

Research Article

Hypergeometric Function Partial Derivatives

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The computation of the hypergeometric function partial derivatives when the hypergeometric function coefficients are function of the same parameter is discussed in this paper. Initially, special cases are considered as the computation of hypergeometric function partial derivatives is straight forward. In the case of Euler integral, Gauss power series (or Binomial expansion without and with maneuvering), and Taylor expansion, the computation of hypergeometric function partial derivatives leads to the creation of new entries in the Table of Integrals, Series, and Products. This paper is based almost entirely on the creation of original analytical derivation but as far as numerical results special cases may be considered as such.

Index Terms—Hypergeometric functions, partial derivatives, Euler integral, Gauss power series, Binomial expansion, maneuvering, Taylor expansion, special cases, Table of Integrals, Series, Products.

1 Introduction

The computation of the hypergeometric function partial derivatives is a very important problem that has occurred and may occur more frequently than we think in the future.

For the first time I came across with the computation of the hypergeometric function partial derivatives as I was preparing Chap. 8 of my pioneer publication on *Indoor Geolocation Systems—Theory and Applications* [1] in 2013.

I needed to compute the statistics of a new distribution model called the exponential generalized beta distribution [1], (see also Progri 2016 [2]) based on some initial work presented in [3].

Although I spent nearly several weeks and a few months with my derivations it appeared to me that the solution seemed too laborious and it would almost lend itself to unknown errors or mistakes which was the main reason why I decided at the time not to publish my book and my derivations on this subject matter.

It was not until last year that I created Giftet Journal of Geolocation, Geo-information, and Geo-intelligence specifically to give me the opportunity to thoroughly investigate and publish problems such as the computation of the hypergeometric function partial derivatives and the computation of exponential generalized beta distribution [2].

The first mention of the hypergeometric function partial derivatives with respect to its coefficients is given by Rassias, Srivastava (2002, [4]). This mention is very brief and it is not connected to any of the applications. The only contribution coming from the mention of Rassias, Srivastava (2002, [4]) is with respect to the generalized harmonic numbers.

Various algorithms have been developed over the years that deal with creation of identities of hypergeometric functions [5] (or even generalized hypergeometric functions [6]) such Gessel et. al. 1982 [7], Gessel 1995 [8], Koepf et al. 1995 [9], and Bagdasaryan 2010 [10], just to name a few.

The work discussed in this paper is entirely original, novel, and innovative. It is not based on any similar work presented in the literature.

The main theme in this paper is to present several methods that discuss *Hypergeometric Function Partial Derivatives* and connections among each method.

The methods discussed in this paper appear to be particularly useful during the computation of probability density functions (pdfs) [11], or cumulative distribution functions (cdfs) [12], or their statistics based on methods discussed in Mathai, Provost 2004, [13] (or see Progi 2016, [2] for a brilliant application of the above).

This paper is organized as follows: in Sect. 2, hypergeometric function partial derivatives computation of special cases is discussed. In Sect. 3, hypergeometric function partial derivatives computation via Euler integral form is presented. Section 4 contains hypergeometric function partial derivatives

computation of special cases via Euler integral form. In Sect. 5, the discussion of the hypergeometric function partial derivatives computation via Gauss power Series (or binomial expansion) is treated. Section 6 contains similar discussion as from Sect. 5 but with maneuvering; i.e., in Sect. 6 hypergeometric function via Gauss power series (or Binomial expansion) with maneuvering is discussed. Section 7 contains the computation of the hypergeometric function partial derivatives when the hypergeometric function is expanded via Taylor power series expansion (of the exponential function). In Sect. 8 a numerical example is considered. Conclusion is provided in Sect. 9 along with a list of references.

2 Hypergeometric Function Partial Derivatives of Special Cases

The computation of the hypergeometric function partial derivatives of [some] special cases is very important because special cases are, in the end, employed to test or validate our analytical derivations [5]. If it were not for special cases, we would have almost no confidence that our analytical derivations are correct and our treatment would almost certainly contain errors.

Before, I discuss special cases let us consider the formulation of the problem, which is the notation that is being used and the condition of the parameters.

Letting $F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]$ denote a hypergeometric function

$$F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \equiv {}_2F_1 \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \equiv {}_2F_1 \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \equiv {}_2F_1 \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \equiv {}_2F_1[a(t), b(t); c(t); z] \quad (1)$$

where $a(t)$, $b(t)$, and $c(t)$ are simple functions of t and $0 \leq z \leq 1$. There are applications that require the computation of the partial derivatives with respect to t and then their evaluation at $t = 0$; i.e.,

$$\left. \frac{\partial F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial t} \right|_{t=0} \equiv \left. \frac{\partial {}_2F_1 \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial t} \right|_0 \quad (2)$$

Or

$$\left. \frac{\partial \log \left\{ F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \right\}}{\partial t} \right|_{t=0} \equiv \left. \frac{\partial \log \left\{ {}_2F_1 \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \right\}}{\partial t} \right|_0 \quad (3)$$

Initially we assume that $a(t = 0) \neq 0$, $b(t = 0) \neq 0$, $c(t = 0) \neq 0$, $\left. \frac{da(t)}{dt} \right|_{t=0} \neq 0$, $\left. \frac{db(t)}{dt} \right|_{t=0} \neq 0$, and $\left. \frac{dc(t)}{dt} \right|_{t=0} \neq 0$.

The immediate questions are: What will this function look like? How do we compute something like this?

Although there exists a short list of identities for particular values of z [7], [8] in general there is no expression that shows how a $F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right]$ can be expressed with the help of other functions which makes very difficult the computation of (2).

Initially, let us compute the special cases solution for $z = 0$, $z = 1/2$ and for $z = 1$ and then approach the most general case.

First, for $z = 0$

$$\left. \frac{\partial F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z=0\right]}{\partial t} \right|_{t=0} = \frac{\partial 1}{\partial t} \Big|_{t=0} = 0 \quad (4)$$

Second, for $z = 1/2$ and for $c(t) = \frac{1+a(t)+b(t)}{2}$ we obtain the Gauss second summation theorem [5] that can be written by the help of a compact notation (Gessel 1995, [8]) as follows

$$F\left[\begin{matrix} a(t), b(t); \\ \frac{1+a(t)+b(t)}{2}; \end{matrix} z = \frac{1}{2}\right] = \Gamma\left[\begin{matrix} \frac{1}{2}; \frac{1+a(t)+b(t)}{2} \\ \frac{1+a(t)}{2}; \frac{1+b(t)}{2} \end{matrix}\right] \equiv \frac{\Gamma\left[\frac{1}{2}\right]\Gamma\left[\frac{1+a(t)+b(t)}{2}\right]}{\Gamma\left[\frac{1+a(t)}{2}\right]\Gamma\left[\frac{1+b(t)}{2}\right]} \quad (5)$$

Taking the partial derivative with respect to t based on the identity

$$\frac{\partial F(t)}{\partial t} = \frac{\partial [e^{\log[F(t)]}]}{\partial t} = \frac{F(t) \partial [\log[F(t)]]}{\partial t} \quad (6)$$

or

$$\frac{\partial F\left[\begin{matrix} a(t), b(t); \\ \frac{1+a(t)+b(t)}{2}; \end{matrix} z = \frac{1}{2}\right]}{\partial t} = \frac{F\left[\begin{matrix} a(t), b(t); \\ \frac{1+a(t)+b(t)}{2}; \end{matrix} z = \frac{1}{2}\right] \partial \log\left\{\Gamma\left[\begin{matrix} \frac{1}{2}; \frac{1+a(t)+b(t)}{2} \\ \frac{1+a(t)}{2}; \frac{1+b(t)}{2} \end{matrix}\right]\right\}}{\partial t} \quad (7)$$

or

$$\left. \frac{\partial F\left[\begin{matrix} a(t), b(t); \\ \frac{1+a(t)+b(t)}{2}; \end{matrix} z = \frac{1}{2}\right]}{\partial t} \right|_{t=0} = \frac{\Gamma\left[\begin{matrix} \frac{1}{2}; \frac{1+a(t)+b(t)}{2} \\ \frac{1+a(t)}{2}; \frac{1+b(t)}{2} \end{matrix}\right] \left\{ \frac{\psi\left[\frac{1+a(t)+b(t)}{2}\right] d[a(t)+b(t)]}{dt} - \frac{\psi\left[\frac{1+a(t)}{2}\right] d[a(t)]}{dt} - \frac{\psi\left[\frac{1+b(t)}{2}\right] d[b(t)]}{dt} \right\}}{2} \Big|_{t=0} \quad (8)$$

Third, for $z = 1/2$ and for $b(t) = 1 - a(t)$ we obtain the Bailey's summation theorem [5] as follows

$$F\left[\begin{matrix} a(t), 1-a(t); \\ c(t); \end{matrix} z = \frac{1}{2}\right] \equiv \Gamma\left[\begin{matrix} \frac{c(t)}{2}, \frac{1+c(t)}{2} \\ \frac{c(t)+a(t)}{2}, \frac{1+c(t)-a(t)}{2} \end{matrix}\right] \quad (9)$$

Similarly,

$$\left. \frac{\partial F\left[\begin{matrix} a(t), 1-a(t); \\ c(t); \end{matrix} z = \frac{1}{2}\right]}{\partial t} \right|_{t=0} = \frac{\Gamma\left[\begin{matrix} \frac{c(t)}{2}, \frac{1+c(t)}{2} \\ \frac{c(t)+a(t)}{2}, \frac{1+c(t)-a(t)}{2} \end{matrix}\right] \left\{ \frac{\psi\left[\frac{c(t)}{2}\right] d[c(t)]}{dt} + \frac{\psi\left[\frac{1+c(t)}{2}\right] d[c(t)]}{dt} - \frac{\psi\left[\frac{c(t)+a(t)}{2}\right] d[c(t)+a(t)]}{dt} - \frac{\psi\left[\frac{1+c(t)-a(t)}{2}\right] d[c(t)-a(t)]}{dt} \right\}}{2} \Big|_{t=0} \quad (10)$$

If we were to assume that in the Gauss summation $b(t) = 1 - a(t)$; i.e., $a(t) + b(t) = 1$ and in the Bailey's summation theorem $c(t) = 1$ then Gauss second summation theorem and Bailey's summation theorem are identical as follows

$$F\left[\begin{matrix} a(t), 1-a(t); \\ 1; \end{matrix} z = \frac{1}{2}\right] = \Gamma\left[\begin{matrix} \frac{1}{2}; 1 \\ \frac{1+a(t)}{2}; 1 - \frac{a(t)}{2} \end{matrix}\right] \equiv \frac{\Gamma\left[\frac{1}{2}\right]}{\Gamma\left[\frac{1+a(t)}{2}\right]\Gamma\left[1 - \frac{a(t)}{2}\right]} \quad (11)$$

And the derivative of $\left. \frac{\partial F\left[\begin{matrix} a(t), 1-a(t); \\ 1; \end{matrix} z = \frac{1}{2}\right]}{\partial t} \right|_{t=0}$ is equal to

$$\left. \frac{\partial F\left[\begin{matrix} a(t), 1-a(t); \\ 1; \end{matrix} \middle| z=\frac{1}{2} \right]}{\partial t} \right|_{t=0} = \frac{\Gamma\left[\begin{matrix} \frac{1}{2}, 1 \\ \frac{1+a(t)}{2}, 1-\frac{a(t)}{2} \end{matrix}\right] \left\{ \psi\left[1-\frac{a(t)}{2}\right] - \psi\left[\frac{1+a(t)}{2}\right] \right\} \frac{d[a(t)]}{dt}}{2} \bigg|_{t=0} \quad (12)$$

