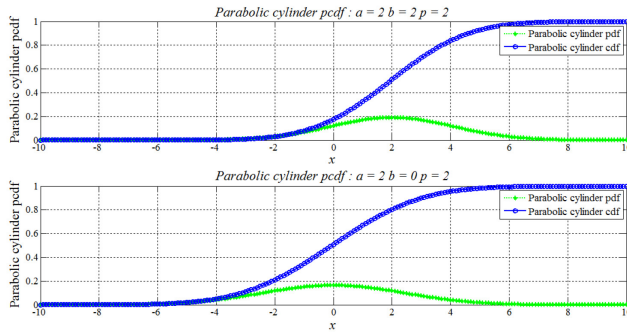
We have been able to produce for the first time the complete discussion of the GPCFD pdf. We have showed that this is a valid pdf function and we have produced the valid normalized coefficient. This work is original and never published before in a journal paper.

We have been able to produce for the first time the closed form expression of the cdf of the GPCFD which contains four Kampé de Fériet functions (Progr 2016, [13]) and six confluent hypergeometric series of two variables [10].

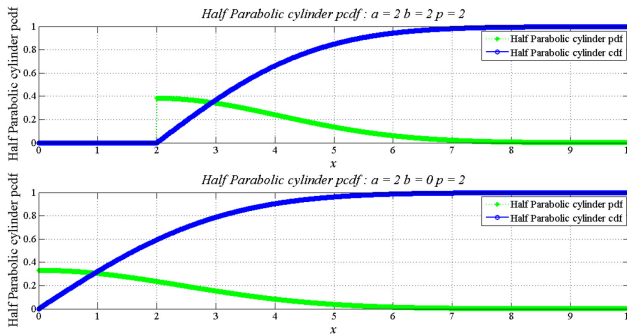
Nevertheless, we do not believe that this closed form expression is simplified enough to enable a fast numerical computation of the cdf of the GPCFD in MATLAB.



(a) GPCFD pdf and cdf for $a = 1$, $p = 1$; and (top) $b = 1$; (bottom) $b = 0$.



(b) GPCFD pdf and cdf for $a = 2$, $p = 2$; and (top) $b = 2$; (bottom) $b = 0$.



(c) GPCFD pdf and cdf for $a = 2$, $p = 2$; and (top) $b = 2$; (bottom) $b = 0$.

FIGURE 1: Numerical calculations of the GPCFD pdf and cdf: (a) $a = 1$, $p = 1$; and (top) $b = 1$, (bottom) $b = 0$; (b) $a = 2$, $p = 2$; and (top) $b = 2$, (bottom) $b = 0$; (c) Half of (b).

We have also shown why more theorems are needed to prove that indeed the closed form expression of the cdf of the GPCFD is indeed a valid cdf.

Once all this work is completed then the differentiation theorem is also needed to be proved to show that indeed the cdf of the GPCFD is exactly the one coming from the pdf of the GPCFD. The differentiation theorem might also lead to the computation of the derivative of the Kampé de Fériet functions (Progri 2016, [13]) and six confluent hypergeometric series of

two variables [10].

Once all this work is completed then the analysis of my initial publication on [16]-[27] need to be reworked to consider scenarios valid for GPCFDs.

6 Acknowledgement

This work was supported by Giftet Inc. executive office.

I want to profoundly thank the MathWorks at Natick, Massachusetts for providing a sponsored MATLAB licence [29] to Giftet Inc. as part of the Indoor Geolocation Systems MATLAB Library development that will enable the results of this work to be published in Dr. Progri pioneer publication *Indoor Geolocation Systems—Theory and Applications. Vol. I* (Not yet available in print) [16].

