

Research Article

Exponential Generalized Beta Distribution

Ilir F. Progri¹

¹*Giftet Inc., 5 Euclid Ave. #3, Worcester, MA 01610, USA*

ORCID: 0000-0001-5197-1278

Correspondence should be addressed to Ilir Progri; iproгри@giftet.com

Received May 28, 2016; Revised May 29, 21-June 7, 2016, Accepted July 16, 2016; Published November 1, 2016.

Scientific Editor-in-Chief/Editor: Ilir F. Progri

Copyright © 2016 Giftet Inc. All rights reserved. This work may not be translated or copied in whole or in part without written permission to the publisher (Giftet Inc., 5 Euclid Ave. #3, Worcester, MA 01610, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software or by similar or dissimilar methodology now known or hereafter developed is forbidden. The use of the publication of trade names, trademarks, service marks, or similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

This paper discusses the exponential generalized Beta distribution (EGBD). For the EGBD model we provide the closed form expression of the cumulative distribution function (cdf), statistics for special cases and the computation of the mean and variance for the general case. Numerical results are derived for each case to validate the theoretical models presented in the paper. As seen from the equations in the paper, for the computation of the mean requires only two digamma functions and one hypergeometric function; however, the computation of the variance requires the computation of two polygamma functions of the first order, two digamma functions, one hypergeometric function, and two Kampé de Fériet functions.

Index Terms—Exponential, generalized, Beta, distribution, EGBD, cumulative distribution function, cdf, probability density function, pdf, Kampé de Fériet function, incomplete beta functions, hypergeometric function.

1 Introduction

I initially developed some interests on a list of four candidates of the probability density function (pdf) [1], the cumulative distribution function (cdf) [2], and its statistics in my Ph.D. dissertation in 2003 [3].

Parts of this research was also enhanced and improved and published in a few more publications in 2005 [4], 2006, [5], 2006, [6], and 2007, [7]. The research that was published during this time was more or less at the Ph.D. student level. Later on, a few years down the road, after I published my first pioneer publication in 2011 [8], and a landmark publication in

2012 [9] in signal design I recognized that my work in channel modeling needed fundamental restructuring and improvements.

This paper discusses the exponential generalized Beta distribution (EGBD) [10]; i.e., the computation of its pdf [1], its cdf [2], and its statistics.

I will leave it up the readers to judge my work whether or not today's research is able to meet or exceed the expectations of the readers. I sincerely hope it is a significant step in the right direction.

I initially encountered EGBD as I was preparing Chap. 8 of *Indoor Geolocation Systems—Theory and Applications* in 2013

[11]. At that time, it became apparent to me that the computation of the EGBD statistics for the general case required special computation. Although I attempted to perform all the required computation to enable one to come up with the closed form expression of the mean and variance of the EGBD it appeared to me that the methodology seemed too laborious and very difficult to verify the validity of the computation.

It was not until last year that I created Giftet Journal of Geolocation, Geo-information, and Geo-intelligence specifically to give me the opportunity to thoroughly investigate and publish problems such as the computation of the hypergeometric function partial derivatives because the computation of EGBD statistics is in fact directly linked with the computation of the hypergeometric function partial derivatives [12]. Although one of the first references that discusses *identities obtained by differentiation* for a class of hypergeometric functions by Gottschalk and Maslen 1986 [13] it does not consider *hypergeometric function partial derivatives* and Progni 2016 does [12] but it is an important reference and it is linked to this work.

As seen from the equations in the paper, for the computation of the mean requires only two digamma functions and one hypergeometric function; however, the computation of the variance requires the computation of two polygamma functions of the first order, two digamma functions, one hypergeometric function [14] and two Kampé de Fériet functions¹ [14].

This paper is organized as follows: in Sect. 2 general discussion of EGBD model pdf and cdf. Section 3 contains details on the computation of EGBD statistics for special cases. Section 4 illustrates EGBD statistics for special cases from moments. Section 4 provides the computation of the mean for

$$f_{\text{EGB}}(x; a, b, d, p, q) = \begin{cases} \frac{x'^d [1-(1-q)x']^{p-1}}{|a|B(d,p)[1+qx']^{d+p}}, & -\infty < x < b'' \\ 0, & b'' \leq x < \infty \end{cases} \quad (4)$$

where $B(d, p)$ is the beta function [15] and,

$$x' = e^{\frac{x-b}{|a|}}, \quad b'' = b + |a| \log\left(\frac{1}{1-q}\right) \quad (5)$$

It should be noted that $f_{\text{EGB}}(x; a, b, d, p, q)$ is continuous for x when $p \neq 1$; else, for $p = 1$ $f_{\text{EGB}}(x; a, b, d, p, q)$ is not continuous for all x . For example,

$$f_{\text{EGB}}(x \rightarrow b''^-; a, b, d, p = 1, q) = \frac{b''^d}{|a|B(d,p)\left(1+\frac{q}{1-q}\right)^{d+p}} \neq 0, \text{ for } b \neq 1, \quad f_{\text{EGB}}(x \rightarrow b''^+; a, b, d, p = 1, q) = 0 \quad (6)$$

The cdf [2] of EGB [10] can be computed from

the most general case. Section 5 presents EGBD Statistics for special cases from MGF. Section 6 demonstrates EGBD statistics for the general case from moments. Section 7 shows EGBD statistics for general case from MGF. Section 8 represents the closed form expression of EGBD variance. In Sect. 9 numerical results are provided. Conclusion is provided in Sect. 10 along with a list of references.

This paper falls into the category of the *marvels in analytical derivations* because the computation of closed from expression of the mean and variance of the EGBD requires out of the box derivations and manipulations.

2 EGBD

In this section we provide the details of the computation of the pdf and cdf of the EGBD that are not provided in any publication.

Let $X \sim f_{\text{GB}}(x; a, c, d, p, q)$ (*), and the random variable $Y = \log X^1$, then we can compute the pdf of Y as follows

$$f_Y(y) = f_X(x) \frac{dx}{dy} = f_X(x)x \quad (1)$$

Furthermore, using the identity

$$\left(\frac{x}{a}\right)^c = e^{c[\log(x) - \log(|a|)]} = e^{\frac{y-b'}{a'}} \quad (2)$$

where:

$$b' = \log(|a|), \quad a' = \frac{1}{c} \quad (3)$$

with re-parameterization, EGB [10] is distributed with the following pdf [10]:

¹ $\log(X) \equiv \ln(X)$ this is the natural logarithm.

$$F_{\text{EGB}}(x; a, b, d, p, q) = \begin{cases} \int_{-\infty}^x \frac{e^{\frac{u-b}{|a|}} \left[1 - (1-q)e^{\frac{u-b}{|a|}} \right]^{p-1}}{|a|B(d,p) \left[1 + qe^{\frac{u-b}{|a|}} \right]^{d+p}} du, & -\infty < x \leq b'' \\ 1, & b'' < x < \infty \end{cases} \quad (7)$$

or

$$F_{\text{EGB}}(x; a, b, d, p, q) = \begin{cases} \int_{-\infty}^x \frac{e^{\frac{u-b}{|a|}} \left[1 - (1-q)e^{\frac{u-b}{|a|}} \right]^{p-1}}{\left[1 + qe^{\frac{u-b}{|a|}} \right]^{d+p}} du \\ |a|B(d,p) \\ 1, \end{cases}, \quad \begin{matrix} -\infty < x \leq b'' \\ b'' < x < \infty \end{matrix} \quad (8)$$

Let $v = e^{\frac{u-b}{|a|}}$; hence, $dv = e^{\frac{u-b}{|a|}} \frac{1}{a} du$; $u \rightarrow |x|_{-\infty}$ then $v \rightarrow |e|_0^{\frac{x-b}{|a|}}$

$$F_{\text{EGB}}(x; a, b, d, p, q) = \begin{cases} \int_0^{x'v^{d-1}[1-(1-q)v]^{p-1}} \frac{dv}{B(d,p) [1+qe^v]^{d+p}}, & -\infty < x \leq b'' \\ 1, & b'' < x < \infty \end{cases} \quad (9)$$

We already know the solution of the above from cdf [2] of Beta [10], [11]; hence

$$F_{\text{EGB}}(x; a, b, d, p, q) = \begin{cases} \frac{B(d,p;x'')}{B(d,p)} = I_{x''}(d,p), & -\infty < x \leq b'' \\ 1, & b'' < x < \infty \end{cases}, \quad x'' = \frac{x'}{1+qx'} \quad (10)$$

This concludes the discussion of the EGBD pdf and cdf. In the next section we show how we can compute the EGBD statistics for the special cases.

3 EGBD for Special Cases

Let us discuss EGBD for special cases for $q = \{0,1\}$. First, for $q = 0$ the EGB pdf and cdf can be obtained from

$$f_{\text{EGB}}(x; a, b, d, p, q = 0) = \begin{cases} \frac{x'^d (1-x')^{p-1}}{|a|B(d,p)}, & -\infty < x < b \\ 0, & b \leq x < \infty \end{cases} \quad (11)$$

$$F_{\text{EGB}}(x; a, b, d, p, q = 0) = \begin{cases} \frac{B(d,p;x')}{B(d,p)} = I_{x'}(d,p), & -\infty < x \leq b \\ 1, & b < x < \infty \end{cases}, \quad x' = e^{\frac{x-b}{|a|}} \quad (12)$$

Second, for $q = 1$ the EGB pdf and cdf can be obtained from

$$f_{\text{EGB}}(x; a, b, d, p, q = 1) = \frac{x'^d (1+x')^{-(d+p)}}{|a|B(d,p)}, \quad -\infty < x < \infty \quad (13)$$

$$F_{\text{EGB}}(x; a, b, d, p, q = 1) = \frac{B(d,p;x'')}{B(d,p)} = I_{x''}(d,p), \quad -\infty < x < \infty, \quad x'' = \frac{x'}{1+x'} \quad (14)$$

This concludes the discussion of EGBD for special cases for $q = \{0,1\}$. Next, let us discuss EGBD statistics for special cases.

4 EGBD Statistics for Special Cases from Moments

Typically, the statistics of a distribution are computed from its moments [10].