Fourth, for $z = 1$ we obtain the Gauss identity as follows [5]

$$F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} \middle| z = 1 \right] = \Gamma\left[\begin{matrix} c(t), c_{ab}(t) \\ c_a(t), c_b(t) \end{matrix}\right] \quad (13)$$

where

$$c_{ab}(t) = c(t) - a(t) - b(t) = c(t) - b(t) - a(t) \equiv c_{ba} \quad (14)$$

$$c_a(t) = c(t) - a(t) \quad (15)$$

$$c_b(t) = c(t) - b(t) \quad (16)$$

Similarly

$$\left. \frac{\partial F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} \middle| z=1 \right]}{\partial t} \right|_{t=0} = \frac{\frac{\psi[c(t)]d[c(t)]}{dt} + \frac{\psi[c_{ab}(t)]d[c_{ab}(t)]}{dt} - \frac{\psi[c_a(t)]d[c_a(t)]}{dt} - \frac{\psi[c_b(t)]d[c_b(t)]}{dt}}{\Gamma^{-1}\left[\begin{matrix} c(t), c_{ab}(t) \\ c_a(t), c_b(t) \end{matrix}\right]}} \bigg|_{t=0} \quad (17)$$

Rearranging the terms of (17) we obtain

$$\left. \frac{\partial F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} \middle| z=1 \right]}{\partial t} \right|_{t=0} = \frac{\frac{\{\psi[c_a(t)] - \psi[c_{ab}(t)]\}d[a(t)]}{dt} + \frac{\{\psi[c_b(t)] - \psi[c_{ab}(t)]\}d[b(t)]}{dt} + \frac{\{\psi[c(t)] + \psi[c_{ab}(t)] - \psi[c_a(t)] - \psi[c_b(t)]\}d[c(t)]}{dt}}{\Gamma^{-1}\left[\begin{matrix} c(t), c_{ab}(t) \\ c_a(t), c_b(t) \end{matrix}\right]}} \bigg|_{t=0} \quad (18)$$

We can see that in this case, a closed form expression of (2) exists for special values of $z = \{0,1\}$ regardless of $a(t), b(t), c(t)$

and for some $z = 1/2$ for a special arrangement of $a(t), b(t), c(t) = \frac{1+a(t)+b(t)}{2}$ or $a(t), 1-b(t), c(t)$.

In general, however, we can only find a complicated closed form expression for $0 < z < 1$ and an exact, less complicated closed form expression for $z = \{0,1\}$ and for $z = 1/2$ for special arrangement of $a(t), b(t), c(t) = \frac{1+a(t)+b(t)}{2}$ or $a(t), 1-b(t), c(t)$.

3 Hypergeometric Function Partial Derivatives Computation via Euler Integral Form

Hypergeometric function via Euler Integral, since, Euler was the first to have studies its integral representation [5].

Let us consider the more general case; hence, $0 < z < 1$, $z \neq 1/2$ the integral representation of the hypergeometric function [5] is

$$B[b(t), c_b(t)] F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} \middle| z \right] = \int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} (1-zx)^{-a(t)} dx \quad (19)$$

If we were to make the substitution $x = e^{-y}$ then we obtain another identity of the integral representation of the hypergeometric function [5]

$$B[b(t), c_b(t)] F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} \middle| z \right] = \int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-a(t)} dy \quad (20)$$

Taking the partial derivative of both sides of either (19) or (20) produces

$$\frac{\partial \left\{ B[b(t), c_b(t)] F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} \middle| z \right] \right\}}{\partial t} = \frac{\partial \left\{ \int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} (1-zx)^{-a(t)} dx \right\}}{\partial t} \quad (21)$$

Or

$$\frac{\partial \left\{ B[b(t), c_b(t)] F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} \middle| z \right] \right\}}{\partial t} = \frac{\partial \left\{ \int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-a(t)} dy \right\}}{\partial t} \quad (22)$$

The left side of either (21) or (22) yields

$$\frac{\partial \{B[b(t), c_b(t)] F\left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix}\right]\}}{\partial t} = \frac{F\left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix}\right] \partial \{B[b(t), c_b(t)]\}}{\partial t} + \frac{B[b(t), c_b(t)] \partial \left\{F\left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix}\right]\right\}}{\partial t} \quad (23)$$

The partial derivative of $\frac{\partial \{B[b(t), c_b(t)]\}}{\partial t}$ can be easily computed as

$$\frac{\partial \{B[b(t), c_b(t)]\}}{\partial t} = B[b(t), c_b(t)] \left[\frac{\psi[b(t)] d[b(t)]}{dt} + \frac{\psi[c_b(t)] d[c_b(t)]}{dt} - \frac{\psi[c(t)] d[c(t)]}{dt} \right] \quad (24)$$

Next, substituting (24) into (23) yields

$$\frac{\partial \{B[b(t), c_b(t)] F\left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix}\right]\}}{\partial t} = \frac{F\left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix}\right] \left[\frac{\psi[b(t)] d[b(t)]}{dt} + \frac{\psi[c_b(t)] d[c_b(t)]}{dt} - \frac{\psi[c(t)] d[c(t)]}{dt} \right] + \frac{\partial \left\{F\left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix}\right]\right\}}{\partial t}}{B^{-1}[b(t), c_b(t)]} \quad (25)$$

The derivative of the right hand side of (21) and (22) reads with the help of Leibniz integral rule [14], or differentiation under the integral sign rule [15], or Reynolds transport theorem [16] as follows

$$\frac{\partial \left[\int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} (1-zx)^{-a(t)} dx \right]}{\partial t} = \int_0^1 \frac{\partial \left[x^{b(t)-1} (1-x)^{c_b(t)-1} (1-zx)^{-a(t)} \right]}{\partial t} dx \quad (26)$$

or

$$\frac{\partial \left[\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-a(t)} dy \right]}{\partial t} = \int_0^\infty \frac{\partial \left[e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-a(t)} \right]}{\partial t} dy \quad (27)$$

Next, taking the partial derivative of (26) and (27) yields

$$\frac{\partial \left[x^{b(t)-1} (1-x)^{c_b(t)-1} (1-zx)^{-a(t)} \right]}{\partial t} = \left[\frac{(1-x)^{c_b(t)-1} (1-zx)^{-a(t)} \partial [x^{b(t)-1}]}{\partial t} + \frac{x^{b(t)-1} (1-zx)^{-a(t)} \partial [(1-x)^{c_b(t)-1}]}{\partial t} + \frac{x^{b(t)-1} (1-x)^{c_b(t)-1} \partial [(1-zx)^{-a(t)}]}{\partial t} \right] \quad (28)$$

or

$$\frac{\partial \left[e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-a(t)} \right]}{\partial t} = \left[\frac{(1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-a(t)} \partial [e^{-b(t)y}]}{\partial t} + \frac{e^{-b(t)y} (1-ze^{-y})^{-a(t)} \partial [(1-e^{-y})^{c_b(t)-1}]}{\partial t} + \frac{e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} \partial [(1-ze^{-y})^{-a(t)}]}{\partial t} \right] \quad (29)$$

Further expansion of (28) and (29) yields

$$\frac{\partial \left[x^{b(t)-1} (1-x)^{c_b(t)-1} (1-zx)^{-a(t)} \right]}{\partial t} = \varphi[a(t), b(t), c(t); x, z] \left\{ \frac{\log(x) d[b(t)]}{dt} + \frac{\log(1-x) d[c_b(t)]}{dt} - \frac{\log(1-zx) d[a(t)]}{dt} \right\} \quad (30)$$

or

$$\frac{\partial \left[e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-a(t)} \right]}{\partial t} = \varphi[a(t), b(t), c(t); x, z] \left\{ -\frac{y d[b(t)]}{dt} + \frac{\log(1-e^{-y}) d[c_b(t)]}{dt} - \frac{\log(1-ze^{-y}) d[a(t)]}{dt} \right\} \quad (31)$$

Where

$$\varphi[a(t), b(t), c(t); x, z] = x^{b(t)-1} (1-x)^{c_b(t)-1} (1-zx)^{-a(t)} \quad (32)$$

$$\delta[a(t), b(t), c(t); y, z] = e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-a(t)} \quad (33)$$

Next, substituting (30) and (31) into (26) and (27) reads

$$\frac{\partial \left[\int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} (1-zx)^{-a(t)} dx \right]}{\partial t} = \int_0^1 \left[\varphi[a(t), b(t), c(t); x, z] \left[\frac{\log(x) d[b(t)]}{dt} + \frac{\log(1-x) d[c_b(t)]}{dt} - \frac{\log(1-zx) d[a(t)]}{dt} \right] \right] dx \quad (34)$$

or

$$\frac{\partial \left[\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-a(t)} dy \right]}{\partial t} = \int_0^\infty \delta[a(t), b(t), c(t); y, z] \left[\begin{array}{c} -\frac{yd[b(t)]}{dt} \\ \frac{\log(1-e^{-y})d[c_b(t)]}{dt} \\ -\frac{\log(1-ze^{-y})d[a(t)]}{dt} \end{array} \right] dy \quad (35)$$

Next, equating (25) with either (34) or (35) yields

$$\frac{F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \left[\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \right] + \frac{\partial \left\{ F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \right\}}{\partial t}}{B^{-1}[b(t), c_b(t)]} = \int_0^1 \left[\varphi[a(t), b(t), c(t); x, z] \left[\begin{array}{c} \frac{\log(x)d[b(t)]}{dt} \\ \frac{\log(1-x)d[c_b(t)]}{dt} \\ -\frac{\log(1-zx)d[a(t)]}{dt} \end{array} \right] \right] dx \quad (36)$$

Or

$$\frac{F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \left[\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \right] + \frac{\partial \left\{ F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \right\}}{\partial t}}{B^{-1}[b(t), c_b(t)]} = \int_0^\infty \delta[a(t), b(t), c(t); y, z] \left[\begin{array}{c} -\frac{yd[b(t)]}{dt} \\ \frac{\log(1-e^{-y})d[c_b(t)]}{dt} \\ -\frac{\log(1-ze^{-y})d[a(t)]}{dt} \end{array} \right] dy \quad (37)$$

Rearranging (36) and (37) and setting the partial derivative of $\frac{\partial \left\{ F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \right\}}{\partial t} \Big|_{t=0}$ produces

$$\frac{\partial \left\{ F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \right\}}{\partial t} \Big|_{t=0} = \frac{\int_0^1 \varphi[a(t), b(t), c(t); x, z] \left[\begin{array}{c} \frac{\log(x)d[b(t)]}{dt} \\ \frac{\log(1-x)d[c_b(t)]}{dt} \\ -\frac{\log(1-zx)d[a(t)]}{dt} \end{array} \right] dx \Big|_{t=0}}{B[b(t), c_b(t)]|_{t=0}} - \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{F^{-1}\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \Big|_{t=0}} \quad (38)$$

Or

$$\frac{\partial \left\{ F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \right\}}{\partial t} \Big|_{t=0} = \frac{\int_0^\infty \delta[a(t), b(t), c(t); y, z] \left[\begin{array}{c} -\frac{yd[b(t)]}{dt} \\ \frac{\log(1-e^{-y})d[c_b(t)]}{dt} \\ -\frac{\log(1-ze^{-y})d[a(t)]}{dt} \end{array} \right] dy \Big|_{t=0}}{B[b(t), c_b(t)]|_{t=0}} - \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{F^{-1}\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \Big|_{t=0}} \quad (39)$$

Equations (38) and (39) represent the general formulas of the partial derivative $\frac{\partial \left\{ F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \right\}}{\partial t}$ in the integral form evaluated at $t = 0$.