7 References

- [1] Anon., “Parabolic cylinder function,” *Wikipedia, the free encyclopedia*, Apr. 2015. https://en.wikipedia.org/wiki/Parabolic_cylinder_function
- [2] A.M. Mathai, R.K. Saxena, “Distributions having the gamma properties,” *Metrika*, vol. 13, no. 1, pp 10-16, Dec. 1968. DOI: <https://doi.org/10.1007/BF02613372>
- [3] R. Skjong, H.O. Madsen, “Practical stochastic fatigue analysis of offshore platforms,” *Ocean Eng.*, vol. 14, no. 4, pp. 313-324, 1987. DOI: [https://doi.org/10.1016/0029-8018\(87\)90030-8](https://doi.org/10.1016/0029-8018(87)90030-8)
- [4] R.A. Shore, T.B. Hansen, “Numerical evaluation of the derivative of the parabolic cylinder function,” *IEEE Trans. Ant. Prop.*, vol. 44, no. 9, pp. 1300-1301, Sep. 1996, DOI: <https://doi.org/10.1109/8.535391>
- [5] P. Dharmawansa, N. Rajatheva, K. Ahmed, “On the distribution of the sum of Nakagami- m random variables,” *IEEE Trans. Commun.*, vol. 55, no. 7, pp. 1407-1416, Jul. 2007. doi: <https://doi.org/10.1109/TCOMM.2007.900621>
- [6] R. Annavajjala, “Comments on “Exact error-rate analysis of diversity 16-QAM with channel-estimation Error” *IEEE Trans. Commun.*, vol. 54, no. 3, pp. 393-396. DOI: <https://doi.org/10.1109/TCOMM.2006.869877>
- [7] E.A.J. Marcatili, “Dielectric guide with curved axis and truncated parabolic index,” *The Bell System Technical Journal*, vol. 49, no. 8, pp. 1645-1663. DOI: <http://doi.org/10.1002/j.1538-7305.1970.tb04283.x>