First, we consider the special case for $q = 0$.

If we were to compute the mean of X in the traditional way we would write

$$E(X) = \int_{-\infty}^b x \frac{x'^d (1-x')^{p-1}}{|a|B(d,p)} dx = \frac{\int_{-\infty}^b [b+|a|\log(x')] x'^d (1-x')^{p-1} dx}{|a|B(d,p)} = \frac{\int_0^1 [b+|a|\log(x')] x'^d (1-x')^{p-1} |a| x'^{-1} dx'}{|a|B(d,p)} \quad (15)$$

Or

$$E(X) = \frac{b \int_0^1 x'^{d-1} (1-x')^{p-1} dx' + |a| \int_0^1 x'^{d-1} (1-x')^{p-1} \log(x') dx'}{B(d,p)} \quad (16)$$

Or from Progri 2016, [12]

$$E(X) = \frac{bB(d,p) + |a|B(d,p)[\psi(d) - \psi(d+p)]}{B(d,p)} = b + |a|[\psi(d) - \psi(d+p)] \quad (17)$$

Similarly, if we were to compute the second moment of X in the traditional way we would write

$$E(X^2) = \int_{-\infty}^b x^2 \frac{x'^d (1-x')^{p-1}}{|a|B(d,p)} dx = \frac{\int_{-\infty}^b [b+|a|\log(x')]^2 x'^d (1-x')^{p-1} dx}{|a|B(d,p)} = \frac{\int_0^1 [b+|a|\log(x')]^2 x'^d (1-x')^{p-1} |a| x'^{-1} dx'}{|a|B(d,p)} \quad (18)$$

Or

$$E(X^2) = \frac{\int_0^1 [b^2 + 2|a|b\log(x') + a^2\log^2(x')] x'^d (1-x')^{p-1} |a| x'^{-1} dx'}{B(d,p)} \quad (19)$$

Or

$$E(X^2) = \frac{b^2 \int_0^1 x'^{d-1} (1-x')^{p-1} dx' + 2|a|b \int_0^1 x'^{d-1} (1-x')^{p-1} \log(x') dx' + a^2 \int_0^1 x'^{d-1} (1-x')^{p-1} \log^2(x') dx'}{B(d,p)} \quad (20)$$

Or from Progri 2016, [12]

$$E(X^2) = \frac{b^2 B(d,p) + 2|a|b B(d,p)[\psi(d) - \psi(d+p)] + a^2 B(d,p)[\psi^2(d) - \psi^2(d+p)]}{B(d,p)} \quad (21)$$

Which is equivalent with

$$E(X^2) = \{b + |a|[\psi(d) - \psi(d+p)]\}^2 + a^2[\psi^2(d) - \psi^2(d+p)] \quad (22)$$

Clearly, the variance of X in the traditional can be computed from

$$Var(X) = E(X^2) - [E(X)]^2 \quad (23)$$

Substituting (17) and (22) into (23) yields

$$Var(X) = \{b + |a|[\psi(d) - \psi(d+p)]\}^2 + a^2[\psi^2(d) - \psi^2(d+p)] - \{b + |a|[\psi(d) - \psi(d+p)]\}^2 \quad (24)$$

Or

$$Var(X) = a^2[\psi^2(d) - \psi^2(d+p)] \quad (25)$$

Second, we consider the special case for $q = 1$.

If we were to compute the mean of X in the traditional way we would write

$$E(X) = \int_{-\infty}^{\infty} x \frac{x'^d (1+x')^{-(d+p)}}{|a|B(d,p)} dx = \frac{\int_{-\infty}^{\infty} [b+|a|\log(x')] x'^d (1+x')^{-(d+p)} dx}{|a|B(d,p)} = \frac{\int_0^{\infty} [b+|a|\log(x')] x'^d (1+x')^{-(d+p)} |a| x'^{-1} dx'}{|a|B(d,p)} \quad (26)$$

Or

$$E(X) = \frac{\int_0^{\infty} [b+|a|\log(x')] x'^{d-1} (1+x')^{-(d+p)} dx'}{B(d,p)} = \frac{b \int_0^{\infty} x'^{d-1} (1+x')^{-(d+p)} dx' + |a| \int_0^{\infty} \log(x') x'^{d-1} (1+x')^{-(d+p)} dx'}{B(d,p)} \quad (27)$$

Next, we make the substitution (14) and we obtain

$$E(X) = \frac{b \int_0^1 x''^{d-1} (1-x'')^{(p-1)} dx''}{B(d,p)} + \frac{|a| \int_0^1 \log\left(\frac{x''}{1-x''}\right) x''^{d-1} (1-x'')^{(p-1)} dx''}{B(d,p)} \quad (28)$$

From Progri 2016, [12]

$$E(X) = \frac{bB(d,p)}{B(d,p)} + \frac{|a|B(d,p)|a|[\psi(d)-\psi(p)]}{B(d,p)} \equiv b + |a|[\psi(d) - \psi(p)] \quad (29)$$

Similarly, if we were to compute the second moment of X in the traditional way we would write

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{x'^d (1+x')^{-(d+p)}}{|a|B(d,p)} dx = \frac{\int_{-\infty}^{\infty} [b+|a|\log(x')]^2 x'^d (1+x')^{-(d+p)} dx}{|a|B(d,p)} = \frac{\int_0^{\infty} [b+|a|\log(x')]^2 x'^d (1+x')^{-(d+p)} |a|x'^{-1} dx'}{|a|B(d,p)} \quad (30)$$

Or

$$E(X^2) = \frac{\int_0^{\infty} [b+|a|\log(x')]^2 x'^{d-1} (1+x')^{-(d+p)} dx'}{B(d,p)} = \frac{\int_0^{\infty} [b^2 + 2b|a|\log(x') + a^2 \log^2(x')] x'^{d-1} (1+x')^{-(d+p)} dx'}{B(d,p)} \quad (31)$$

Or

$$E(X^2) = \frac{b^2 \int_0^{\infty} x'^{d-1} (1+x')^{-(d+p)} dx'}{B(d,p)} + 2b|a| \frac{\int_0^{\infty} \log(x') x'^{d-1} (1+x')^{-(d+p)} dx'}{B(d,p)} + a^2 \frac{\int_0^{\infty} \log^2(x') x'^{d-1} (1+x')^{-(d+p)} dx'}{B(d,p)} \quad (32)$$

Substituting (29) into (32) produces

$$E(X^2) = \frac{b^2 B(d,p)}{B(d,p)} + 2b|a| \frac{B(d,p)[\psi(d)-\psi(p)]}{B(d,p)} + a^2 \frac{\int_0^{\infty} \log^2(x') x'^{d-1} (1+x')^{-(d+p)} dx'}{B(d,p)} \quad (33)$$

From Progri 2016, [12]

$$E(X^2) = b^2 + 2b|a|[\psi(d) - \psi(p)] + a^2 \frac{B(d,p)[\psi(d)-\psi(p)]^2 + B(d,p)[\psi^{(1)}(d) + \psi^{(1)}(p)]}{B(d,p)} \quad (34)$$

Or

$$E(X^2) = \{b + |a|[\psi(d) - \psi(p)]\}^2 + a^2[\psi^{(1)}(d) + \psi^{(1)}(p)] \quad (35)$$

Clearly, the variance of X in the traditional can be computed from substituting (29) and (35) into (23) yields

$$Var(X) = a^2[\psi^{(1)}(d) + \psi^{(1)}(p)] \quad (36)$$

This concludes the discussion of EGBD statistics for special cases for $q = \{0,1\}$. Next, let us discuss EGBD statistics for special cases based on the computation of the moment generating function (MGF).

5 EGBD Statistics for Special Cases from MGF

As seen from the previous section, the computation of the moments of the EGBD is laborious. Employing an indirect method for computing the statistics of the EGBD based on the computation of the MGF is far less computationally intensive.

Moment generating function: It can be shown that the t th MGF of the EGBD can be expressed as follows [10]:

$$E_{\text{EGB}}(e^{Xt}) = \frac{e^{btB(d+|a|t,p)} {}_2F_1\left[\begin{matrix} d+t|a|t|a| \\ d+p+t|a| \end{matrix}; q\right]}{B(d,p)} \quad (37)$$

First, let us compute the special cases solution for $q = \{0,1\}$ then approach the most general case.

Therefore, for $q = 0$

$$E_{\text{EGB}}(e^{Xt}) = \frac{e^{btB(d+|a|t,p)} {}_2F_1\left[\begin{matrix} d+t|a|t|a| \\ d+p+t|a| \end{matrix}; q=0\right]}{B(d,p)} = \frac{e^{btB(d+|a|t,p)}}{B(d,p)} \quad (38)$$

Or employing the compact notation (Gessel 1995, [18])

$$E_{\text{EGB}}(e^{Xt}) = \frac{e^{btB(d+|a|t,p)}}{B(d,p)} = \frac{e^{bt\Gamma(d+|a|t)\Gamma(p)\Gamma(d+p)}}{\Gamma(d+|a|t+p)\Gamma(d)\Gamma(p)} = e^{bt\Gamma\left[\begin{matrix} d+p & d+|a|t \\ d & d+|a|t+p \end{matrix}\right]} \quad (39)$$

Taking the log

$$\log[E_{\text{EGB}}(e^{Xt})] = bt + \log\left[\frac{\Gamma(d+p)}{\Gamma(d)}\right] + \log[\Gamma(d+|a|t)] - \log[\Gamma(d+|a|t+p)] \quad (40)$$

The k th cumulant, K_k , can be determined as follows (Mathai, Provost 2004, [20])

$$K_k = \frac{\partial^k \log[E_{\text{EGB}}(e^{Xt})]}{\partial t^k} \Big|_{t=0} = \frac{\partial^{k-1} [b + |a|\psi(d + |a|t) - |a|\psi(d + |a|t + p)]}{\partial t^{k-1}} \Big|_{t=0} \quad (41)$$

The first cumulant, K_1 , can be determined as follows (Mathai, Provost 2004, [20])

$$K_1 = \frac{\partial \log[E_{\text{EGB}}(e^{Xt})]}{\partial t} \Big|_{t=0} = b + |a|\psi(d + |a|t) - |a|\psi(d + |a|t + p) \Big|_{t=0}, \text{ (Mean)} \quad (42)$$

or

$$K_1 = b + |a|[\psi(d) - \psi(d + p)], \text{ (Mean)} \quad (43)$$

$$K_2 = \frac{\partial^2 \log[E_{\text{EGB}}(e^{Xt})]}{\partial t^2} \Big|_{t=0} = \frac{\partial \{b + |a|\psi(d + |a|t) - |a|\psi(d + |a|t + p)\}}{\partial t} \Big|_{t=0}, \text{ (Variance)} \quad (44)$$

or

$$K_2 = a^2 \psi^{(1)}(d + |a|t) - a^2 \psi^{(1)}(d + |a|t + p) \Big|_{t=0}, \text{ (Variance)} \quad (45)$$

or

$$K_2 = a^2 [\psi^{(1)}(d) - \psi^{(1)}(d + p)], \text{ (Variance)} \quad (46)$$

Equations (43) and (46) are identical to (17) and (25).