The first question is: Are these equations unique? From the properties of the hypergeometric function [5] we have

$$F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \equiv F\left[\begin{matrix} b(t), a(t); \\ c(t); \end{matrix} z\right] \quad (40)$$

Hence, in (38) and (39) we can interchange $a(t)$ with $b(t)$ and vice versa and we can obtain a new set of identical equations as follows

$$\frac{\partial \left\{ F\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \right\}}{\partial t} \Big|_{t=0} = \frac{\int_0^1 \varphi[b(t), a(t), c(t); x, z] \left[\begin{array}{c} \frac{\log(x)d[a(t)]}{dt} \\ \frac{\log(1-x)d[c_b(t)]}{dt} \\ -\frac{\log(1-zx)d[b(t)]}{dt} \end{array} \right] dx \Big|_{t=0}}{B[a(t), c_b(t)]|_{t=0}} - \frac{\frac{\psi[a(t)]d[a(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{F^{-1}\left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z\right] \Big|_{t=0}} \quad (41)$$

Or

$$\left. \frac{\partial \left\{ {}_F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \right\}}{\partial t} \right|_{t=0} = \frac{\int_0^\infty \delta[b(t), a(t), c(t); y, z] \left| \begin{matrix} -\frac{y d[a(t)]}{dt} \\ + \frac{\log(1-e^{-y}) d[c_b(t)]}{dt} \\ - \frac{\log(1-ze^{-y}) d[b(t)]}{dt} \end{matrix} dy \right|_{t=0}}{B[a(t), c_a(t)]|_{t=0}} - \frac{\frac{\psi[a(t)] d[a(t)]}{dt} + \frac{\psi[c_b(t)] d[c_b(t)]}{dt} - \frac{\psi[c(t)] d[c(t)]}{dt}}{F^{-1} \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \Big|_{t=0}} \quad (42)$$

So, I believe that (38), (39), (41), and (42) present the complete set of identical equations of $\frac{\partial \left\{ {}_F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \right\}}{\partial t}$ in the integral form evaluated at $t = 0$.

4 Hypergeometric Function Partial Derivatives Computation of Special Cases via Euler Integral Form

There are several special cases that we can recognize in either (38) or (39).

First, if $a(t) \equiv a$, $b(t) \equiv b$, $c(t) \equiv c$ are constants then from either (38) or (39) we obtain

$$\left. \frac{\partial \left\{ {}_F \left[\begin{matrix} a(t) \equiv a, b(t) \equiv b; \\ c(t) \equiv c; \end{matrix} z \right] \right\}}{\partial t} \right|_{t=0} = 0 \quad (43)$$

as it should be.

Second, if $b(t) \equiv b$, $c(t) \equiv c$ are constants then from either (38) or (39) we obtain

$$\left. \frac{\partial \left\{ {}_F \left[\begin{matrix} a(t), b; \\ c; \end{matrix} z \right] \right\}}{\partial t} \right|_{t=0} = \frac{-\int_0^1 x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a(t)} \log(1-zx) \frac{d[a(t)]}{dt} dx \Big|_{t=0}}{B[b, c-b]} \quad (44)$$

Or

$$\left. \frac{\partial \left\{ {}_F \left[\begin{matrix} a(t), b; \\ c; \end{matrix} z \right] \right\}}{\partial t} \right|_{t=0} = \frac{-\int_0^\infty e^{-by} (1-e^{-y})^{c-b-1} (1-ze^{-y})^{-a} \log(1-ze^{-y}) \frac{d[a(t)]}{dt} dy \Big|_{t=0}}{B[b, c-b]} \quad (45)$$

And from (41) and (42) we obtain

$$\left. \frac{\partial \left\{ {}_F \left[\begin{matrix} a(t), b; \\ c; \end{matrix} z \right] \right\}}{\partial t} \right|_{t=0} = \frac{\int_0^1 x^{a(t)-1} (1-x)^{c-a(t)-1} (1-zx)^{-b} [\log(x) - \log(1-x)] \frac{d[a(t)]}{dt} dx \Big|_{t=0}}{B[a(t), c-a(t)]|_{t=0}} - \frac{\{\psi[a(t)] - \psi[c-a(t)]\} \frac{d[a(t)]}{dt}}{F^{-1} \left[\begin{matrix} a(t), b; \\ c; \end{matrix} z \right] \Big|_{t=0}} \Big|_{t=0} \quad (46)$$

Or

$$\left. \frac{\partial \left\{ {}_F \left[\begin{matrix} a(t), b; \\ c; \end{matrix} z \right] \right\}}{\partial t} \right|_{t=0} = \frac{-\int_0^\infty e^{-a(t)y} (1-e^{-y})^{c-a(t)-1} (1-ze^{-y})^{-b} [y + \log(1-e^{-y})] \frac{d[a(t)]}{dt} dy \Big|_{t=0}}{B[a(t), c-a(t)]|_{t=0}} - \frac{\{\psi[a(t)] - \psi[c-a(t)]\} \frac{d[a(t)]}{dt}}{F^{-1} \left[\begin{matrix} a(t), b; \\ c; \end{matrix} z \right] \Big|_{t=0}} \Big|_{t=0} \quad (47)$$

Third, let us consider that $\frac{d[c_b(t)]}{dt} \equiv 0$ or $\frac{d[c(t)]}{dt} \equiv \frac{d[b(t)]}{dt} \neq \frac{d[a(t)]}{dt}$ then from either (38) or (39) we obtain

$$\left. \frac{\partial \left\{ {}_F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \right\}}{\partial t} \right|_{t=0} = \frac{\int_0^1 \varphi[a(t), b(t), c(t); x, z] \left\{ \frac{\log(x) d[b(t)]}{dt} - \frac{\log(1-zx) d[a(t)]}{dt} \right\} dx \Big|_{t=0}}{B[b(t), c_b(t)]|_{t=0}} - \frac{\{\psi[b(t)] - \psi[c(t)]\} \frac{d[b(t)]}{dt}}{F^{-1} \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \Big|_{t=0}} \Big|_{t=0} \quad (48)$$

Or

$$\left. \frac{\partial \left\{ F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] \right\}}{\partial t} \right|_{t=0} = \frac{-\int_0^\infty \delta[a(t), b(t), c(t); y, z] \left\{ \frac{y d[b(t)]}{dt} + \frac{\log(1-ze^{-y}) d[a(t)]}{dt} \right\} dy}{B[b(t), c_b(t)]|_{t=0}} - \frac{\{\psi[b(t)] - \psi[c(t)]\} \frac{d[b(t)]}{dt}}{F^{-1} \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] |_{t=0}} \quad (49)$$

Fourth, let us consider that $\frac{d[c_a(t)]}{dt} \equiv 0$ or $\frac{d[c(t)]}{dt} \equiv \frac{d[a(t)]}{dt} \neq \frac{d[b(t)]}{dt}$ then from either (41) or (42) we obtain

$$\left. \frac{\partial \left\{ F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] \right\}}{\partial t} \right|_{t=0} = \frac{\int_0^1 \varphi[b(t), a(t), c(t); x, z] \left\{ \frac{\log(x) d[a(t)]}{dt} - \frac{\log(1-zx) d[b(t)]}{dt} \right\} dx}{B[a(t), c_a(t)]|_{t=0}} - \frac{\{\psi[a(t)] - \psi[c(t)]\} \frac{d[a(t)]}{dt}}{F^{-1} \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] |_{t=0}} \quad (50)$$

Or

$$\left. \frac{\partial \left\{ F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] \right\}}{\partial t} \right|_{t=0} = \frac{-\int_0^\infty \delta[b(t), a(t), c(t); y, z] \left\{ \frac{y d[a(t)]}{dt} + \frac{\log(1-ze^{-y}) d[b(t)]}{dt} \right\} dy}{B[a(t), c_a(t)]|_{t=0}} - \frac{\{\psi[a(t)] - \psi[c(t)]\} \frac{d[a(t)]}{dt}}{F^{-1} \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] |_{t=0}} \quad (51)$$

Next, let us consider the case when $\frac{d[a(t)]}{dt} \equiv \frac{d[b(t)]}{dt} \equiv \frac{d[c(t)]}{dt}$ then from either (38) or (39) or (41) or (42) we obtain

$$\left. \frac{\partial \left\{ F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] \right\}}{\partial t} \right|_{t=0} = \frac{\int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} (1-zx)^{-a(t)} \log\left(\frac{x}{1-zx}\right) \frac{d[a(t)]}{dt} dx}{B[b(t), c_b(t)]|_{t=0}} - \frac{\{\psi[b(t)] - \psi[c(t)]\} \frac{d[a(t)]}{dt}}{F^{-1} \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] |_{t=0}} \quad (52)$$

Or

$$\left. \frac{\partial \left\{ F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] \right\}}{\partial t} \right|_{t=0} = \frac{-\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-a(t)} \frac{y d[a(t)]}{dt} dy}{B[b(t), c_b(t)]|_{t=0}} - \frac{\{\psi[b(t)] - \psi[c(t)]\} \frac{d[a(t)]}{dt}}{F^{-1} \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] |_{t=0}} \quad (53)$$

Or

$$\left. \frac{\partial \left\{ F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] \right\}}{\partial t} \right|_{t=0} = \frac{\int_0^1 x^{a(t)-1} (1-x)^{c_b(t)-1} (1-zx)^{-b(t)} \log\left(\frac{x}{1-zx}\right) \frac{d[a(t)]}{dt} dx}{B[a(t), c_b(t)]|_{t=0}} - \frac{\{\psi[a(t)] - \psi[c(t)]\} \frac{d[a(t)]}{dt}}{F^{-1} \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] |_{t=0}} \quad (54)$$

Or

$$\left. \frac{\partial \left\{ F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] \right\}}{\partial t} \right|_{t=0} = \frac{-\int_0^\infty e^{-a(t)y} (1-e^{-y})^{c_b(t)-1} (1-ze^{-y})^{-b(t)} [y + \log(1-ze^{-y})] \frac{y d[a(t)]}{dt} dy}{B[a(t), c_b(t)]|_{t=0}} - \frac{\{\psi[a(t)] - \psi[c(t)]\} \frac{d[a(t)]}{dt}}{F^{-1} \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] |_{t=0}} \quad (55)$$

By inspecting (38) through (55) we arrive at the conclusion that the computation of $\left. \frac{\partial \left\{ F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] \right\}}{\partial t} \right|_{t=0}$ by means of integral

equations requires the computation of a laborious integrals; hence, we must seek alternate means to evaluate $\left. \frac{\partial \left\{ F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); z \end{smallmatrix} \right] \right\}}{\partial t} \right|_{t=0}$.

Next, we can find several closed form expressions of these integrals for special values of z .

First, substituting $z = 0$ into (38) and (39) yields

$$\int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} \left[\frac{\log(x) d[b(t)]}{dt} + \frac{\log(1-x) d[c_b(t)]}{dt} \right] dx \Big|_{t=0} \equiv \frac{\frac{\psi[b(t)] d[b(t)]}{dt} + \frac{\psi[c_b(t)] d[c_b(t)]}{dt} - \frac{\psi[c(t)] d[c(t)]}{dt}}{B^{-1}[b(t), c_b(t)]|_{t=0}} \quad (56)$$

Or

$$\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} \left[\frac{\log(1-e^{-y}) d[c_b(t)]}{dt} - \frac{y d[b(t)]}{dt} \right] dy \Big|_{t=0} \equiv \frac{\frac{\psi[b(t)] d[b(t)]}{dt} + \frac{\psi[c_b(t)] d[c_b(t)]}{dt} - \frac{\psi[c(t)] d[c(t)]}{dt}}{B^{-1}[b(t), c_b(t)]|_{t=0}} \quad (57)$$

Substituting $z = 0$ into (41) and (42) produces identical expression of (56) and (57) with $b(t)$ interchanged with $a(t)$.