- [8] Anon., "Parabolic cylinder function," Copyright © 1999-2016 Wolfram Research, Inc., All rights reserved, <http://mathworld.wolfram.com/ParabolicCylinderFunction.html>.
- [9] E. Cojocaru, "Parabolic cylinder functions," *MATLAB Toolbox*, Department of Theoretical Physics, Horia Hulubei, National Institute of Physics and Nuclear Engineering, Romania, Jan. 2009.
- [10] I.S. Gradshteyn, I.M. Ryzhik, (A. Jeffrey, D. Zwillinger, editors) *Table of Integrals, Series, and Products*, 7th ed., Burlington, MA: Academic Press, 1171 pp., 2007.
- [11] Anon., "Confluent hypergeometric function," *Wikipedia, the free encyclopedia*, Jan. 2016. https://en.wikipedia.org/wiki/Confluent_hypergeometric_function
- [12] G.B. Arfken, H.J. Weber, *Mathematical Methods for Physicists*, San Diego, CA: Academic Press, 1995.
- [13] I. Proгри, "Generalized Bessel function distributions," *J. Geol. Geoinfo. Geointel.*, vol. 2016, article ID 2016071602, 14 pg., Nov. 2016. DOI: <https://doi.org/10.18610/JG3.2016.071602>
- [14] I.F. Proгри, P. Huang, Y. Pi, X. Xia, "Indoor adaptive GNSS signal acquisition—theory and simulations," in *Proc. ION ITM 2006*, Monterey, CA, pg. 902-931, Jan. 2016.
- [15] I.F. Proгри, P. Huang, Y. Pi, X. Xia, "Indoor adaptive GNSS signal acquisition. Part 1: theory and simulations," *J. Geol. Geoinfo. Geointel.*, vol. 2016, article ID 2016071601, 19 pg., Nov. 2016. DOI: <https://doi.org/10.18610/JG3.2016.071601>
- [16] I. Proгри, *Indoor Geolocation Systems—Theory and Applications. Vol. I*, 1st ed., Worcester, MA: Giftet Inc., ~800 pp., ~2017 (not yet available in print).
- [17] I. Proгри, "On generalized multi-dimensional geolocation modulation waveforms," in *Proc. IEEE/ION-PLANS 2012*, Myrtle Beach, SC, pp. 919-951, Apr. 2012, DOI: <https://doi.org/10.1109/PLANS.2012.6236835>
- [18] I. Proгри, M.C. Bromberg, W.R. Michalson, J. Wang, "A theoretical survey of the spreading modulation of the new GPS signals (L1C, L2C, and L5)," in *Proc. ION-NTM 2007*, San Diego, CA, pp. 561-569, Jan. 2007.
- [19] I. Proгри, "GPS L5 signal acquisition and tracking under unintentional interference or jamming," in *Proc. ION-NTM 2006*, Monterey, CA, pp. 112-121 Jan. 2006.
- [20] M.C. Bromberg, I. Proгри, "An anti-jam GPS receiver, using Markov Chain, Monte Carlo integration," in *Proc. IEEE/ION PLANS 2006*, Coronado, CA, pp. 957-962, Apr. 2006. DOI: <https://doi.org/10.1109/PLANS.2006.1650696>
- [21] M.C. Bromberg, and I. Proгри, "Bayesian parameter estimation for time and frequency synchronization," in *Proc. WTS 2005*, Pomona, CA, pp. 127-130, Apr. 2005. DOI: <https://doi.org/10.1109/WTS.2005.1524776>
- [22] I. Proгри, M.C. Bromberg, W.R. Michalson, "Maximum likelihood GPS parameter estimation," *Navigation*, vol. 52, no. 4, pp. 229-238, winter 2005-2006. DOI: <http://doi.org/10.1002/j.2161-4296.2005.tb00365.x>
- [23] I. Proгри, M.C. Bromberg, J. Wang, "Markov Chain, Monte Carlo global search and integration for Bayesian, GPS, parameter estimation," *Navigation*, vol. 56, no. 3, pp. 195-204, fall 2009. DOI: <http://doi.org/10.1002/j.2161-4296.2009.tb01755.x>
- [24] W.R. Michalson, I. Proгри, "Reconfigurable geolocation system," US Patent 7,079,025, July 2006.
- [25] I.F. Proгри, "An assessment of indoor geolocation systems," *Ph.D. Dissertation*, Worcester Polytechnic Institute, May 2003.
- [26] I. Proгри, W.R. Michalson, J. Wang, and M.C. Bromberg, "Indoor geolocation using FCDMA pseudolites: signal structure and performance analysis," *Navigation*, vol. 54, no. 3, pp. 242-256, fall 2007. DOI: <http://doi.org/10.1002/j.2161-4296.2007.tb00407.x>
- [27] I. Proгри, *Geolocation of RF Signals—Principles and Simulations*. 1st ed., New York, NY: Springer SBM, LLC, 330 pp., Jan. 2011. DOI: <http://doi.org/10.1007/978-1-4419-7952-0>
- [28] Anon, "MATLAB 2016b," *The MathWorks, Inc.*, Natick, MA, Copyright © 1994-2016, The MathWorks, Inc., <https://mathworks.com/products/matlab.html>.

ⁱ The parabolic cylinder functions are a class of functions sometimes called Weber functions. There are a number of slightly different definitions in use by various authors [8]. The credit for introducing this function in the "Indoor adaptive GNSS signal acquisition. Part 1: theory and simulations" paper (see Proгри et al. 2016, [14], [15]) goes to my co-author P. Huang since 2010 or early 2011. The credit for actually naming the function and performing the computations and the research goes to Dr. Proгри since January of 2016. The PCFs are used to compute the PCFDs which are a special class of a multivariate Gaussian distribution. In this paper we will only show that PCFDs produces

valid pdfs. We will neither show how to compute the main statistics nor will we show how to compute the closed form expression of the PCFDs cdf and later its inverse. The computation of the PCFDs pdf was performed using the MATLAB toolbox developed by Cojocaru [9] in January 2009. As far as I can tell the PCFDs are introduced for the first time in the navigation community and as of January 2016 there were only four papers [4]-[7] that had discussion on parabolic cylinder functions in the IEEE Xplore; hence, Proгри et al. 2016 [14], [15] presented a unique opportunity to introduce and discuss the PCFDs in a navigation related topic namely adaptive GPS signal detection based on

DCAC.

ⁱⁱ Reformatted in 2021 due to enumeration errors of the equation numbers.