Next, for $q = 1$

$$E_{\text{EGB}}(e^{Xt}) = \frac{e^{bt} B(d + |a|t, p) {}_2F_1 \left[\begin{matrix} d + t|a|, t|a| \\ d + p + t|a| \end{matrix}; q=1 \right]}{B(d, p)} \quad (47)$$

Or employing Gauss theorem for $q = 1$ we obtain [10]

$$E_{\text{EGB}}(e^{Xt}) = \frac{e^{bt} B(d + |a|t, p) \Gamma \left[\begin{matrix} d + p + t|a| \\ d + p + t|a| - d - t|a| \end{matrix} \right]}{B(d, p)} = \frac{e^{bt} B(d + |a|t, p) \Gamma \left[\begin{matrix} d + p + t|a| & p - t|a| \\ p & d + p \end{matrix} \right]}{B(d, p)} \quad (48)$$

Or employing the compact notation (Gessel 1995, [18])

$$E_{\text{EGB}}(e^{Xt}) = e^{bt} \Gamma \left[\begin{matrix} d + p & d + |a|t \\ d & d + |a|t + p \end{matrix} \right] \Gamma \left[\begin{matrix} d + p + t|a| & p - t|a| \\ p & d + p \end{matrix} \right] \quad (49)$$

or

$$E_{\text{EGB}}(e^{Xt}) = e^{bt} \Gamma \left[\begin{matrix} d + |a|t & p - t|a| \\ d & p \end{matrix} \right] \quad (50)$$

The k th cumulant, K_k , can be determined as follows [20]

$$K_k = \frac{\partial^k \log[E_{\text{EGB}}(e^{Xt})]}{\partial t^k} \Big|_{t=0} = \frac{\partial^{k-1} [b + |a|\psi(d + |a|t) - a\psi(p - |a|t)]}{\partial t^{k-1}} \Big|_{t=0} \quad (51)$$

The mean and the variance of the EGB distribution are respectively given by K_1 and K_2

$$K_1 = \frac{\partial \log[E_{\text{EGB}}(e^{Xt})]}{\partial t} \Big|_{t=0} = b + |a|\psi(d + |a|t) - |a|\psi(p - |a|t) \Big|_{t=0}, \text{ (Mean)} \quad (52)$$

or

$$K_1 = \frac{\partial \log[E_{\text{EGB}}(e^{Xt})]}{\partial t} \Big|_{t=0} = b + |a|[\psi(d) - \psi(p)], \text{ (Mean)} \quad (53)$$

$$K_2 = \frac{\partial^2 \log[E_{\text{EGB}}(e^{Xt})]}{\partial t^2} \Big|_{t=0} = \frac{\partial [b + |a|\psi(d + |a|t) - |a|\psi(p - |a|t)]}{\partial t} \Big|_{t=0}, \text{ (Variance)} \quad (54)$$

or

$$K_2 = a^2 \psi^{(1)}(d + |a|t) + a^2 \psi^{(1)}(p - |a|t) \Big|_{t=0}, \text{ (Variance)} \quad (55)$$

or

$$K_2 = a^2 \psi^{(1)}(d) + a^2 \psi^{(1)}(p), \text{ (Variance)} \quad (56)$$

Again, we see that (52) and (56) are identical to (29) and (36).

The computation of statistics of the EGBD based on the computation of the MGF for special cases is straight forward. In the following section we discuss the general case.

6 EGBD Statistics for the General Case from Moments

Typically, the statistics of a distribution are computed from its moments [10].

If we are going to compute the mean of X in the traditional way we can write

$$E(X) = \int_{-\infty}^{b''} x \frac{x'^d [1-(1-q)x']^{p-1}}{|a|B(d,p)(1+qx')^{d+p}} dx = \frac{\int_{-\infty}^{b''} [b+|a|\log(x')] x'^d [1-(1-q)x']^{p-1} dx}{|a|B(d,p)} = \frac{\int_0^{\frac{1}{1-q}} [b+|a|\log(x')] x'^d [1-(1-q)x']^{p-1} |a|x'^{-1} dx'}{(1+qx')^{d+p}} \quad (57)$$

Or

$$E(X) = \frac{\int_0^{\frac{1}{1-q}} [b+|a|\log(x')] x'^{d-1} [1-(1-q)x']^{p-1} dx'}{(1+qx')^{d+p}} = \frac{b \int_0^{\frac{1}{1-q}} x'^{d-1} [1-(1-q)x']^{p-1} dx' + |a| \int_0^{\frac{1}{1-q}} \log(x') x'^{d-1} [1-(1-q)x']^{p-1} dx'}{B(d,p)} \quad (58)$$

Next, we perform the substitution (10a) in (58) and we obtain

$$E(X) = \frac{b \int_0^1 x''^{d-1} (1-x'')^{p-1} dx''}{B(d,p)} + |a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log\left(\frac{x''}{1-qx''}\right) dx''}{B(d,p)} \quad (59)$$

Or

$$E(X) = \frac{b \int_0^1 x''^{d-1} (1-x'')^{p-1} dx''}{B(d,p)} + |a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') dx''}{B(d,p)} - |a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-qx'') dx''}{B(d,p)} \quad (60)$$

Which is equivalent with

$$E(X) = b + |a|[\psi(d) - \psi(d+p)] - |a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-qx'') dx''}{B(d,p)} \quad (61)$$

We will leave the integral in (61) for now as is. In the following section we shall see what this integral is equal to in a closed form expression. However, we can check the solution (61) for special cases of $q = \{0,1\}$. We can clearly see that $q = \{0,1\}$ (61) is equal to (17) and (29).

Similarly, let us compute the second moment as follows

$$E(X^2) = \int_{-\infty}^{b''} x^2 \frac{x'^d [1-(1-q)x']^{p-1}}{|a|B(d,p)(1+qx')^{d+p}} dx = \frac{\int_{-\infty}^{b''} [b+|a|\log(x')]^2 x'^d [1-(1-q)x']^{p-1} dx}{|a|B(d,p)} = \frac{\int_0^{\frac{1}{1-q}} [b+|a|\log(x')]^2 x'^d [1-(1-q)x']^{p-1} dx'}{(1+qx')^{d+p}} \quad (62)$$

Or

$$E(X^2) = \frac{b^2 \int_0^{\frac{1}{1-q}} x'^{d-1} [1-(1-q)x']^{p-1} dx' + 2|a|b \int_0^{\frac{1}{1-q}} \log(x') x'^{d-1} [1-(1-q)x']^{p-1} dx' + a^2 \int_0^{\frac{1}{1-q}} \log^2(x') x'^{d-1} [1-(1-q)x']^{p-1} dx'}{(1+qx')^{d+p}} \quad (63)$$

Next, we perform the substitution (10) in (63) and we obtain

$$E(X^2) = \frac{b^2 \int_0^1 x''^{d-1} (1-x'')^{p-1} dx''}{B(d,p)} + 2b|a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log\left(\frac{x''}{1-qx''}\right) dx''}{B(d,p)} + a^2 \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2\left(\frac{x''}{1-qx''}\right) dx''}{B(d,p)} \quad (64)$$

Or

$$E(X^2) = \left| \frac{b^2 \int_0^1 x''^{d-1} (1-x'')^{p-1} dx''}{B(d,p)} + 2b|a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') dx''}{B(d,p)} - 2b|a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-qx'') dx''}{B(d,p)} \right. \\ \left. + a^2 \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2\left(\frac{x''}{1-qx''}\right) dx''}{B(d,p)} \right. \quad (65)$$

Which is equivalent with

$$E(X^2) = \left| \frac{b^2 + 2b|a|[\psi(d) - \psi(d+p)] - 2b|a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-qx'') dx''}{B(d,p)}}{+a^2 \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(x'') dx''}{B(d,p)} - 2 \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') \log(1-qx'') dx''}{B(d,p)} + \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(1-qx'') dx''}{B(d,p)}} \right. \quad (66)$$

From (66) there are three more integrals that we do not currently know the closed form expression.

However, we do know that $q = 0$ (66) must equal to (22) or

$$E(X^2) = b^2 + 2b|a|[\psi(d) - \psi(d+p)] + a^2 \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(x'') dx''}{B(d,p)} \quad (67)$$

Or

$$\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(x'') dx'' = \frac{\psi^{(1)}(d) - \psi^{(1)}(d+p) + [\psi(d) - \psi(d+p)]^2}{B^{-1}(d,p)} \quad (68)$$

Substituting (68) into (66) produces

$$E(X^2) = \left| \{b + |a|[\psi(d) - \psi(d+p)]\}^2 + a^2[\psi^{(1)}(d) - \psi^{(1)}(d+p)] - 2b|a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-qx'') dx''}{B(d,p)} \right. \\ \left. + a^2 \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(1-qx'') dx'' - 2 \int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') \log(1-qx'') dx''}{B(d,p)} \right. \quad (69)$$

From (69) it only remains to solve two integrals in closed form expression to compute the second moment of X or $E(X^2)$.

Moreover, for $q = 1$ we already know that the answer of (69) is which is equal to (35).