Equations (56) and (57) can be employed to create an infinite number of new entries into the Table of Integrals, Series, and Products, (see Gradshteyn and Ryzhik, 2007 [17]).

After the separation of variables, we can obtain the following entries:

$$\int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} \log\left(\frac{x}{1-x}\right) dx \equiv \frac{\psi[b(t)] - \psi[c_b(t)]}{B^{-1}[b(t), c_b(t)]} \quad (58)$$

$$\int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} \log(1-x) dx \equiv \frac{\psi[c_b(t)] - \psi[c(t)]}{B^{-1}[b(t), c_b(t)]} \quad (59)$$

Or

$$\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} [y + \log(1-e^{-y})] dy \equiv \frac{\psi[c_b(t)] - \psi[b(t)]}{B^{-1}[b(t), c_b(t)]} \quad (60)$$

$$\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} \log(1-e^{-y}) dy \equiv \frac{\psi[c_b(t)] - \psi[c(t)]}{B^{-1}[b(t), c_b(t)]} \quad (61)$$

Next, substituting (59) into (58) and (61) into (60) we can create a new entry as follows:

$$\int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} \log(x) dx \equiv \frac{\psi[b(t)] - \psi[c(t)]}{B^{-1}[b(t), c_b(t)]} \quad (62)$$

Or

$$\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_b(t)-1} y dy \equiv \frac{\psi[c(t)] - \psi[b(t)]}{B^{-1}[b(t), c_b(t)]} \quad (63)$$

The entry in the Table of Integrals, Series, and Products (Gradshteyn and Ryzhik, 2007 [17] pg. 540 ex. 4.253 1.) is a special case of (62) by taking $b(t) = \mu/r$ and $c(t) - b(t) = \nu$ after initially a substitution in ([17] pg. 540 ex. 4.253 1.) is performed for setting $x^r = y$ from where we get exactly the answer in (62) scaled by $1/r^2$. Also the exactly same entry in the Table of Integrals, Series, and Products (Gradshteyn and Ryzhik, 2007, [17] pg. 559 ex. 4.294 13.) can be obtained from (61) by setting $b(t) = \mu$ and $c(t) - b(t) = \nu$.

Next, substituting $z = 1$ into (37) and (38) yields

$$\left. \frac{\partial \left\{ F \left[\begin{matrix} a(t), b(t); \\ c(t); 1 \end{matrix} \right] \right\}}{\partial t} \right|_{t=0} = \frac{\int_0^1 \varphi[a(t), b(t), c(t); x, 1] \left\{ \frac{\log(x)d[b(t)]}{dt} + \frac{\log(1-x)d[c_b(t)]}{dt} \right\} dx \Big|_{t=0}}{B[b(t), c_b(t)]|_{t=0}} - \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{F^{-1} \left[\begin{matrix} a(t), b(t); \\ c(t); 1 \end{matrix} \right] \Big|_{t=0}} \quad (64)$$

Or

$$\left. \frac{\partial \left\{ F \left[\begin{matrix} a(t), b(t); \\ c(t); 1 \end{matrix} \right] \right\}}{\partial t} \right|_{t=0} = \frac{\int_0^\infty \delta[a(t), b(t), c(t); y, 1] \left\{ -\frac{y d[b(t)]}{dt} + \frac{\log(1-e^{-y}) d[c_b(t)]}{dt} \right\} dy \Big|_{t=0}}{B[b(t), c_b(t)]|_{t=0}} - \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{F^{-1} \left[\begin{matrix} a(t), b(t); \\ c(t); 1 \end{matrix} \right] \Big|_{t=0}} \quad (65)$$

Or making some rearrangements we obtain

$$\int_0^1 \varphi[a(t), b(t), c(t); x, 1] \left\{ \frac{\log(x)d[b(t)]}{dt} + \frac{\log(1-x)d[c_b(t)]}{dt} \right\} dx \Big|_{t=0} = \frac{\left. \frac{\partial \left\{ F \left[\begin{matrix} a(t), b(t); \\ c(t); 1 \end{matrix} \right] \right\}}{\partial t} \right|_{t=0} + \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{F^{-1} \left[\begin{matrix} a(t), b(t); \\ c(t); 1 \end{matrix} \right] \Big|_{t=0}}}{B^{-1}[b(t), c_b(t)]|_{t=0}} \quad (66)$$

Or

$$\int_0^\infty \delta[a(t), b(t), c(t); y, 1] \left\{ \frac{\log(1-e^{-y})d[c_b(t)]}{dt} - \frac{y d[b(t)]}{dt} \right\} dy \Big|_{t=0} = \frac{\left. \frac{\partial \left\{ F \left[\begin{matrix} a(t), b(t); \\ c(t); 1 \end{matrix} \right] \right\}}{\partial t} \right|_{t=0} + \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{F^{-1} \left[\begin{matrix} a(t), b(t); \\ c(t); 1 \end{matrix} \right] \Big|_{t=0}}}{B^{-1}[b(t), c_b(t)]|_{t=0}} \quad (67)$$

On the other hand, substituting (17) in both (66) and (67) produces

$$\int_0^1 x^{b(t)-1} (1-x)^{c_{ba}(t)-1} \left\{ \frac{\log(x)d[b(t)]}{dt} + \frac{\log(1-x)d[c_{ba}(t)]}{dt} \right\} dx \Big|_{t=0} = \frac{\frac{\psi[c_{ba}(t)]d[c_{ba}(t)]}{dt} + \frac{\psi[b(t)]d[b(t)]}{dt} - \frac{\psi[c_a(t)]d[c_a(t)]}{dt} \Big|_{t=0}}{B^{-1}[b(t), c_{ab}(t)] \Big|_{t=0}} \quad (68)$$

Or

$$\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_{ba}(t)-1} \left\{ \frac{\log(1-e^{-y})d[c_{ba}(t)]}{dt} - \frac{yd[b(t)]}{dt} \right\} dy \Big|_{t=0} = \frac{\frac{\psi[c_{ab}(t)]d[c_{ab}(t)]}{dt} + \frac{\psi[b(t)]d[b(t)]}{dt} - \frac{\psi[c_a(t)]d[c_a(t)]}{dt} \Big|_{t=0}}{B^{-1}[b(t), c_{ab}(t)] \Big|_{t=0}} \quad (69)$$

Equations (68) and (69) are identical with (56) and (57) in that $c(t) - a(t)$ in (68) and (69) was replaced with $c(t)$ in (56) and (57).

After the separation of variables (68) and (69) is identical with the following entries

$$\int_0^1 x^{b(t)-1} (1-x)^{c_{ba}(t)-1} \log(1-x) dx \equiv \frac{\psi[c_{ba}(t)] - \psi[c_a(t)]}{B^{-1}[b(t), c_{ab}(t)]} \quad (70)$$

$$\int_0^1 x^{b(t)-1} (1-x)^{c_{ba}(t)-1} \log\left(\frac{x}{1-x}\right) dx \equiv \frac{\psi[b(t)] - \psi[c_{ba}(t)]}{B^{-1}[b(t), c_{ab}(t)]} \quad (71)$$

Or

$$\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_{ba}(t)-1} \log(1-e^{-y}) dy \equiv \frac{\psi[c_{ab}(t)] - \psi[c_a(t)]}{B^{-1}[b(t), c_{ab}(t)]} \quad (72)$$

$$\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_{ba}(t)-1} [y + \log(1-e^{-y})] dy \equiv \frac{\psi[c_{ab}(t)] - \psi[b(t)]}{B^{-1}[b(t), c_{ab}(t)]} \quad (73)$$

After, substituting (71) into (70) and (73) into (72) produces

$$\int_0^1 x^{b(t)-1} (1-x)^{c_{ba}(t)-1} \log(x) dx \equiv \frac{\psi[b(t)] - \psi[c_a(t)]}{B^{-1}[b(t), c_{ab}(t)]} \quad (74)$$

Or

$$\int_0^\infty e^{-b(t)y} (1-e^{-y})^{c_{ba}(t)-1} y dy \equiv \frac{\psi[c_a(t)] - \psi[b(t)]}{B^{-1}[b(t), c_{ab}(t)]} \quad (75)$$

It appears that (70) through (75) appear to be identical with (58) through (63) in which $c_{ba}(t)$, $c_a(t)$ are interchanged with $c_b(t)$, $c(t)$.

Similarly, substituting $z = 1$ into (39) and (40) and then substituting (17) produces equations identical with (68) and (69) in which $a(t)$ and $b(t)$ are interchanged with each other. Moreover, separation of variables will produce identical entries with (70) through (75) in which $a(t)$ and $b(t)$ are interchanged with each other; hence, we leave this as an exercise to the reader.

Next, substituting $z = 1/2$ into (38) and (39) yields

$$\frac{\partial \left\{ F \left[\begin{matrix} a(t), b(t); \frac{1}{2} \\ c(t); \frac{1}{2} \end{matrix} \right] \right\}}{\partial t} \Big|_{t=0} = \frac{\int_0^1 \varphi \left[a(t), b(t), c(t); x; \frac{1}{2} \right] \left\{ \frac{\log(x)d[b(t)]}{dt} + \frac{\log(1-x)d[c_b(t)]}{dt} - \frac{\log(1-\frac{1}{2}x)d[a(t)]}{dt} \right\} dx \Big|_{t=0}}{B[b(t), c_b(t)] \Big|_{t=0}} - \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{F^{-1} \left[\begin{matrix} a(t), b(t); \frac{1}{2} \\ c(t); \frac{1}{2} \end{matrix} \right] \Big|_{t=0}} \quad (76)$$

Or

$$\frac{\partial \left\{ F \left[\begin{matrix} a(t), b(t); \frac{1}{2} \\ c(t); \frac{1}{2} \end{matrix} \right] \right\}}{\partial t} \Big|_{t=0} = \frac{\int_0^\infty \delta \left[a(t), b(t), c(t); y; \frac{1}{2} \right] \left\{ \frac{-yd[b(t)]}{dt} + \frac{\log(1-e^{-y})d[c_b(t)]}{dt} - \frac{\log(1-\frac{1}{2}e^{-y})d[a(t)]}{dt} \right\} dy \Big|_{t=0}}{B[b(t), c_b(t)] \Big|_{t=0}} - \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{F^{-1} \left[\begin{matrix} a(t), b(t); \frac{1}{2} \\ c(t); \frac{1}{2} \end{matrix} \right] \Big|_{t=0}} \quad (77)$$

Equations (76) and (77) can be rearranged as

$$\int_0^1 \varphi \left[a(t), b(t), c(t); x, \frac{1}{2} \right] \left| \frac{\frac{\log(x)d[b(t)]}{dt} + \frac{\log(1-x)d[c_b(t)]}{dt}}{\log\left(1-\frac{1}{2}x\right)d[a(t)]} \right|_{t=0} dx = \frac{\frac{\partial \left\{ F \left[\frac{a(t), b(t); 1}{c(t); \frac{1}{2}} \right] \right\}}{\partial t} \Big|_{t=0}}{B^{-1}[b(t), c_b(t)]|_{t=0}} + \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{B^{-1}[b(t), c_b(t)]|_{t=0} F^{-1} \left[\frac{a(t), b(t); 1}{c(t); \frac{1}{2}} \right] \Big|_{t=0}} \quad (78)$$

Or

$$\int_0^\infty \delta \left[a(t), b(t), c(t); y, \frac{1}{2} \right] \left| \frac{\frac{-yd[b(t)]}{dt} + \frac{\log(1-e^{-y})d[c_b(t)]}{dt}}{\log\left(1-\frac{1}{2}e^{-y}\right)d[a(t)]} \right|_{t=0} dy = \frac{\frac{\partial \left\{ F \left[\frac{a(t), b(t); 1}{c(t); \frac{1}{2}} \right] \right\}}{\partial t} \Big|_{t=0}}{B^{-1}[b(t), c_b(t)]|_{t=0}} + \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi[c_b(t)]d[c_b(t)]}{dt} - \frac{\psi[c(t)]d[c(t)]}{dt} \Big|_{t=0}}{B^{-1}[b(t), c_b(t)]|_{t=0} F^{-1} \left[\frac{a(t), b(t); 1}{c(t); \frac{1}{2}} \right] \Big|_{t=0}} \quad (79)$$