From Progni 2016, ([12] (58)) we have

$$\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-x'') dx'' \equiv \frac{\psi(p) - \psi(d+p)}{B^{-1}(d,p)} \quad (70)$$

Substituting (70) into (69) for $q = 1$ we obtain

$$E(X^2) = \left| \{b + |a|[\psi(d) - \psi(p)]\}^2 - a^2[\psi(d) - \psi(p)]^2 + a^2[\psi(d) - \psi(d+p)]^2 + a^2[\psi^{(1)}(d) - \psi^{(1)}(d+p)] \right. \\ \left. + a^2 \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(1-x'') dx'' - 2 \int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') \log(1-x'') dx''}{B(d,p)} \right. \quad (71)$$

Since, (71) must equal (35) then we have

$$\left| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(1-x'') dx''}{-2 \int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') \log(1-x'') dx''} = \frac{\psi^{(1)}(p) + \psi^{(1)}(d+p) + [\psi(d) - \psi(p)]^2 - [\psi(d) - \psi(d+p)]^2}{B^{-1}(d,p)} \right. \quad (72)$$

Furthermore, from (68) it is easy to see that

$$\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(1-x'') dx'' = \int_0^1 x''^{p-1} (1-x'')^{d-1} \log^2(x'') dx'' = \frac{\psi^{(1)}(p) - \psi^{(1)}(d+p) + [\psi(p) - \psi(d+p)]^2}{B^{-1}(d,p)} \quad (73)$$

Substituting (73) into (72) yields

$$\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') \log(1-x'') dx'' = \frac{[\psi(d+p)]^2 + \psi(d)\psi(p) - \psi(p)\psi(d+p) - \psi(d)\psi(d+p) - \psi^{(1)}(d+p)}{B^{-1}(d,p)} \quad (74)$$

Equations (72)-(74) are new integrals that do not currently exists in the Table of Integrals, Series, and Products (Gradshteyn and Ryzhik, 2007 [19]).

The variance of X can be computed from

$$Var(X) = \left| \begin{aligned} & \{b + |a|[\psi(d) - \psi(d+p)]\}^2 + a^2 [\psi^{(1)}(d) - \psi^{(1)}(d+p)] - 2b|a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-qx'') dx''}{B(d,p)} \\ & + a^2 \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(1-qx'') dx'' - 2 \int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') \log(1-qx'') dx''}{B(d,p)} \\ & - \left\{ b + |a|[\psi(d) - \psi(d+p)] - |a| \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-qx'') dx''}{B(d,p)} \right\}^2 \end{aligned} \right| \quad (75)$$

Or

$$Var(X) = a^2 \left| \begin{aligned} & \psi^{(1)}(d) - \psi^{(1)}(d+p) + \frac{2[\psi(d) - \psi(d+p)] \int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-qx'') dx''}{B(d,p)} \\ & + \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(1-qx'') dx'' - 2 \int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') \log(1-qx'') dx''}{B(d,p)} \\ & - \left[\frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-qx'') dx''}{B(d,p)} \right]^2 \end{aligned} \right| \quad (76)$$

Substituting the result from the next section we have

$$Var(X) = a^2 \left| \begin{aligned} & \psi^{(1)}(d) - \psi^{(1)}(d+p) - \frac{2[\psi(d) - \psi(d+p)] d \left[\frac{d+1,1,1;}{d+p+1,2;}, q \right] q}{d+p} F - \left[\frac{dF \left[\frac{d+1,1,1;}{d+p+1,2;}, q \right] q}{d+p} \right]^2 \\ & + \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(1-qx'') dx'' - 2 \int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') \log(1-qx'') dx''}{B(d,p)} \end{aligned} \right| \quad (77)$$

This concludes the EGBD statistics for the general case from moments. Next, we compute the EGBD statistics for general case from MGF.

7 EGBD Statistics for General Case from MGF

In general, however, we can only find a complicated closed form expression for $0 < q < 1$ and an exact, less complicated closed form expression for $q = \{0,1\}$. Let us consider the more general case; hence, $0 < q < 1$

$$E_{\text{EGB}}(e^{Xt}) = \frac{e^{bt} B(d+|a|t, p) {}_2F_1 \left[\frac{d+t|a|, t|a|;}{d+p+t|a|;}, q \right]}{B(d,p)} \quad (78)$$

where (see Progi 2016, [12] Progi's expansion of the hypergeometric function for computation of the hypergeometric function partial derivatives)

$${}_2F_1 \left[\frac{a(t), b(t)=0;}{c(t);}, q \right] \Big|_{t=0} = 1 + b(t) \frac{\frac{\partial F \left[\frac{a(t), b(t)=0;}{c(t);}, q \right]}{\partial t} \Big|_{t=0}}{\frac{d[b(t)]}{dt} \Big|_{t=0}} = 1 + b(t) \frac{a(t) \left[\frac{a(t)+1, b(t)+1, 1;}{c(t)+1, 2;}, q \right] q}{c(t)} F \quad (79)$$

and where

$$a(t) = d + |a|t \quad (80)$$

$$b(t) = |a|t \quad (81)$$

$$c(t) = d + p + |a|t \quad (82)$$

$$E_{\text{EGB}}(e^{Xt}) = \frac{e^{bt} B(d+at, p) \left[1 + b(t) \frac{a(t)}{c(t)} F \left[\frac{a(t)+1, b(t)+1, 1;}{c(t)+1, 2;}, q \right] q \right]}{B(d,p)} \quad (83)$$

Taking the log of (83) yields

$$\log[E_{\text{EGB}}(e^{Xt})] = bt + \log[\Gamma(d + |a|t)] - \log[\Gamma(d + p + |a|t)] + \log[1 + b(t)g(t)] + \log \left[\frac{\Gamma(d+p)}{\Gamma(d)} \right] \quad (84)$$

where

$$g(t) = \frac{a(t)^F \left[\frac{a(t)+1, b(t)+1, 1;}{c(t)+1, 2; } q \right]}{c(t)} \quad (85)$$

The k th cumulant, K_k , can be determined as follows [18]

$$K_k = \frac{\partial^k \log[E_{\text{EGB}}(e^{Xt})]}{\partial t^k} \Big|_{t=0} = \frac{\partial^{k-1} \left\{ b + a[\psi(d+|a|t) - \psi(d+p+|a|t) + \frac{\partial \log[1+b(t)g(t)]}{\partial t}] \right\}}{\partial t^{k-1}} \Big|_{t=0} \quad (86)$$

or

$$\frac{\partial \log[1+b(t)g(t)]}{\partial t} \Big|_{t=0} = \frac{\frac{\partial [1+b(t)g(t)]}{\partial t}}{1+b(t)g(t)} \Big|_{t=0} = \frac{\frac{db(t)}{dt}g(t) + b(t)\frac{\partial g(t)}{\partial t}}{1+b(t)g(t)} \Big|_{t=0} = \frac{db(t)}{dt}g(t) \Big|_{t=0} \quad (87)$$

Substituting (87) into (86) produces

$$K_k = \frac{\partial^k \log[E_{\text{EGB}}(e^{Xt})]}{\partial t^k} \Big|_{t=0} = \frac{\partial^{k-1} \left\{ b + |a|[\psi(d+|a|t) - \psi(d+p+|a|t) + \frac{db(t)}{dt}g(t)] \right\}}{\partial t^{k-1}} \Big|_{t=0} \quad (88)$$

Finally, the k th cumulant, K_k , from (88) can be determined as follows [20]

$$K_k = \frac{\partial^k \log[E_{\text{EGB}}(e^{Xt})]}{\partial t^k} \Big|_{t=0} = \frac{\partial^{k-1} \left\{ b + |a|[\psi(d+|a|t) - \psi(d+p+|a|t) + \frac{db(t)a(t)}{dt} \frac{a(t)^F \left[\frac{a(t)+1, b(t)+1, 1;}{c(t)+1, 2; } q \right]}{c(t)}] \right\}}{\partial t^{k-1}} \Big|_{t=0} \quad (89)$$

The first cumulant, K_1 , can be determined as follows

$$K_1 = b + |a|[\psi(d) - \psi(d+p)] + \frac{db(t)}{dt} \frac{a(t)^F \left[\frac{a(t)+1, b(t)+1, 1;}{c(t)+1, 2; } q \right]}{c(t)} \Big|_{t=0} \quad (90)$$

or

$$K_1 = b + |a| \left[\psi(d) - \psi(d+p) + \frac{dF \left[\frac{d+1, 1, 1;}{d+p+1, 2; } q \right]}{d+p} \right], (\text{Mean}) \quad (91)$$

By comparing and contrasting (91) with (61) we obtain

$$\int_0^1 x''^{d-1} (1-x'')^{p-1} \log(1-qx'') dx'' = - \frac{B(d,p) dF \left[\frac{d+1, 1, 1;}{d+p+1, 2; } q \right]}{d+p} \quad (92)$$

Before, we proceed any further is (91) the correct closed form expression of the first cumulant, K_1 ?

If we were to substitute $q = 0$ in (91) we obtain

$$K_1|_{q=0} = b + |a|[\psi(d) - \psi(d+p)], (\text{Mean}) \quad (93)$$

We can see that (93) is identical to (17) as it was expected.

Next, if we were to substitute $q = 1$ in (91) and employ Progri's identity (Progri 2016, [12] (125)) we obtain

$$K_1|_{q=1} = b + |a|[\psi(d) - \psi(d+p) + \psi(d+p) - \psi(p)] = b + |a|[\psi(d) - \psi(p)], (\text{Mean}) \quad (94)$$

Again, (94) is identical to (29) as expected.

Equation (91) provides the correct closed form expression for the computation of the mean of EGBD for $0 \leq q \leq 1$.

The second cumulant, K_2 , can be determined as follows

$$K_2 = \frac{\partial^2 \log[E_{\text{EGB}}(e^{Xt})]}{\partial t^2} \Big|_{t=0} = \frac{\partial \left\{ b + |a|[\psi(d+at) - \psi(d+p+at) + \frac{\partial \log[1+b(t)g(t)]}{\partial t}] \right\}}{\partial t} \Big|_{t=0} \quad (95)$$

or

$$K_2 = a^2 [\psi^{(1)}(d+at) - \psi^{(1)}(d+p+at)] + \frac{\partial^2 \log[1+b(t)g(t)]}{\partial t^2} \Big|_{t=0} \quad (96)$$

where

$$\frac{\partial^2 \log[1+b(t)g(t)]}{\partial t^2} = \frac{\partial}{\partial t} \left[\frac{\frac{\partial[1+b(t)g(t)]}{\partial t}}{1+b(t)g(t)} \right] = \frac{\frac{\partial^2[1+b(t)g(t)]}{\partial t^2} [1+b(t)g(t)] - \left[\frac{\partial[1+b(t)g(t)]}{\partial t} \right]^2}{[1+b(t)g(t)]^2} \quad (97)$$

Or (97) can be further written as

$$\frac{\partial^2 \log[1+b(t)g(t)]}{\partial t^2} = \frac{\frac{\partial}{\partial t} \left[\frac{d[b(t)]}{dt} g(t) + b(t) \frac{\partial[g(t)]}{\partial t} \right] [1+b(t)g(t)] - \left[\frac{\partial[1+b(t)g(t)]}{\partial t} \right]^2}{[1+b(t)g(t)]^2} \quad (98)$$

Which is equivalent with

$$\frac{\partial^2 \log[1+b(t)g(t)]}{\partial t^2} = \frac{\left\{ \frac{d^2[b(t)]}{dt^2} g(t) + \frac{d[b(t)]}{dt} \frac{\partial[g(t)]}{\partial t} + \frac{d[b(t)]}{dt} \frac{\partial[g(t)]}{\partial t} + b(t) \frac{\partial^2[g(t)]}{\partial t^2} \right\} [1+b(t)g(t)] - \left\{ \frac{d[b(t)]}{dt} g(t) + b(t) \frac{\partial[g(t)]}{\partial t} \right\}^2}{[1+b(t)g(t)]^2} \quad (99)$$

Finally,

$$\frac{\partial^2 \log[1+b(t)g(t)]}{\partial t^2} \Big|_{t=0} = 2|a| \frac{\partial[g(t)]}{\partial t} \Big|_{t=0} - a^2 [g(t)]^2 \Big|_{t=0} \quad (100)$$

Substituting (100) into (96) yields

$$K_2 = a^2 [\psi^{(1)}(d) - \psi^{(1)}(d+p)] + 2|a| \frac{\partial[g(t)]}{\partial t} \Big|_{t=0} - a^2 [g(t)]^2 \Big|_{t=0} \quad (101)$$

where

$$\frac{\partial[g(t)]}{\partial t} \Big|_{t=0} = \frac{\partial \left[\frac{a(t)}{c(t)} F \left[\frac{a(t)+1, b(t)+1, 1; q}{c(t)+1, 2; q} \right] \right]}{\partial t} \Big|_{t=0} = \frac{\frac{a(t)}{c(t)} F \left[\frac{a(t)+1, b(t)+1, 1; q}{c(t)+1, 2; q} \right] q}{\partial t} + \frac{\frac{a(t)}{c(t)} \partial \left[F \left[\frac{a(t)+1, b(t)+1, 1; q}{c(t)+1, 2; q} \right] \right] q}{\partial t} \Big|_{t=0} \quad (102)$$

Substituting (102) into (101) produces

$$K_2 = a^2 [\psi^{(1)}(d) - \psi^{(1)}(d+p)] + 2|a| \left[\frac{\frac{a(t)}{c(t)} F \left[\frac{a(t)+1, b(t)+1, 1; q}{c(t)+1, 2; q} \right] q}{\partial t} + \frac{\frac{a(t)}{c(t)} \partial \left[F \left[\frac{a(t)+1, b(t)+1, 1; q}{c(t)+1, 2; q} \right] \right] q}{\partial t} \right] \Big|_{t=0} - a^2 [g(t)]^2 \Big|_{t=0} \quad (103)$$

Next, we compute

$$\frac{\partial \left[\frac{a(t)}{c(t)} \right]}{\partial t} \Big|_{t=0} = \frac{\frac{d[a(t)]}{dt} c(t) - a(t) \frac{d[c(t)]}{dt}}{[c(t)]^2} \Big|_{t=0} = |a| \frac{d+p-d}{(d+p)^2} = |a| \frac{p}{(d+p)^2} \quad (104)$$

Substituting (104) into (103) produces

$$K_2 = a^2 [\psi^{(1)}(d) - \psi^{(1)}(d+p)] + 2|a| \left[|a| \frac{p F \left[\frac{d+1, 1, 1; q}{d+p+1, 2; q} \right] q}{(d+p)^2} + \frac{\frac{a(t)}{c(t)} \partial \left[F \left[\frac{a(t)+1, b(t)+1, 1; q}{c(t)+1, 2; q} \right] \right] q}{\partial t} \right] \Big|_{t=0} - a^2 \frac{d^2 \left\{ F \left[\frac{d+1, 1, 1; q}{d+p+1, 2; q} \right] \right\}^2 q^2}{(d+p)^2} \quad (105)$$

Or

$$K_2 = a^2 \left[\psi^{(1)}(d) - \psi^{(1)}(d+p) + \frac{F \left[\frac{d+1, 1, 1; q}{d+p+1, 2; q} \right] q [2p-d^2 F \left[\frac{d+1, 1, 1; q}{d+p+1, 2; q} \right] q]}{(d+p)^2} + \frac{2|a| d q}{d+p} \frac{\partial \left[F \left[\frac{a(t)+1, b(t)+1, 1; q}{c(t)+1, 2; q} \right] \right]}{\partial t} \right] \Big|_{t=0} \quad (106)$$

Before we proceed any further, we must check if (106) is the correct expression of the second cumulant, K_2 . We must check it for values of $q = \{0, 1\}$. First, for $q = 0$ we have

$$K_2|_{q=0} = a^2 [\psi^{(1)}(d) - \psi^{(1)}(d+p)] \quad (107)$$

Equation (107) is identical to (25) as it was expected. Second, for $q = 1$ we have

$$K_2|_{q=1} = a^2 [\psi^{(1)}(d) - \psi^{(1)}(d+p)] + 2|a| \frac{\partial[g(t)]}{\partial t} \Big|_{t=0, q=1} - a^2 [g(t)]^2 \Big|_{t=0, q=1} \quad (108)$$

Which is equivalent with

$$K_2|_{q=1} = a^2[\psi^{(1)}(d) - \psi^{(1)}(d+p)] + 2a^2 \frac{\psi^{(1)}(d+p) + \psi^{(1)}(p) + [\psi(d+p) - \psi(p)]^2}{2} - a^2[\psi(d+p) - \psi(p)]^2 \quad (109)$$

Or

$$K_2|_{q=1} = a^2[\psi^{(1)}(d) - \psi^{(1)}(d+p) + \psi^{(1)}(d+p) + \psi^{(1)}(p)] = a^2[\psi^{(1)}(d) + \psi^{(1)}(p)] \quad (110)$$

Where

$$\left. \frac{\partial[g(t)]}{\partial t} \right|_{t=0, q=1} = \frac{|a|[\psi^{(1)}(d+p) + \psi^{(1)}(p) + [\psi(d+p) - \psi(p)]^2]}{2} \quad (111)$$

Or

$$\left| a \left[\frac{{}_pF\left[\begin{smallmatrix} d+1, 1, 1; \\ d+p+1, 2; \end{smallmatrix} 1\right]}{(d+p)^2} + \frac{\frac{a(t)}{c(t)} \partial \left[{}_F\left[\begin{smallmatrix} a(t)+1, b(t)+1, 1; \\ c(t)+1, 2; \end{smallmatrix} 1\right] \right]}{\partial t} \right] \right|_{t=0} = \frac{|a|[\psi^{(1)}(d+p) + \psi^{(1)}(p) + [\psi(d+p) - \psi(p)]^2]}{2} \quad (112)$$

Or

$$\left| a \left[\frac{p[\psi(d+p) - \psi(p)]}{d(d+p)} + \frac{d}{d+p} \frac{\partial \left[{}_F\left[\begin{smallmatrix} a(t)+1, b(t)+1, 1; \\ c(t)+1, 2; \end{smallmatrix} 1\right] \right]}{\partial t} \right] \right|_{t=0} = \frac{|a|[\psi^{(1)}(d+p) + \psi^{(1)}(p) + [\psi(d+p) - \psi(p)]^2]}{2} \quad (113)$$

Finally,

$$\left. \frac{\partial \left[{}_F\left[\begin{smallmatrix} a(t)+1, b(t)+1, 1; \\ c(t)+1, 2; \end{smallmatrix} 1\right] \right]}{\partial t} \right|_{t=0} = |a| \frac{d(d+p)[\psi^{(1)}(d+p) + \psi^{(1)}(p) + [\psi(d+p) - \psi(p)]^2] - 2p[\psi(d+p) - \psi(p)]}{2d^2} \quad (114)$$

Equation (114) is a new identity that we just obtained.

In order to obtain (106) in closed form expression we need to obtain $\frac{\partial \left[{}_F\left[\begin{smallmatrix} a(t)+1, b(t)+1, 1; \\ c(t)+1, 2; \end{smallmatrix} q\right] \right]}{\partial t}$ is closed form expression for all values of $0 < q < 1$ which is not easily obtainable.