Next, substituting (5) and (8) into (78) and (79) and $c(t) = \frac{1+a(t)+b(t)}{2}$ yields

$$\int_0^1 \frac{x^{b(t)-1}(1-x)^{\frac{a(t)-b(t)-1}{2}}}{\left(1-\frac{x}{2}\right)^{a(t)}} \left| \frac{\frac{\log(x)d[b(t)]}{dt} + \frac{\log(1-x)d[a(t)-b(t)]}{2dt}}{\log\left(1-\frac{x}{2}\right)d[a(t)]} \right|_{t=0} dx = \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi\left[\frac{1+a(t)-b(t)}{2}\right]d[a(t)-b(t)]}{dt} - \frac{\psi\left[\frac{1+a(t)}{2}\right]d[a(t)]}{dt} - \frac{\psi\left[\frac{1+b(t)}{2}\right]d[b(t)]}{dt} \Big|_{t=0}}{\frac{2}{\Gamma\left[\frac{1}{2}\right]}\Gamma\left[\frac{1+a(t)}{2}, \frac{1+b(t)}{2}\right] \Big|_{t=0}} \quad (80)$$

Or

$$\int_0^\infty \frac{e^{-b(t)y}(1-e^{-y})^{\frac{a(t)-b(t)-1}{2}}}{\left(1-\frac{e^{-y}}{2}\right)^{a(t)}} \left| \frac{\frac{-yd[b(t)]}{dt} + \frac{\log(1-e^{-y})d[a(t)-b(t)]}{2dt}}{\log\left(1-\frac{e^{-y}}{2}\right)d[a(t)]} \right|_{t=0} dy = \frac{\frac{\psi[b(t)]d[b(t)]}{dt} + \frac{\psi\left[\frac{1+a(t)-b(t)}{2}\right]d[a(t)-b(t)]}{dt} - \frac{\psi\left[\frac{1+a(t)}{2}\right]d[a(t)]}{dt} - \frac{\psi\left[\frac{1+b(t)}{2}\right]d[b(t)]}{dt} \Big|_{t=0}}{\frac{2}{\Gamma\left[\frac{1}{2}\right]}\Gamma\left[\frac{1+a(t)}{2}, \frac{1+b(t)}{2}\right] \Big|_{t=0}} \quad (81)$$

After the separation of variables in (80) and (81) we obtain

$$\int_0^1 \frac{x^{b(t)-1}(1-x)^{\frac{a(t)-b(t)-1}{2}}}{\left(1-\frac{x}{2}\right)^{a(t)}} \left| \frac{\log(1-x)}{2} - \log\left(1-\frac{x}{2}\right) \right| dx \equiv \frac{\psi\left[\frac{1+a(t)-b(t)}{2}\right] - \psi\left[\frac{1+a(t)}{2}\right]}{\frac{2}{\Gamma\left[\frac{1}{2}\right]}\Gamma\left[\frac{1+a(t)}{2}, \frac{1+b(t)}{2}\right] \Big|_{t=0}} \quad (82)$$

$$\int_0^1 \frac{x^{b(t)-1}(1-x)^{\frac{a(t)-b(t)-1}{2}}}{\left(1-\frac{x}{2}\right)^{a(t)}} \left| \log(x) - \frac{\log(1-x)}{2} \right| dx \equiv \frac{\psi[b(t)] - \psi\left[\frac{1+a(t)-b(t)}{2}\right] - \psi\left[\frac{1+b(t)}{2}\right]}{\frac{2}{\Gamma\left[\frac{1}{2}\right]}\Gamma\left[\frac{1+a(t)}{2}, \frac{1+b(t)}{2}\right] \Big|_{t=0}} \quad (83)$$

Or

$$\int_0^\infty \frac{e^{-b(t)y}(1-e^{-y})^{\frac{a(t)-b(t)-1}{2}}}{\left(1-\frac{e^{-y}}{2}\right)^{a(t)}} \left| \frac{\log(1-e^{-y})}{2} - \log\left(1-\frac{e^{-y}}{2}\right) \right| dy \equiv \frac{\psi\left[\frac{1+a(t)-b(t)}{2}\right] - \psi\left[\frac{1+a(t)}{2}\right]}{\frac{2}{\Gamma\left[\frac{1}{2}\right]}\Gamma\left[\frac{1+a(t)}{2}, \frac{1+b(t)}{2}\right] \Big|_{t=0}} \quad (84)$$

$$\int_0^\infty \frac{e^{-b(t)y}(1-e^{-y})^{\frac{a(t)-b(t)-1}{2}}}{\left(1-\frac{e^{-y}}{2}\right)^{a(t)}} \left| y + \frac{\log(1-e^{-y})}{2} \right| dy \equiv \frac{\psi\left[\frac{1+a(t)-b(t)}{2}\right] + \psi\left[\frac{1+b(t)}{2}\right] - \psi[b(t)]}{\frac{2}{\Gamma\left[\frac{1}{2}\right]}\Gamma\left[\frac{1+a(t)}{2}, \frac{1+b(t)}{2}\right] \Big|_{t=0}} \quad (85)$$

Between (82)-(85) we can recover another equivalent pair of relations as follows

$$\int_0^1 \frac{x^{b(t)-1}(1-x)^{\frac{a(t)-b(t)-1}{2}} \log \frac{x}{1-x/2}}{\left(1-\frac{x}{2}\right)^{a(t)}} dx \equiv \frac{\psi[b(t)] - \psi\left[\frac{1+a(t)}{2}\right] - \psi\left[\frac{1+b(t)}{2}\right]}{\frac{2}{\Gamma\left[\frac{1}{2}\right]}\Gamma\left[\frac{1+a(t)}{2}, \frac{1+b(t)}{2}\right] \Big|_{t=0}} \quad (86)$$

Or

$$\int_0^\infty \frac{e^{-b(t)y(1-e^{-y})} \frac{a(t)-b(t)-1}{2} \left[y + \log\left(1 - \frac{e^{-y}}{2}\right) \right] dy}{\left(1 - \frac{e^{-y}}{2}\right)^{a(t)}} \equiv \frac{\psi\left[\frac{1+a(t)}{2}\right] + \psi\left[\frac{1+b(t)}{2}\right] - \psi[b(t)]}{\frac{2}{\Gamma\left[\frac{1}{2}\right]} \Gamma\left[\frac{1+a(t)}{2}, \frac{1+b(t)}{2}\right]} \quad (87)$$

Similar expressions may be obtained by substituting $z = \frac{1}{2}$ into (41) and (42) and then $c(t) = \frac{1+a(t)+b(t)}{2}$ yields expressions that are identical with (80) and (81) by interchanging $a(t)$ with $b(t)$. Finally, if we assume that $\frac{d[a(t)-b(t)]}{dt} \equiv 0$ or $\frac{d[a(t)]}{dt} \equiv \frac{d[b(t)]}{dt}$ yields identical to (82) and (87) in which $a(t)$ is interchanged with $b(t)$.

Next, if we were to apply Bailey's summation theorem [5] and $b(t) = 1 - a(t)$ into (78) and (79) we obtain

$$\int_0^1 \varphi\left[a(t), 1 - a(t), c(t); x, \frac{1}{2}\right] \left| \frac{\log\left(\frac{1-x}{x(1-\frac{x}{2})}\right) d[a(t)]}{dt} + \frac{\log(1-x) d[c(t)]}{dt} \right| dx = \frac{\left\{ \psi\left[\frac{1+c(t)-a(t)}{2}\right] - \psi\left[\frac{c(t)+a(t)}{2}\right] + 2\{\psi[c_b(t)] - \psi[b(t)]\} \right\} \frac{d[a(t)]}{dt}}{\left\{ \psi\left[\frac{c(t)}{2}\right] + \psi\left[\frac{1+c(t)}{2}\right] - \psi\left[\frac{c(t)+a(t)}{2}\right] - \psi\left[\frac{1+c(t)-a(t)}{2}\right] - 2\psi[c(t)] + 2\psi[c_b(t)] \right\} \frac{d[c(t)]}{dt}} \quad (88)$$

Or

$$\int_0^\infty \delta\left[a(t), 1 - a(t), c(t); y, \frac{1}{2}\right] \left| \frac{\left[y + \log\left(\frac{1-e^{-y}}{1-\frac{e^{-y}}{2}}\right) \right] d[a(t)]}{dt} + \frac{\log(1-e^{-y}) d[c(t)]}{dt} \right| dy = \frac{\left\{ \psi\left[\frac{1+c(t)-a(t)}{2}\right] - \psi\left[\frac{c(t)+a(t)}{2}\right] + 2\{\psi[c_b(t)] - \psi[b(t)]\} \right\} \frac{d[a(t)]}{dt}}{\left\{ \psi\left[\frac{c(t)}{2}\right] + \psi\left[\frac{1+c(t)}{2}\right] - \psi\left[\frac{c(t)+a(t)}{2}\right] - \psi\left[\frac{1+c(t)-a(t)}{2}\right] - 2\psi[c(t)] + 2\psi[c_b(t)] \right\} \frac{d[c(t)]}{dt}} \quad (89)$$

After the separation of variables in (88) and (89) we obtain

$$\int_0^1 \frac{x^{-a(t)} (1-x)^{c(t)+a(t)} \log\left(\frac{1-x}{x(1-\frac{x}{2})}\right)}{\left(1-\frac{x}{2}\right)^{a(t)}} dx = \frac{\psi\left[\frac{1+c(t)-a(t)}{2}\right] - \psi\left[\frac{c(t)+a(t)}{2}\right] + 2\{\psi[c_b(t)] - \psi[b(t)]\}}{2B^{-1}[1-a(t), c(t)+a(t)-1] \Gamma^{-1}\left[\frac{c(t)}{2}, \frac{1+c(t)}{2}\right]} \quad (90)$$

$$\int_0^1 \frac{x^{-a(t)} (1-x)^{c(t)+a(t)} \log(1-x)}{\left(1-\frac{x}{2}\right)^{a(t)}} dx = \frac{\psi\left[\frac{c(t)}{2}\right] + \psi\left[\frac{1+c(t)}{2}\right] - \psi\left[\frac{c(t)+a(t)}{2}\right] - \psi\left[\frac{1+c(t)-a(t)}{2}\right] - 2\psi[c(t)] + 2\psi[c_b(t)]}{2B^{-1}[1-a(t), c(t)+a(t)-1] \Gamma^{-1}\left[\frac{c(t)}{2}, \frac{1+c(t)}{2}\right]} \quad (91)$$

Or

$$\int_0^\infty \frac{e^{[a(t)-1]y(1-e^{-y})} \frac{c(t)+a(t)-1}{2} y + \log\left(\frac{1-e^{-y}}{1-\frac{e^{-y}}{2}}\right)}{\left(1-\frac{e^{-y}}{2}\right)^{a(t)}} dy = \frac{\psi\left[\frac{1+c(t)-a(t)}{2}\right] - \psi\left[\frac{c(t)+a(t)}{2}\right] + 2\{\psi[c_b(t)] - \psi[b(t)]\}}{2B^{-1}[1-a(t), c(t)+a(t)-1] \Gamma^{-1}\left[\frac{c(t)}{2}, \frac{1+c(t)}{2}\right]} \quad (92)$$

$$\int_0^\infty \frac{e^{[a(t)-1]y(1-e^{-y})} \frac{c(t)+a(t)-1}{2} \log(1-e^{-y})}{\left(1-\frac{e^{-y}}{2}\right)^{a(t)}} dy = \frac{\psi\left[\frac{c(t)}{2}\right] + \psi\left[\frac{1+c(t)}{2}\right] - \psi\left[\frac{c(t)+a(t)}{2}\right] - \psi\left[\frac{1+c(t)-a(t)}{2}\right] - 2\psi[c(t)] + 2\psi[c_b(t)]}{2B^{-1}[1-a(t), c(t)+a(t)-1] \Gamma^{-1}\left[\frac{c(t)}{2}, \frac{1+c(t)}{2}\right]} \quad (93)$$

Similar expressions may be obtained by substituting $z = \frac{1}{2}$ into (40) and (41) and then $a(t) = 1 - b(t)$ yields expressions that are identical with (90) and (93) by interchanging $a(t)$ with $b(t)$. Finally, if we assume that $\frac{d[c(t)-b(t)]}{dt} \equiv 0$ or $\frac{d[c(t)]}{dt} \equiv \frac{d[b(t)]}{dt}$ yields other identical expression in which $a(t)$ is interchanged with $b(t)$; hence, we leave it as an exercise to the reader.

I believe that this technique is sufficiently explained and one can create many more entries in the *Table of Integrals, Series, and Products* (Gradshteyn and Ryzhik, 2007 [17]) by substituting particular values of z into either (38) or (39) or (41) or (42) and special arrangements of $a(t)$, $b(t)$, $c(t)$, and particular values of $F\left[\frac{a(t), b(t)}{c(t)}; z\right]$, (see [7]-[8]), at $t = 0$.

Before we introduce a new method let us discuss the pros and cons of the current method. The main advantage of the current method is that very effective for particular values of z of (38), (39), (41), or (42) and special arrangements of $a(t)$, $b(t)$, $c(t)$, and particular values of $F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]$, (see [7]-[8]), at $t = 0$.

The main disadvantage is that it only depends on depends or particular values of z and it requires new computation (or is laborious) as every time it requires new analytical and numerical computations depending on particular values of z .

5 Hypergeometric Function Partial Derivatives Computation via Gauss Power Series (or Binomial Expansion)

Hypergeometric function via Gauss power series, since, Gauss was the first to have systematically studied its series representation [5]. In this section we show that in fact Gauss power series is nothing more that the Binomial expansion (Arfken and Weber, 1995, pg. 317, [20]).

For this reason, let us employ Gauss definition of the hypergeometric function $F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]$ for $|z| < 1$ as follows [8]

$$F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\Gamma[c(t)]}{\Gamma[a(t)]\Gamma[b(t)]\Gamma[c(t)+n]} \frac{z^n}{n!} \quad (94)$$

Equation (94) can be written as

$$F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] = \frac{\Gamma[c(t)]}{\Gamma[a(t)]\Gamma[b(t)]} \sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]}{\Gamma[c(t)+n]} \frac{z^n}{n!} \quad (95)$$

Taking the partial derivative of both sides of (95) with respect to t produces

$$\frac{\partial F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial t} = \frac{\partial \left\{ \frac{\Gamma[c(t)]}{\Gamma[a(t)]\Gamma[b(t)]} \right\}}{\partial t} \sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]}{\Gamma[c(t)+n]} \frac{z^n}{n!} + \frac{\Gamma[c(t)]}{\Gamma[a(t)]\Gamma[b(t)]} \frac{\partial}{\partial t} \sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]}{\Gamma[c(t)+n]} \frac{z^n}{n!} \quad (96)$$

Next, substituting (96) into (97) yields

$$\frac{\partial F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial t} = \frac{\partial \left\{ \frac{\Gamma[c(t)]}{\Gamma[a(t)]\Gamma[b(t)]} \right\}}{\partial t} F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] + \frac{\Gamma[c(t)]}{\Gamma[a(t)]\Gamma[b(t)]} \sum_{n=0}^{\infty} \frac{\partial}{\partial t} \left\{ \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]}{\Gamma[c(t)+n]} \right\} \frac{z^n}{n!} \quad (97)$$

After a few more development into the expressions of (97) we have

$$\left. \frac{\partial F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial t} \right|_{t=0} = \frac{\left. \frac{\psi[c(t)]d[c(t)]}{dt} - \frac{\psi[a(t)]d[a(t)]}{dt} - \frac{\psi[b(t)]d[b(t)]}{dt} \right|_{t=0}}{F^{-1} \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \Big|_{t=0}} + \frac{\sum_{n=0}^{\infty} \frac{\psi[a(t)+n] \frac{d[a(t)]}{dt} + \psi[b(t)+n] \frac{d[b(t)]}{dt} - \psi[c(t)+n] \frac{d[c(t)]}{dt}}{\Gamma[a(t)+n]\Gamma[b(t)+n]} \frac{z^n}{n!} \Big|_{t=0}}{\left. \frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} \right|_{t=0}} \quad (98)$$

Separating the variables in (98) produces

$$\left. \frac{\partial F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial t} \right|_{t=0} = \left[\begin{aligned} & \left\{ \frac{\Gamma[c(t)]}{\Gamma[a(t)]\Gamma[b(t)]} \sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[a(t)+n]}{\Gamma[c(t)+n]} \frac{z^n}{n!} - F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \psi[a(t)] \right\} \frac{d[a(t)]}{dt} \Big|_{t=0} \\ & + \left\{ \frac{\Gamma[c(t)]}{\Gamma[a(t)]\Gamma[b(t)]} \sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[b(t)+n]}{\Gamma[c(t)+n]} \frac{z^n}{n!} - F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \psi[b(t)] \right\} \frac{d[b(t)]}{dt} \Big|_{t=0} \\ & + \left\{ F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \psi[c(t)] - \frac{\Gamma[c(t)]}{\Gamma[a(t)]\Gamma[b(t)]} \sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[c(t)+n]}{\Gamma[c(t)+n]} \frac{z^n}{n!} \right\} \frac{d[c(t)]}{dt} \Big|_{t=0} \end{aligned} \right] \quad (99)$$

Equation (99) provides the series representation of the partial derivative of the hypergeometric function with respect to t evaluated at $t = 0$. We cannot employ this formula directly to compute the case when either $a, b(t = 0) = 0$ since it contains singularities. In the Sect. 6 we show how to perform maneuvering first and then compute the partial derivative.

Nevertheless, before we do that let us consider a special case for the general series formula of the partial derivatives.

Next, substituting $z = 1$ into (99) and then separating the variables produces

$$\left. \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[a(t)+n]1^n}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} n!} - \frac{\psi[a(t)]}{F^{-1}\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right]} + \frac{B[b(t), c_{ab}(t)]\{\psi[c_{ab}(t)] - \psi[c(t)]\}}{B[b(t), c_b(t)]} \right|_{t=0} \equiv 0 \quad (100)$$

$$\left. \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[b(t)+n]1^n}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} n!} - \frac{B[b(t), c_{ab}(t)]\{\psi[b(t)] - \psi[c_{ab}(t)]\}}{B[b(t), c_b(t)]} - \frac{\psi[c_b(t)]}{F^{-1}\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right]} \right|_{t=0} \equiv 0 \quad (101)$$

$$\left. - \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[c(t)+n]1^n}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} n!} - \frac{B[b(t), c_{ab}(t)]\{\psi[c_{ab}(t)] - \psi[c(t)]\}}{B[b(t), c_b(t)]} + \frac{\psi[c_b(t)]}{F^{-1}\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right]} \right|_{t=0} \equiv 0 \quad (102)$$

Equations (100) through (102) are identical with the following equations

$$\left. \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[a(t)+n]1^n}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} n!} + \frac{B[b(t), c_{ab}(t)]\{\psi[c_{ab}(t)] - \psi[c(t)]\}}{B[b(t), c_b(t)]} - \frac{\psi[a(t)]}{F^{-1}\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right]} \right|_{t=0} \equiv 0 \quad (103)$$

$$\left. \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[b(t)+n]1^n}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} n!} + \frac{B[b(t), c_{ab}(t)]\{\psi[b(t)] - \psi[c_{ab}(t)]\}}{B[b(t), c_b(t)]} - \frac{\psi[c_b(t)]}{F^{-1}\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right]} \right|_{t=0} \equiv 0 \quad (104)$$

$$\left. - \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[c(t)+n]1^n}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} n!} - \frac{B[b(t), c_{ab}(t)]\{\psi[c_{ab}(t)] - \psi[c(t)]\}}{B[b(t), c_b(t)]} + \frac{\psi[c_b(t)]}{F^{-1}\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right]} \right|_{t=0} \equiv 0 \quad (105)$$

Since (100) through (105) must have exactly the same denominator then

$$-\frac{1}{F^{-1}\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right]} + \frac{B[b(t), c_{ab}(t)]}{B[b(t), c_b(t)]} \equiv 0 \quad (106)$$

Or

$$F\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right] \equiv \frac{B[b(t), c_{ab}(t)]}{B[b(t), c_b(t)]} \equiv \Gamma\left[\begin{smallmatrix} c(t), c_{ab}(t) \\ c_a(t), c_b(t) \end{smallmatrix}\right] \quad (107)$$

Equation (107) is an indirect proof of the Gaussian identity at $z = 1$ that we employed in (11).

Finally, rearranging the terms in (100) through (105) and employing (107) produces

$$\left. \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[a(t)+n]1^n}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} n!} \right|_{t=0} \equiv \frac{\psi[a(t)] + \psi[c(t)] - \psi[c_{ab}(t)]}{F^{-1}\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right]} \Big|_{t=0} \quad (108)$$

$$\left. \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[b(t)+n]1^n}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} n!} \right|_{t=0} \equiv \frac{\psi[b(t)] + \psi[c_b(t)] - \psi[c_{ab}(t)]}{F^{-1}\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right]} \Big|_{t=0} \quad (109)$$

$$\left. \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]\psi[c(t)+n]1^n}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} n!} \right|_{t=0} \equiv \frac{\psi[c(t)] + \psi[c_b(t)] - \psi[c_{ab}(t)]}{F^{-1}\left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} 1\right]} \Big|_{t=0} \quad (110)$$

Again, this technique gives the creation of new series identities.]

Next, let us consider the term $(1 - zx)^{-a(t)}$ in (19) closely. Employing the Binomial expansion, we obtain [20]

$$(1 - zx)^{-a(t)} = \sum_{n=0}^{\infty} \binom{n + a(t) - 1}{n} z^n x^n \quad (111)$$

Substituting (111) into (19) we obtain

$$\frac{\int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} \left[\sum_{n=0}^{\infty} \binom{n + a(t) - 1}{n} z^n x^n \right] dx}{B[b(t), c(t) - b(t)]} \equiv \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]z^n}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]} n!}}{B[b(t), c(t) - b(t)]} \quad (112)$$

Changing the order of summation and integration produces

$$\frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]z^n}{\Gamma[c(t)+n]n!}}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]}} \equiv \frac{\sum_{n=0}^{\infty} \binom{n+a(t)-1}{n} z^n \int_0^1 x^{b(t)+n-1} (1-x)^{c_b(t)-1} dx}{B[b(t), c_b(t)]} \quad (113)$$

Since, the integral in (113) is a Beta function [18] we obtain

$$\int_0^1 x^{b(t)+n-1} (1-x)^{c_b(t)-1} dx \equiv B[b(t) + n, c_b(t)] \quad (114)$$

Finally, substituting (114) into (113) produces

$$\frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]z^n}{\Gamma[c(t)+n]n!}}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]}} \equiv \frac{\sum_{n=0}^{\infty} \binom{n+a(t)-1}{n} B[b(t)+n, c_b(t)] z^n}{B[b(t), c_b(t)]} \quad (115)$$

The identity (114) is true iff (if and only if)

$$F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \equiv \frac{\sum_{n=0}^{\infty} \binom{n+a(t)-1}{n} B[b(t)+n, c_b(t)] z^n}{B[b(t), c_b(t)]} \equiv \frac{\sum_{n=0}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]z^n}{\Gamma[c(t)+n]n!}}{\frac{\Gamma[a(t)]\Gamma[b(t)]}{\Gamma[c(t)]}} \quad (116)$$

Thus, we have arrived at the definition of the hypergeometric function. Therefore, the original definition of the hypergeometric function is based on the binomial expansion. There are several shortcomings of this expansion. First, since we are dealing with additional singularizes at $a, b(t=0) = 0$; however, $d[a, b(t)]/dt|_{t=0} \neq 0$ it requires additional maneuvering. Second, binomial expansion is very slow in terms of convergence. These two reasons alone are sufficient enough for us to look into a different type of expansion of the hypergeometric function.