However, by comparing and contrasting (106) with (77) we obtain

$$\left. \frac{\partial \left[{}_F\left[\begin{smallmatrix} a(t)+1, b(t)+1, 1; \\ c(t)+1, 2; \end{smallmatrix} q\right] \right]}{\partial t} \right|_{t=0} = \frac{a^2(d+p)}{2|a|dq} \left[-2 \frac{\{d(d+p)[\psi(d) - \psi(d+p)] + p\} {}_F\left[\begin{smallmatrix} d+1, 1, 1; \\ d+p+1, 2; \end{smallmatrix} q\right]}{(d+p)^2} + \frac{\int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(1-qx'') dx'' - 2 \int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') \log(1-qx'') dx''}{B(d, p)} \right] \quad (115)$$

Which is equivalent with

$$\left. \frac{\partial \left[{}_F\left[\begin{smallmatrix} a(t)+1, b(t)+1, 1; \\ c(t)+1, 2; \end{smallmatrix} q\right] \right]}{\partial t} \right|_{t=0} = \frac{|a|}{d} \left[- \frac{\{d(d+p)[\psi(d) - \psi(d+p)] + p\} {}_F\left[\begin{smallmatrix} d+1, 1, 1; \\ d+p+1, 2; \end{smallmatrix} q\right]}{d+p} + \frac{(d+p) \int_0^1 x''^{d-1} (1-x'')^{p-1} \log^2(1-qx'') dx'' - 2 \int_0^1 x''^{d-1} (1-x'')^{p-1} \log(x'') \log(1-qx'') dx''}{2q B(d, p)} \right] \quad (116)$$

Equation (116) gives for the first-time relation between the partial derivative of ${}_F\left[\begin{smallmatrix} a(t)+1, b(t)+1, 1; \\ c(t)+1, 2; \end{smallmatrix} q\right]$ with respect to t evaluated at $t = 0$ and for $q \neq 0$. For $q = 1$ we can substitute (72) into (116) and employ Progrid's identity (Progrid 2016, [12] (125)) we can obtain (114).

This concludes the discussion on the EGBD statistics for general case. The exact closed form expression of (106) is considered next.

8 Closed Form Expression of EGBD Variance

This section discusses the possibility of producing the closed form expression of (106). In order to produce the closed form expression of (106) we must produce the closed form expression of (116). From (116) we have two integrals that we want to produce the closed form expression. First, let us consider the closed form expression of the first integral. From Proгри 2016 ([12], 138)

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log^2(1-qx) dx}{B(d,p)} = \frac{\int_0^1 x^{d-1}(1-x)^{p-1} (-1) \sum_{k=1}^{\infty} \frac{q^k x^k}{k} (-1) \sum_{n=1}^{\infty} \frac{q^n x^n}{n} dx}{B(d,p)} \quad (117)$$

Rearranging the order of summation and integration we can obtain

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log^2(1-qx) dx}{B(d,p)} = \frac{(-1)^2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{k+n}}{kn} \int_0^1 x^{d+k+n-1} (1-x)^{p-1} dx}{B(d,p)} = \frac{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{k+n} B(d+k+n,p)}{kn}}{B(d,p)} \quad (118)$$

Which can be written as

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log^2(1-qx) dx}{B(d,p)} = \frac{q^2 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(d+k+n+2) \Gamma(p) q^{k+n}}{\Gamma(d+k+n+2+p) (k+1)(n+1)}}{B(d,p)} = \frac{q^2 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(d+2+k+n) \Gamma(p) q^k q^n}{\Gamma(d+p+2+k+n) (k+1)(n+1)}}{B(d,p)} \quad (119)$$

Which is equivalent with

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log^2(1-qx) dx}{B(d,p)} = \frac{q^2 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(d+2)(d+2)_{k+n} \Gamma(p) q^k q^n}{\Gamma(d+p+2)(d+p+2)_{k+n} (k+1)(n+1)}}{B(d,p)} = \frac{q^2 \Gamma(d+2) \Gamma(p)}{\Gamma(d+p+2)} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(d+2)_{k+n} q^k q^n}{(d+p+2)_{k+n} (k+1)(n+1)} \quad (120)$$

Which is equivalent with

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log^2(1-qx) dx}{B(d,p)} = \frac{q^2 \frac{(d+1)d \Gamma(d) \Gamma(p)}{(d+p+1)(d+p) \Gamma(d+p)} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(d+2)_{k+n} q^k q^n}{(d+p+2)_{k+n} (k+1)(n+1)}}{B(d,p)} = q^2 \frac{(d+1)d \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(d+2)_{k+n} q^k q^n}{(d+p+2)_{k+n} (k+1)(n+1)}}{(d+p+1)(d+p)} \quad (121)$$

Finally, we can produce the closed form expression of (121) with the help of Kampé de Fériet function (See Proгри 2016, [14])

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log^2(1-qx) dx}{B(d,p)} = \frac{q^2 (d+1)d \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(d+2)_{k+n} (1)_k (1)_n q^k q^n}{(d+p+2)_{k+n} (2)_k (2)_n k! n!}}{(d+p+1)(d+p)} = \frac{q^2 (d+1) d F_{1:2;2}^{1:1;1} \left[\begin{matrix} d+2; 1, 1 \\ d+p+2; 2, 2 \end{matrix} ; q, q \right]}{(d+p+1)(d+p)} \quad (122)$$

Next, let us consider the closed form expression of the second integral of (116). Again from (Gradshteyn and Ryzhik, 2007 [19], pg. 53 ex. 1.512 1.) and Proгри 2016 ([12], 138)

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log(x) \log(1-qx) dx}{B(d,p)} = \frac{\int_0^1 x^{d-1}(1-x)^{p-1} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{q^k (x-1)^k}{k} (-1) \sum_{n=1}^{\infty} \frac{q^n x^n}{n} dx}{B(d,p)} \quad (123)$$

Which is equivalent with

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log(x) \log(1-qx) dx}{B(d,p)} = \frac{\int_0^1 x^{d-1}(1-x)^{p-1} \sum_{k=1}^{\infty} (-1)^{k+1} (-1)^k \frac{q^k (1-x)^k}{k} (-1) \sum_{n=1}^{\infty} \frac{q^n x^n}{n} dx}{B(d,p)} \quad (124)$$

Or after interchanging the order of summation and integration produces

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log(x) \log(1-qx) dx}{B(d,p)} = \frac{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_0^1 x^{d+n-1} (1-x)^{p+k-1} dx \frac{q^k q^n}{k n}}{B(d,p)} = \frac{\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} B(d+n, p+k) \frac{q^k q^n}{k n}}{B(d,p)} \quad (125)$$

Which is equivalent with

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log(x) \log(1-qx) dx}{B(d,p)} = \frac{\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B(d+n+1, p+k+1) \frac{q^{k+1} q^{n+1}}{k+1 n+1}}{B(d,p)} = q^2 \frac{\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(d+n+1) \Gamma(p+k+1) q^k q^n}{\Gamma(d+p+2+n+k) k+1 n+1}}{B(d,p)} \quad (126)$$

Which can be further written as

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log(x) \log(1-qx) dx}{B(d,p)} = q^2 \frac{\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(d+1)(d+1)_n \Gamma(p+1)(p+1)_k q^k q^n}{\Gamma(d+p+2)(d+p+2)_{n+k} k+1 n+1}}{B(d,p)} = q^2 \frac{\Gamma(d+1) \Gamma(p+1)}{\Gamma(d+p+2)} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(d+1)_n (p+1)_k q^k q^n}{(d+p+2)_{n+k} k+1 n+1} \quad (127)$$

Or

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log(x) \log(1-qx) dx}{B(d,p)} = q^2 \frac{\frac{d \Gamma(d) p \Gamma(p)}{(d+p+1)(d+p) \Gamma(d+p)} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(p+1)_k (d+1)_n k! n! k! n!}{(d+p+2)_{n+k} (k+1)! (n+1)! k! n!} q^k q^n}{B(d,p)} \quad (128)$$

Finally, we can produce the closed form expression of (128) with the help of Kampé de Fériet function (See Proгри 2016, [14])

$$\frac{\int_0^1 x^{d-1}(1-x)^{p-1} \log(x) \log(1-qx) dx}{B(d,p)} = \frac{q^2 dp \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(p+1)_k (1)_k (1)_k (d+1)_n (1)_n (1)_n q^k q^n}{(d+p+2)_{(n+k)} (2)_k (2)_n k! n!}}{(d+p+1)(d+p)} = \frac{q^2 dp F_{1:1;1}^{0:3;3} \left[\begin{matrix} -p+1, 1, 1; d+1, 1, 1 \\ d+p+2, 2, 2 \end{matrix} ; q, q \right]}{(d+p+1)(d+p)} \quad (129)$$

Substituting (122) and (129) into (116) yields,

$$\left. \frac{\partial}{\partial t} \left[F_{1:1;1}^{a(t)+1, b(t)+1, 1; c(t)+1, 2; } \left[\begin{matrix} d+1, 1, 1; \\ d+p+1, 2; \end{matrix} ; q \right] \right] \right|_{t=0} = \frac{|a|}{d} - \frac{\{d(d+p)[\psi(d)-\psi(d+p)]+p\} F_{1:1;1}^{d+1, 1, 1; d+p+1, 2; } \left[\begin{matrix} d+1, 1, 1; \\ d+p+1, 2; \end{matrix} ; q \right]}{d+p} + \frac{qd}{2} \frac{\left\{ (d+1) F_{1:1;1}^{1:2;2} \left[\begin{matrix} d+2:1, 1, 1; \\ d+p+2:2, 2; \end{matrix} ; q, q \right] - 2p F_{1:1;1}^{0:3;3} \left[\begin{matrix} -p+1, 1, 1; d+1, 1, 1; \\ d+p+2:2, 2; \end{matrix} ; q, q \right] \right\}}{d+p+1} \quad (130)$$

Finally, the closed form expression of (106) is as follows

$$K_2 \equiv Var(X) = a^2 \left[\psi^{(1)}(d) - \psi^{(1)}(d+p) - \frac{\left[\frac{dq F_{1:1;1}^{d+1, 1, 1; d+p+1, 2; } \left[\begin{matrix} d+1, 1, 1; \\ d+p+1, 2; \end{matrix} ; q \right]}{d+p} \right]^2}{d+p} - \frac{2[\psi(d)-\psi(d+p)] dq F_{1:1;1}^{d+1, 1, 1; d+p+1, 2; } \left[\begin{matrix} d+1, 1, 1; \\ d+p+1, 2; \end{matrix} ; q \right]}{d+p} + \frac{dq^2 \left\{ (d+1) F_{1:1;1}^{1:2;2} \left[\begin{matrix} d+2:1, 1, 1; \\ d+p+2:2, 2; \end{matrix} ; q, q \right] - 2p F_{1:1;1}^{0:3;3} \left[\begin{matrix} -p+1, 1, 1; d+1, 1, 1; \\ d+p+2:2, 2; \end{matrix} ; q, q \right] \right\}}{(d+p)(d+p+1)} \right] \quad (131)$$

Equation (131) represents for the first time the closed form expression of the variance of the EGBD.