6 Hypergeometric Function via Gauss Power Series (or Binomial Expansion) with Maneuvering

The main problem for the Gauss Power Series (or Binomial Expansion) is that it has already produced an enormous expression (see (98)) just for computing the first order partial derivative and we still have to do more maneuvering to eliminate a particular singularity at $\exists a, b(t=0) = 0, \Gamma[a, b(t=0) = 0] = \infty \psi[a, b(t=0) = 0] = -\infty$; however, $d[a, b(t)]/dt|_{t=0} \neq 0$. Imagine what happens for higher order derivatives.

Hence, the main disadvantage of the Gauss power series expansion is that it is not suitable for taking partial derivatives with respect to its coefficients subject to the set of singularizes $\exists a, b(t=0) = 0, \Gamma[a, b(t=0) = 0] = \infty \psi[a, b(t=0) = 0] = -\infty$; however, $d[a, b(t)]/dt|_{t=0} \neq 0$ without maneuvering.

Maneuvering is the substitution of a mathematical identity that explicitly removes the singularity from the computation of the gamma function. For example, if $\Gamma[a(t)]$ contains a singularity for $a(t=0) = 0$ then if we substitute an identity $\Gamma[a(t)] = \Gamma[a(t) + 1]/a(t)$ then the singularity is no longer in the computation of the gamma function.

Let us discuss the maneuvering that we have to perform assuming that $a(t=0) = 0$,

$$F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] = 1 + \frac{a(t)\Gamma[c(t)]}{\Gamma[a(t)+1]\Gamma[b(t)]} \sum_{n=1}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]z^n}{\Gamma[c(t)+n]n!} = 1 + a(t)g(t) \quad (117)$$

where

$$g(t) = \frac{\Gamma[c(t)]}{\Gamma[a(t)+1]\Gamma[b(t)]} \sum_{n=1}^{\infty} \frac{\Gamma[a(t)+n]\Gamma[b(t)+n]z^n}{\Gamma[c(t)+n]n!} \quad (118)$$

Taking the partial derivative of $\frac{\partial F[a(t), b(t); z]}{\partial t}$ based on (117) expansion yields

$$\left. \frac{\partial F[a(t)=0, b(t); z]}{\partial t} \right|_{t=0} = \left. \frac{g(t)d[a(t)]}{dt} + \frac{a(t)\partial[g(t)]}{\partial t} \right|_{t=0} = \left. \frac{g(t)d[a(t)]}{dt} \right|_{t=0} \quad (119)$$

Where

$$g(t=0) = \frac{\Gamma[c(t)]}{\Gamma[b(t)]} \sum_{n=1}^{\infty} \frac{\Gamma[n]\Gamma[b(t)+n]z^n}{\Gamma[c(t)+n]n!} \Big|_{t=0} = \frac{\Gamma[c(t)]}{\Gamma[b(t)]} \sum_{n=0}^{\infty} \frac{\Gamma[n+1]\Gamma[b(t)+1+n]z^{n+1}}{\Gamma[c(t)+1+n](n+1)!} \Big|_{t=0} \neq 0 \quad (120)$$

Or

$$g(t=0) = \frac{b(t)}{c(t)} z \sum_{n=0}^{\infty} \frac{(b(t)+1)_n (1)_n (1)_n}{(c(t)+1)_n (2)_n} \frac{z^n}{n!} \Big|_{t=0} = \frac{b(t)}{c(t)} F \left[\begin{matrix} b(t)+1, 1, 1 \\ c(t)+1, 2 \end{matrix}; z \right] z \quad (121)$$

Substituting (121) into (119) produces

$$\frac{\partial F \left[\begin{matrix} a(t)=0, b(t); \\ c(t); \end{matrix}; z \right]}{\partial t} \Big|_{t=0} = \frac{b(t)}{c(t)} F \left[\begin{matrix} b(t)+1, 1, 1 \\ c(t)+1, 2 \end{matrix}; z \right] \frac{d[a(t)]}{dt} \Big|_{t=0} z \quad (122)$$

Assuming that $\frac{d[a(t)]}{dt} \Big|_{t=0} = a$ we obtain

$$\frac{\partial F \left[\begin{matrix} a(t)=0, b(t); \\ c(t); \end{matrix}; z \right]}{\partial t} \Big|_{t=0} = a \frac{b(t)}{c(t)} F \left[\begin{matrix} b(t)+1, 1, 1 \\ c(t)+1, 2 \end{matrix}; z \right] \Big|_{t=0} z \quad (123)$$

The main question is: Is (123) the correct equations of the first order partial derivative? Let us evaluate (123) for values of $z = \{0, \frac{1}{2}, 1\}$. For $z = 0$ we already know that $F \left[\begin{matrix} a(t)=0, b(t); \\ c(t); \end{matrix}; z \right] \Big|_{t=0} = 1$; hence,

$$\frac{\partial F \left[\begin{matrix} a(t)=0, b(t); \\ c(t); \end{matrix}; z=0 \right]}{\partial t} \Big|_{t=0} 0 \equiv a \frac{b(t)}{c(t)} z F \left[\begin{matrix} b(t)+1, 1, 1 \\ c(t)+1, 2 \end{matrix}; z \right] \Big|_{t=0, z=0} = 0 \quad (124)$$

For $z = 1$ we already know the answer from Gauss theorem

$$\frac{\partial F \left[\begin{matrix} a(t)=0, b(t); \\ c(t); \end{matrix}; z=1 \right]}{\partial t} \Big|_{t=0} = a \{\psi[c(t)] - \psi[c_b(t)]\} \Big|_{t=0} \quad (125)$$

Hence, (123) is identical to (125) iff

$$\frac{b(t)}{c(t)} F \left[\begin{matrix} b(t)+1, 1, 1 \\ c(t)+1, 2 \end{matrix}; 1 \right] = \psi[c(t)] - \psi[c_b(t)] \quad (126)$$

Or

$$F \left[\begin{matrix} b(t)+1, 1, 1 \\ c(t)+1, 2 \end{matrix}; 1 \right] = \frac{c(t)}{b(t)} \{\psi[c(t)] - \psi[c_b(t)]\} \quad (127)$$

Is there any data from the literature that looks kind of similar to (127)? It appears from Gottschalk and Maslen (1988, [19] (2a) if we substitute $a = 1$) we have

$$F \left[\begin{matrix} b(t)+1, 1, 1 \\ b(t)+2, 2 \end{matrix}; 1 \right] = \frac{b(t)+1}{b(t)} \{\psi[b(t)+1] - \psi(1)\} \quad (128)$$

Equation (128) is identical to (127) iff $c(t) = b(t) + 1$.

Next, for $z = \frac{1}{2}$ we already know the answer from the Gauss second summation theorem as follows

$$\frac{\partial F \left[\begin{matrix} a(t)=0, b(t); \\ \frac{1+b(t)}{2}; \end{matrix}; z=\frac{1}{2} \right]}{\partial t} \Big|_{t=0} = a \frac{\psi \left[\frac{1+b(t)}{2} \right] - \psi \left(\frac{1}{2} \right)}{2} \Big|_{t=0} \quad (129)$$

Equation (123) is identical to (129) iff

$$F \left[\begin{matrix} b(t)+1, 1, 1 \\ \frac{1+b(t)}{2}+1, 2 \end{matrix}; \frac{1}{2} \right] \equiv \frac{1+b(t)}{2b(t)} \left\{ \psi \left[\frac{1+b(t)}{2} \right] - \psi \left(\frac{1}{2} \right) \right\} \quad (130)$$

Next, for $z = \frac{1}{2}$ we already know the answer from Bailey's summation theorem as follows

$$\frac{\partial F \left[\begin{matrix} a(t)=0, 1-a(t)=1; \\ c(t); \end{matrix}; z=\frac{1}{2} \right]}{\partial t} \Big|_{t=0} = a \frac{\psi \left[\frac{1+c(t)}{2} \right] - \psi \left[\frac{c(t)}{2} \right]}{2} \Big|_{t=0} \quad (131)$$

Equation (123) is identical to (131) iff

$$F \left[\begin{matrix} 2, 1, 1; \\ c(t) + 1, 2; \end{matrix} \frac{1}{2} \right] \equiv F \left[\begin{matrix} 1, 1; \\ c(t) + 1; \end{matrix} \frac{1}{2} \right] \equiv c(t) \left\{ \psi \left[\frac{1+c(t)}{2} \right] - \psi \left[\frac{c(t)}{2} \right] \right\} \quad (132)$$

If we were to substitute $b(t) = 1$ in (130) and $c(t) = 1$ in (132) then we would get an identical answer equal to

$$F \left[\begin{matrix} 2, 1, 1; \\ 2, 2; \end{matrix} \frac{1}{2} \right] \equiv F \left[\begin{matrix} 1, 1; \\ 2; \end{matrix} \frac{1}{2} \right] \equiv \psi(1) - \psi \left(\frac{1}{2} \right) \quad (133)$$

This concludes the discussion on the Gauss Power Series (or Binomial Expansion) with maneuvering. The computation of the hypergeometric function partial derivatives of the second and higher orders are much more laborious and much more difficult to come up with closed form expressions; hence, they will be considered in future publications.

7 Hypergeometric Function via Taylor Power Series Expansion (of Exponential Function)

Hypergeometric function via Taylor power series (Arfken and Weber, 1995, pg. 313, [20]) (of the exponential function) is an original expansion of the hypergeometric function that has never been published before and it appears to be suitable for elimination of singularities at $\exists a, b(t=0) = 0, \Gamma[a, b(t=0) = 0] = \infty, \psi[a, b(t=0) = 0] = -\infty$; however, $d[a, b(t)]/dt|_{t=0} \neq 0$ without any additional maneuvering.