It remains to perform the final check. First, clearly for $q = 0$, K_2 from (131) is equal to (25), (46), and (107). Second, for $q = 1$, K_2 must equal to (36)

$$\frac{F_{1:1;1}^{1:2;2} \left[\begin{matrix} d+2:1, 1, 1; \\ d+p+2:2, 2; \end{matrix} ; 1, 1 \right]}{(d+1)^{-1}} - \frac{F_{1:1;1}^{0:3;3} \left[\begin{matrix} -p+1, 1, 1; d+1, 1, 1; \\ d+p+2:2, 2; \end{matrix} ; 1, 1 \right]}{(2p)^{-1}} = \frac{(d+p)(d+p+1) \{ \psi^{(1)}(p) + \psi^{(1)}(d+p) + [\psi(d+p) - \psi(p)] [2\psi(d) - \psi(p) - \psi(d+p)] \}}{d} \quad (132)$$

Or from (73) and (74) and (122) and (129) we can obtain

$$F_{1:1;1}^{1:2;2} \left[\begin{matrix} d+2:1, 1, 1; \\ d+p+2:2, 2; \end{matrix} ; 1, 1 \right] = \frac{(d+p+1)(d+p) \{ \psi^{(1)}(p) - \psi^{(1)}(d+p) + [\psi(p) - \psi(d+p)]^2 \}}{d(d+1)} \quad (133)$$

$$F_{1:1;1}^{0:3;3} \left[\begin{matrix} -p+1, 1, 1; d+1, 1, 1; \\ d+p+2:2, 2; \end{matrix} ; 1, 1 \right] = \frac{(d+p)(d+p+1) \{ [\psi(d+p)]^2 + \psi(d)\psi(p) - \psi(p)\psi(d+p) - \psi(d)\psi(d+p) - \psi^{(1)}(d+p) \}}{dp} \quad (134)$$

One can verify that substituting (133) and (134) turns (132) into an identity. We can see that the closed form expression of $K_2 \equiv Var(X)$ is a complicated expression because it requires the computation of two Kampé de Fériet functions (See Proгри 2016, [14]) for which even MATLAB 2016a does not have a function to do so. We can only compute Kampé de Fériet functions indirectly as part of the integration or integral. The direct computation of the Kampé de Fériet function will be considered in a future publication.

9 Numerical, Theoretical Results

The parameters of the EGBD are as follows: $a = 1$, $b = 2$, $c = 0$, $d = 2$, and $p = 3$. Let us consider EGBD statistics for values of $q = \{0, 0.5, 1\}$ which consists of examples 1-3. Moreover, we show the pdf and cdf for convolution EGB for $\alpha = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$ and exponentiated EGB for $\alpha = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$ and $\alpha = [0.5 \ 1 \ 1 \ 1 \ 1 \ 1]$.

9.1 Example 1: EGBD statistics for $q = 0$

In order to compute the mean of X where X EGBD given the parameters $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, and $q = 0$ we employed three formulas: (17), (57), and (91).

There was no surprise that the answer from the three were

identical. $E(X) = 0.9166666666666667$ from (17) and (91) and $E(X) = 0.9166666666666666$ from (57). The error between (57) and either (17) or (91) was 10^{-16} .

In order to compute the variance of X where X EGBD given the parameters $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, and $q = 0$ we employed three formulas: (25), ((17), (62), and (23)) and (77). In the computation of the variance, we got $Var(X) = 0.650854139658888$ and no error or 0 error.

Figure 1 shows the generalized convolution EGB of 3rdkind pdf for $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$ and $\alpha = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$.

The pdf and cdf in Fig. 1 is supposed to be zero mean and unit variance Gaussian or normal.

Figure 2 displays the generalized exponentiated EGB of 3rdkind pdf $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, $q = 0$

and $\alpha = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$.

The EGB pdf looks very sharp and the cdf is very steep.

Figure 3 illustrates exactly the same information as Fig. 2 but for $\alpha = [0.5 \ 1 \ 1 \ 1 \ 1 \ 1]$. We can see how sharpness changes for the pdf and the cdf becomes less steep. It is a way of smoothing the rough edges.

9.2 Example 2: EGBD statistics for $q = 0.5$

In order to compute the mean of X where X EGBD given the parameters $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, and $q = 0.5$ we employed two formulas: (91) and (61).

There was no surprise that the answer from the two were identical. $E(X) = 1.147969736080383$ from both (91) and (61). The error between (91) and (61) is 0.

In order to compute the variance of X where X EGBD given the parameters $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, and $q = 0.5$ we employed two formulas: ((61), (62), and (23)) and (75). In the computation of the variance, we got $Var(X) = 0.768888432699550$ from (75) and $Var(X) = 0.768888432699559$ from ((17), (62), and (23)) with an absolute error of 10^{-16} .

Figures 4 through 6 portray exactly the same scenarios as Figs. 1 through 3 but for $q = 0.5$.

There is no surprise that Fig. 4 looks almost identical to Fig. 1 and Figs. 5 and 6 pdfs/cdfs are less sharper/steeper than Figs. 1 and 3 pdfs/cdfs because Figs. 5 and 6 pdfs correspond to $q = 0.5$ as opposed to Figs. 1 and 3 pdfs correspond to $q = 0$.

We observe that there are two ways to reduce the sharpness or steepness of the function: one by changing the exponentiated vector α and the other by changing the sharpness or steepness parameter q .

9.3 Example 3: EGBD statistics for $q = 1$

In order to compute the mean of X where X EGBD given the parameters $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, and $q = 1$ we employed three formulas: (29), (61), and (91). There was no surprise that the answer from the three were identical. $E(X) = 1.95$ from (29), (61), and (91). The error between (29), (61), and (91) was 0.

In order to compute the variance of X where X EGBD given the parameters $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, and $q = 1$ we employed three formulas: (36), ((29), (62), and (23)) and (77).

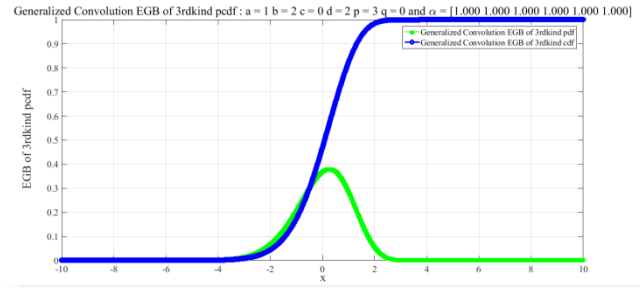


FIGURE 1: Generalized convolution EGB of 3rdkind pdf: $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, $q = 0$ and $\alpha = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$.

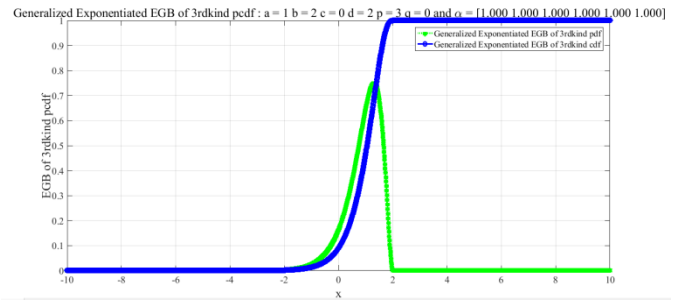


FIGURE 2: Generalized exponentiated EGB of 3rdkind pdf $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, $q = 0$ and $\alpha = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$.

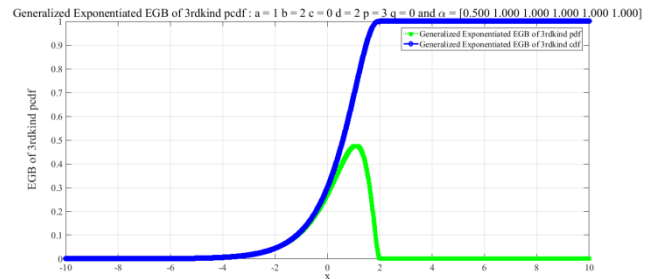


FIGURE 3: Generalized exponentiated EGB of 3rdkind pdf $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, $q = 0$ and $\alpha = [0.5 \ 1 \ 1 \ 1 \ 1 \ 1]$.

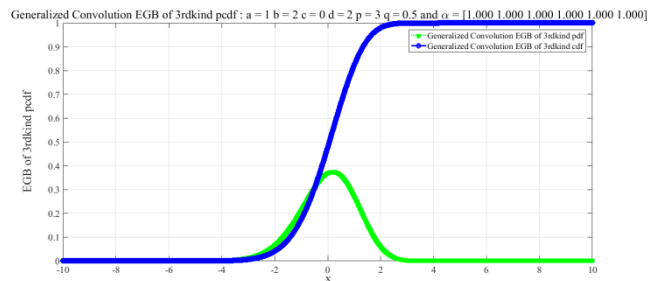


FIGURE 4: Generalized convolution EGB of 3rdkind pdf: $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, $q = 0.5$ and $\alpha = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$.

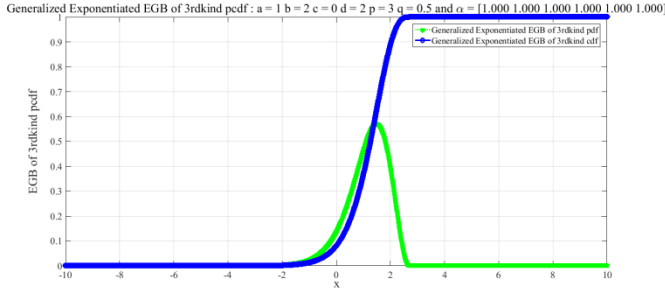


FIGURE 5: Generalized exponentiated EGB of 3rdkind pdf $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, $q = 0.5$ and $\alpha = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$.