Let us consider a different expansion of the of the term in (19) based on Taylor series expansion of the exponential function as follows

$$(1 - zx)^{-a(t)} = e^{-a(t)\log(1-zx)} = \sum_{n=0}^{\infty} \frac{[-a(t)]^n \log^n(1-zx)}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n [a(t)]^n \log^n(1-zx)}{n!} \quad (134)$$

Substituting (134) into (19) produces

$$B[b(t), c_b(t)] F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] = \int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} \sum_{n=0}^{\infty} \frac{(-1)^n [a(t)]^n \log^n(1-zx)}{n!} dx \quad (135)$$

Changing the order of summation and integration in (135) yields

$$B[b(t), c_b(t)] F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(-1)^n [a(t)]^n \int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} \log^n(1-zx) dx}{n!} \quad (136)$$

Where

$$B[b(t), c_b(t)] \frac{\partial^n F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial [a(t)]^n} \Big|_{a(t)=0} = \int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} \log^n(1-zx) dx \quad (137)$$

Substituting (137) into (136) produces the Taylor series expansion of $F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]$ evaluated at $a(t) = 0$ or

$$F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \Big|_{t=0} \equiv \sum_{n=0}^{\infty} \frac{(-1)^n [a(t)]^n \frac{\partial^n F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial [a(t)]^n} \Big|_{a(t)=0}}{n!} \equiv 1 + \sum_{n=1}^{\infty} \frac{(-1)^n [a(t)]^n \frac{\partial^n F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial [a(t)]^n} \Big|_{a(t)=0}}{n!} \quad (138)$$

Because the Taylor series converges much faster than the binomial expansion, the main question is how we compute

$$\frac{\partial^n F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial [a(t)]^n} \Big|_{a(t)=0}.$$

Next, let us examine the series expansion of $\log(1 - zx)$ and then $\log^n(1 - zx)$ as follows

$$\log(1 - zx) = - \sum_{k=1}^{\infty} \frac{z^k x^k}{k} \quad (139)$$

$$\log^n(1 - zx) = (-1)^n \sum_{k_1=1}^{\infty} \frac{z^{k_1} x^{k_1}}{k_1} \sum_{k_2=1}^{\infty} \frac{z^{k_2} x^{k_2}}{k_2} \cdots \sum_{k_n=1}^{\infty} \frac{z^{k_n} x^{k_n}}{k_n} \quad (140)$$

Or

$$\log^n(1 - zx) = (-1)^n \sum_{k_{1:n}=1}^{\infty} \frac{z^{k_{\Sigma 1:n}} x^{k_{\Sigma 1:n}}}{k_{\prod 1:n}} \quad (141)$$

Where

$$k_{1:n} = k_1, k_2, \dots, k_n \quad (142)$$

$$k_{\Sigma 1:n} = k_1 + k_2 + \dots + k_n \quad (143)$$

$$k_{\prod 1:n} = k_1 k_2 \dots k_n \quad (144)$$

Next, substituting (138) into (137) produces

$$\left. \frac{\partial^n F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial [a(t)]^n} \right|_{a(t)=0} = \frac{\int_0^1 x^{b(t)-1} (1-x)^{c_b(t)-1} (-1)^n \sum_{k_{1:n}=1}^{\infty} \frac{z^{k_{\Sigma 1:n}} x^{k_{\Sigma 1:n}}}{k_{\prod 1:n}} dx}{B[b(t), c_b(t)]} \quad (145)$$

Interchanging the order of summation and integration in (145) yields

$$\left. \frac{\partial^n F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial [a(t)]^n} \right|_{a(t)=0} = \frac{(-1)^n \sum_{k_{1:n}=1}^{\infty} \frac{z^{k_{\Sigma 1:n}} \int_0^1 x^{b(t)+k_{\Sigma 1:n}-1} (1-x)^{c_b(t)-1} dx}{k_{\prod 1:n}}}{B[b(t), c_b(t)]} \quad (146)$$

Or

$$\left. \frac{\partial^n F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial [a(t)]^n} \right|_{a(t)=0} = \frac{(-1)^n \sum_{k_{1:n}=1}^{\infty} \frac{z^{k_{\Sigma 1:n}} B[b(t)+k_{\Sigma 1:n}, c_b(t)]}{k_{\prod 1:n}}}{B[b(t), c_b(t)]} \quad (147)$$

Finally, substituting (147) into (138) gives the power series expansion of the Hypergeometric function of the coefficient $[a(t)]^n$

$$F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \equiv 1 + B^{-1}[b(t), c_b(t)] \sum_{n=1}^{\infty} \frac{[a(t)]^n \left\{ u_n(t) \equiv \sum_{k_{1:n}=1}^{\infty} \frac{z^{k_{\Sigma 1:n}} B[b(t)+k_{\Sigma 1:n}, c_b(t)]}{k_{\prod 1:n}} \right\}}{n!} \quad (148)$$

An identical expression may be obtained if we interchange $a(t)$ with $b(t)$ as follows

$$F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] \equiv 1 + B^{-1}[a(t), c_a(t)] \sum_{n=1}^{\infty} \frac{[b(t)]^n \sum_{k_{1:n}=1}^{\infty} \frac{z^{k_{\Sigma 1:n}} B[a(t)+k_{\Sigma 1:n}, c_a(t)]}{k_{\prod 1:n}}}{n!} \quad (149)$$

The nice thing about either (136) or (138) is that it simplifies tremendously the computation of the partial derivatives of

$$\left. \frac{\partial^n F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial t^n} \right|_{t=0} \quad \text{when either } a(t) = 0 \text{ or } b(t) = 0. \text{ Moreover, when } a(t), b(t), \text{ and } c(t) \text{ are linear functions of } t \text{ then}$$

$$\frac{d^n a(t)}{dt^n} \equiv \frac{d^n b(t)}{dt^n} \equiv \frac{d^n c(t)}{dt^n} = 0 \quad \forall n \geq 2 \quad (150)$$

Let us take the partial derivative of (138) as follows

$$\left. \frac{\partial F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right]}{\partial t} \right|_{t=0} \equiv \frac{\left[\frac{\psi[c(t)] d[c(t)]}{dt} \frac{\psi[b(t)] d[b(t)]}{dt} \frac{\psi[c_b(t)] d[c_b(t)]}{dt} \left\{ F \left[\begin{matrix} a(t), b(t); \\ c(t); \end{matrix} z \right] - 1 \right\} + \sum_{n=1}^{\infty} \frac{n[a(t)]^{n-1} \frac{d[a(t)]}{dt} u_n(t) + [a(t)]^n \frac{\partial u_n(t)}{\partial t}}{n!} \right]}{B[b(t), c_b(t)]|_{t=0}} \quad (151)$$

Since, $a(t) = 0$ then

$$F \left[\begin{matrix} a(t) = 0, b(t); \\ c(t); \end{matrix} z \right] \Big|_{t=0} \equiv 1 \quad (152)$$

$$\sum_{n=1}^{\infty} \frac{n[a(t)]^{n-1} \frac{d[a(t)]}{dt} u_n(t)}{n!} \Big|_{t=0} \equiv \frac{d[a(t)]}{dt} \left\{ u_1(t) \equiv \sum_{k=1}^{\infty} \frac{z^k B[b(t)+k, c_b(t)]}{k} \right\} \Big|_{t=0} \quad (153)$$

$$\sum_{n=1}^{\infty} \frac{[a(t)]^n \frac{\partial u_n(t)}{\partial t}}{n!} \equiv 0 \quad (154)$$

Substituting (152) through (154) into (150) produces

$$\left. \frac{\partial F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} z \right]}{\partial t} \right|_{t=0} \equiv \left. \frac{\frac{d[a(t)]}{dt} u_1(t)}{B[b(t), c_b(t)]} \right|_{t=0} \quad (155)$$

Since, we have also assumed that $a(t)$, $b(t)$, and $c(t)$ a linear function of t and of the same slope, a then we have

$$\frac{d[a(t)]}{dt} \equiv \frac{d[b(t)]}{dt} \equiv \frac{d[c(t)]}{dt} = a \quad (156)$$

Finally, substituting (156) into (155) produces

$$\left. \frac{\partial F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} z \right]}{\partial t} \right|_{t=0} \equiv a \left. \frac{\sum_{k=1}^{\infty} \frac{z^k B[b(t)+k, c_b(t)]}{k}}{B[b(t), c_b(t)]} \right|_{t=0} \quad (157)$$

The final step is the interpretation of the new series terms. The series term in (157) has the following closed form expression

$$\sum_{k=1}^{\infty} \frac{z^k B[b(t)+k, c_b(t)]}{k} = \Gamma[c_b(t)] \sum_{k=1}^{\infty} \frac{\Gamma[b(t)+k]}{k \Gamma[c(t)+k]} z^k = \Gamma[c_b(t)] \sum_{k=1}^{\infty} \frac{\Gamma[b(t)+k](k-1)!}{\Gamma[c(t)+k]} \frac{z^k}{k!} \quad (158)$$

Or

$$\sum_{k=1}^{\infty} \frac{z^k B[b(t)+k, c_b(t)]}{k} = B[b(t), c_b(t)] \sum_{k=1}^{\infty} \frac{b(t)[b(t)]_k (k-1)!}{c(t)[c(t)]_k} \frac{z^k}{k!} = B[b(t), c_b(t)] \frac{b(t)}{c(t)} \sum_{k=0}^{\infty} \frac{[b(t)]_{k+1} k!}{[c(t)]_{k+1}} \frac{z^{k+1}}{(k+1)!} \quad (159)$$

Or

$$\sum_{k=1}^{\infty} \frac{z^k B[b(t)+k, c_b(t)]}{k} = B[b(t), c_b(t)] \frac{b(t)}{c(t)} z \sum_{k=0}^{\infty} \frac{[b(t)+1]_k [1]_k [1]_k}{[c(t)+1]_k [2]_k} \frac{z^k}{k!} = B[b(t), c_b(t)] z F \left[\begin{smallmatrix} b(t) + 1, 1, 1; \\ c(t) + 1, 2; \end{smallmatrix} z \right] \quad (160)$$

Substituting (160) into (157) produces

$$\left. \frac{\partial F \left[\begin{smallmatrix} a(t), b(t); \\ c(t); \end{smallmatrix} z \right]}{\partial t} \right|_{t=0} \equiv a \frac{b(t)}{c(t)} z F \left[\begin{smallmatrix} b(t) + 1, 1, 1; \\ c(t) + 1, 2; \end{smallmatrix} z \right] \Big|_{t=0} \quad (161)$$

As expected, (161) is identical to (123).

This concludes the discussion on the Taylor Power Series Expansion (of Exponential Function) which is in fact a more sophisticated form of maneuvering. The computation of the hypergeometric function partial derivatives of the second and higher orders are much more laborious and much more difficult to come up with closed form expressions; hence, they will be considered in future publications.

8 Numerical Example

Before we conclude this paper, we consider one last numerical example.

Consider when $z = 1$, $c(t = 0) = 3$, and $b(t = 0) = 1$. From (118) we obtain

$$g(t = 0) = \frac{\Gamma(3)}{\Gamma(1)\Gamma(1)} \sum_{n=1}^{\infty} \frac{\Gamma(n)\Gamma(1+n)}{\Gamma(3+n)} \frac{1^n}{n!} \quad (162)$$

On the other hand, from Gauss theorem and Progi's (126) we have

$$g(t = 0) = \frac{1}{3} F \left[\begin{smallmatrix} 1 + 1, 1, 1; \\ 3 + 1, 2; \end{smallmatrix} 1 \right] = \psi(3) - \psi(2) = \frac{1}{2} \quad (163)$$

Since (162) and (163) are identical then

$$g(t = 0) = 2 \sum_{n=1}^{\infty} \frac{\Gamma(n)}{\Gamma(3+n)} = \frac{1}{2} \quad (164)$$

Or

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \equiv \frac{1}{4} \quad (165)$$

I believe that (165) is a new series entry or at least I have not seen (165) published anywhere. Nevertheless, (165) was also verified in MATLAB and it was proved numerically to be an identity.

I believe that one can create many more numerical series like (165) employing the technique that we just described.

9 Conclusions

In conclusion, I have offered several techniques for the computation of the hypergeometric function partial derivatives when the hypergeometric function coefficients are function of the same parameter. Initially, special cases are considered as the computation of hypergeometric function partial derivatives is straight forward. In the case of Euler integral, Gauss power series (or Binomial expansion without and with maneuvering), and Taylor expansion, the computation of hypergeometric function partial derivatives leads to the creation of new entries in the Table of Integrals, Series, and Products. This paper is based almost entirely on the creation of original analytical derivation but as far as numerical results special cases may be considered as such.

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11 References

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