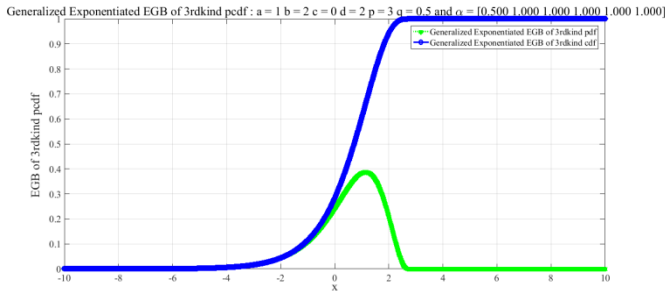


FIGURE 6: Generalized exponentiated EGB of 3rdkind pdf $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, $q = 0.5$ and $\alpha = [0.5 \ 1 \ 1 \ 1 \ 1 \ 1]$.

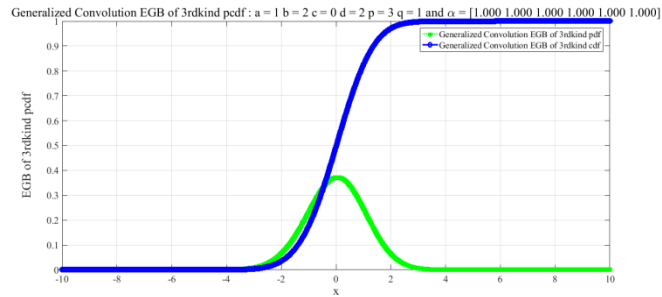


FIGURE 7: Generalized convolution EGB of 3rdkind pdf: $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, $q = 1$ and $\alpha = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$.

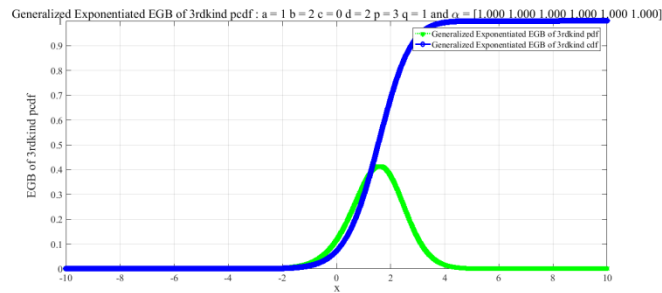


FIGURE 8: Generalized exponentiated EGB of 3rdkind pdf $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, $q = 1$ and $\alpha = [1 \ 1 \ 1 \ 1 \ 1 \ 1]$.

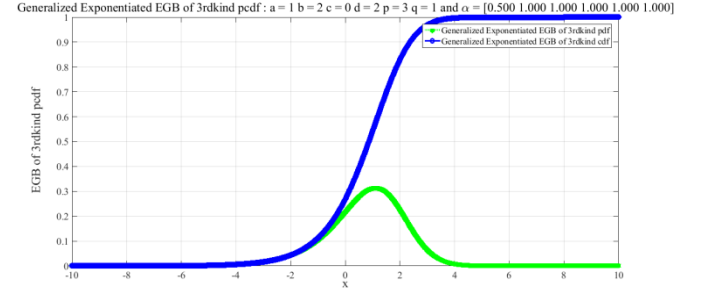


FIGURE 9: Generalized exponentiated EGB of 3rdkind pdf $a = 1$, $b = 2$, $c = 0$, $d = 2$, $p = 3$, $q = 1$ and $\alpha = [0.5 \ 1 \ 1 \ 1 \ 1 \ 1]$.

In the computation of the variance we got $Var(X) = 1.019739247894506$ from (36), $Var(X) = 1.019739247894503$ from (77) and $Var(X) = 1.019739247891255$ from ((29), (62), and (23)) with an absolute error of 3×10^{-16} between the first two and 3.248×10^{-12} between the last two.

Figures 7 through 9 present exactly the same scenarios as Figs. 1 through 3 or Fig1. 4 through 6, but for $q = 1$.

There is no surprise that Fig. 7 looks almost identical to Fig. 1 or Fig. 4, and Figs. 8 and 9 pdfs/cdfs are less sharper/steeper than Figs. 1 and 3 (or Figs. 4 and 5) pdfs/cdfs because Figs. 8 and 9 pdfs correspond to $q = 1$ as opposed to Figs. 1 and 3 (or Figs. 4 and 5) pdfs correspond to $q = 1$ (or $q = 0.5$).

We observe that there are two ways to reduce the sharpness or steepness of the function: one by changing the exponentiated vector α and the other by changing the sharpness or steepness parameter q .

10 Conclusions

I believe that I may be either the first or one of the first people to have computed the closed form expression of the mean and variance of the EGBD for the general case.

As seen from the equations in the paper, for the computation of the mean requires only two digamma functions and one hypergeometric function; however, the computation of the variance requires the computation of two polygamma functions of the first order, two digamma functions, one hypergeometric function, and two Kampé de Fériet functions.

The computation of the closed form expression of the variance of the EGBD is exponentially more difficult than the computation of the mean.

Numerical results validate our theoretical models within the

numerical precision offered by MATLAB 2016a.

This paper falls into the category of the “Marvels in Analytical Derivations” because the computation of closed form expression of the mean and variance of the EGBD requires out of the box derivations and manipulations.

11 Acknowledgement

This work was supported by Giftet Inc. executive office.

12 References

- [1] Anon., “Probability density function,” *Wikipedia*, the free encyclopedia, Jul. 2013. http://en.wikipedia.org/wiki/Probability_density_function
- [2] Anon., “Cumulative distribution function,” *Wikipedia*, the free encyclopedia, Jul. 2013. http://en.wikipedia.org/wiki/Cumulative_distribution_function
- [3] I.F. Progri, “An assessment of indoor geolocation systems,” *Ph.D. Dissertation*, Worcester Polytechnic Institute, May 2003.
- [4] I. Progri, “A unified geolocation channel model—part I (path loss),” in *Proc. ION-NTM 2005*, San Diego, CA, pp. 1148-1161, Jan. 2005.
- [5] I. Progri, “A unified geolocation channel model—part II (multipath distribution),” in *Proc. WTS-2006*, Pomona, CA, pp. 1-8, Apr. 2006. DOI: <https://doi.org/10.1109/WTS.2006.334542>
- [6] W.R. Michalson, I. Progri, “Reconfigurable geolocation system,” US Patent 7,079,025, July 2006.
- [7] I. Progri, W.R. Michalson, J. Wang, M.C. Bromberg, “Theoretical data on support of a unified indoor geolocation channel model,” in *Proc. ION-NTM 2007*, San Diego, CA, pp. 577-584, Jan. 2007.
- [8] I. Progri, *Geolocation of RF Signals—Principles and Simulations*. 1st ed., New York, NY: Springer SBM, LLC, 330 pp., Jan. 2011. DOI: <https://doi.org/10.1007/978-1-4419-7952-0>
- [9] I. Progri, “On generalized multi-dimensional geolocation modulation waveforms,” in *Proc. IEEE/ION-PLANS 2012*, Myrtle Beach, SC, pp. 919-951, Apr. 2012. DOI: <https://doi.org/10.1109/PLANS.2012.6236835>
- [10] Anon., “Generalized beta distribution,” *Wikipedia*, the free encyclopedia, June 2013. http://en.wikipedia.org/wiki/Generalized_beta_distribution
- [11] I. Progri, *Indoor Geolocation Systems—Theory and Applications. Vol. I*, 1st ed., Worcester, MA: Giftet Inc., ~800 pp., ~2017. (not yet available in print)
- [12] I. Progri, “Hypergeometric function partial derivatives,” *J. Geol. Geoinfo. Geointel.*, vol. 2016, article ID 2016071604, 21 pg., Nov. 2016. DOI: <https://doi.org/10.18610/JG3.2016.071604>
- [13] J.E. Gottschalk, E.N. Maslen, “Reduction formulae for generalised hypergeometric functions of one variable,” *J. Phys. A: Math. Gen.*, vol. 21 pp. 1983-1998, 1988. DOI: <https://doi.org/10.1088/0305-4470/21/9/015>
- [14] I. Progri, “Generalized Bessel function distributions,” *J. Geol. Geoinfo. Geointel.*, vol. 2016, article ID 2016071602, 14 pg., Nov. 2016. DOI: <https://doi.org/10.18610/JG3.2016.071602>
- [15] Anon., “Beta function,” *From Wikipedia, the free encyclopedia*, Apr. 2016. https://en.wikipedia.org/wiki/Beta_function
- [16] S. Nadarajah, S. Kotz, “A comment on “Modeling to support reliability enhancement during product development with applications in the U.K. aerospace industry”,” in *IEEE Trans. Engin. Manag.*, vol. 55, no. 2, pp. 338-339, May 2008. DOI: <https://doi.org/10.1109/TEM.2008.919731>
- [17] L. Walls, J. Quigley, J. Marshall, “Modeling to support reliability enhancement during product development with applications in the U.K. Aerospace industry,” *IEEE Trans. Eng. Manage.*, vol. 53, no. 2, pp. 263-274, May 2006. DOI: <https://doi.org/10.1109/TEM.2006.872342>
- [18] I. Gessel, “Finding identities with the WZ method,” *J. Sym. Com.*, vol. 20, no 5-6, pp. 537-566, Nov. 1995. DOI: <https://doi.org/10.1006/jsco.1995.1064>
- [19] I.S. Gradshteyn, I.M. Ryzhik, (A. Jeffrey, D. Zwillinger, editors) *Table of Integrals, Series, and Products*, 7th ed., Burlington, MA: Academic Press, 1171 pg., 2007.
- [20] A.M. Mathai, S.B. Provost, “On the distribution of order statistics from generalized logistic samples,” *METRON - Inter. J. Statis.*, vol. LXII, no. 1, pp. 63-71, 2004.

ⁱ The only mention of the Kampé de Fériet functions in IEEE Xplore comes

from Nadarajah and Kotz, 2008 [16]. This is not even a real paper it is a comment on a paper Walls et al. 2006 [17]. I should mention that the work

presented in this paper is developed entirely independent and original with no connection what so ever from Nadarajah and Kotz, 2008 [16].