

Research Article

Generalized Bessel Function Distributions

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This paper discussed two generalized Bessel function distributions one of which is original and never presented before in the literature. For each generalized Bessel distribution model the closed form expression of the cumulative distribution function (cdf) is given by means of series expansion of the modified Bessel functions which leads to incomplete gamma functions, hypergeometric series, and the Kampé de Fériet function. Numerical results are derived for each case to validate the theoretical models presented in the paper.

Index Terms—Bessel functions, modified Bessel functions, cumulative distribution function, Kampé de Fériet function, series expansion, closed form expression, signal analysis, generalized functions, time-domain analysis, incomplete gamma functions, hypergeometric series.

1 Introduction

There are several publications that discuss the Bessel functions distributions of the first and second kind [1]-[13] due to its importance of modeling “heavier” than Gaussian tails observed in the data [7].

The issue, however, is not whether we try to approximate something “heavier” than Gaussian tails observed in the data as Bessel distributions of the first and second kind [7]. The main issue is the following: Are the Bessel distributions of the first and second kind valid probability density functions (pdf)? And if yes is there proof to confirm that they indeed are valid pdfs?

The Bessel function distribution was first proposed by McKay 1938 [1]. The proposed Bessel function distribution by McKay 1938 [1] works well for positive x s; however, McKay 1938 [1] appears to be wrong for negative x s.

McKay original error has propagated to almost every publication Miller 1964 [2], McNolty 1967 [3], 1974 [4], Lingappaiah 1978 [5], Arnaut 2003 [6], Nadarajah 2007 [7], [8], 2008 [9], Arnaut 2009 [10], and Ahsanullah et al. 2015 [11], just to name a few authors.

It was not until Progri et al. 2016 [12], [13] that the mistakes (or normalized coefficient errors and negative values)

associated with Bessel function distributions of the first and second kind were identified and fixed.

In almost all publications with the exception of Arnaut 2003 [6] the closed form expression of the cdf were not shown.

The computation of the closed form expression of the cdfs of the Bessel function distributions of the first and second kinds was accomplished via the series expansion [14]-[21] and the application of special function known as the Kampé de Fériet function [22]-[25]. As such this paper presents an original work that falls into the category of *Marvels in analytical derivations series* never published anywhere else.

This work presents a significant advancement of the pdfs and cdfs of indoor geolocation channel models [26]-[41].

In the numerical results section, we present both the pdfs and cdfs of the generalized Bessel functions of the first and second kinds for values of x from $-\infty$ to ∞ . The methodology is consistent with the works presented by Arnaut 2003 [6] even though the work of the latter is only valid for positive x s and only for a special class of Bessel function distributions.

Since, the generalized Bessel functions cdfs require the computation of the Kampé de Fériet function [22]-[25] and the latter is not yet available in MATLAB 2016a [42], Giftet Inc. is also taking on the task to compute the Kampé de Fériet function [22]-[25] as part of the Indoor Geolocation Systems MATLAB Library and publish the results of this work in Dr. Progrid pioneer publication [26].

This paper is organized as follows: in Sect. 2 generalized Bessel distributions of the first kind and second kinds are discussed. The series expansion of the generalized Bessel distribution is discussed in Sect. 3. Section 4 contains numerical, theoretical results; Conclusion is provided in Sect. 5 along with a list of references.

2 Generalized Bessel Distributions

Generalized Bessel distributions consist of the generalized Bessel distributions of the first kind and second kinds. The generalized Bessel distribution of the first kind is discussed first and the generalized Bessel distribution of the second kind is discussed second.

2.1 Generalized Bessel Distribution of the First Kind

Generalized Bessel distribution of the first kind has its pdf given by

$$f_{\text{GBessel1}}(x; a, d, p) = \begin{cases} 0, & x < 0 \\ \frac{\sqrt{\pi}|1-d^2|^{p_1} x^p e^{-\frac{d}{a}x} I_p\left(\frac{x}{a}\right)}{2^p \Gamma(p_1) a^{p_2}} & x \geq 0 \end{cases} \quad (1)$$

for $a > 0$, $d > 1$, and $p > 1$ where [1]

$$I_p(x) = \frac{x^p}{\sqrt{\pi} 2^p \Gamma(p_1)} \int_{-1}^1 (1-t^2)^{p_1-1} e^{\pm xt} dt \quad (2)$$

is the modified Bessel function of the first kind; and where

$$p_i = p + \frac{i}{2}; \quad i = \{1, 2, \dots\} \quad (2a)$$

Since, there appears to be an error in the formation of the pdf of the Generalized Bessel distribution of the first kind in (1) in [8], [9] the $\sqrt{\pi}$ in formulation of the univariate Bessel distribution of the first kind in [8], [9] is in the denominator as opposed to in (1) in the numerator; hence, here we provide a complete proof that indeed (1) is a valid pdf.

Since,

$$f_{\text{GBessel1}}(x; a, d, p) \geq 0 \quad (3)$$

for $a > 0$, $d > 1$, and $p > 0$; it only remains to show that

$$\int_0^\infty f_{\text{GBessel1}}(x; a, d, p) dx = 1 \quad (4)$$

Taking the integral of (1) yields

$$\int_0^\infty \frac{\sqrt{\pi}|1-d^2|^{p_1} x^p e^{-\frac{d}{a}x} I_p\left(\frac{x}{a}\right) dx}{2^p \Gamma(p_1) a^{p_2}} = \frac{\int_0^\infty \left(\frac{x}{a}\right)^p e^{-\frac{d}{a}x} I_p\left(\frac{x}{a}\right) d\left(\frac{x}{a}\right)}{C_1(p, d)} \quad (5)$$

where

$$C_1(p, d) = \frac{2^p \Gamma(p_1)}{\sqrt{\pi}|1-d^2|^{p_1}} \quad (6)$$

If we make the substitution $x' \equiv \frac{x}{a}$ we have $dx' = \frac{dx}{a}$ and $x \rightarrow |_0^\infty$ then $x' \rightarrow |_0^\infty$; hence, (5) can be written as

$$\frac{\int_0^\infty \left(\frac{x}{a}\right)^p e^{-\frac{d}{a}x} I_p\left(\frac{x}{a}\right) d\left(\frac{x}{a}\right)}{C_1(p, d)} = \frac{\int_0^\infty x'^p e^{-x'd} I_p(x') dx'}{C_1(p, d)} \quad (7)$$

We can prove (7) is equal to one in three different ways. First, using the table of integrals (see [14] pg. 701, ex. 6.623 1.) we have

$$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^\nu dx = \frac{(2\beta)^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}(\alpha^2 + \beta^2)^{\nu + \frac{1}{2}}} \quad (8)$$

for $\text{Real}(\nu) > -\frac{1}{2}$, $\text{Real}(\alpha) > \text{Imag}(|\beta|)$ where

$$J_\nu(x) = \frac{x^\nu}{\sqrt{\pi} 2^\nu \Gamma\left(\nu + \frac{1}{2}\right)} \int_{-1}^1 (1-t^2)^{\nu - \frac{1}{2}} e^{ixt} dt \quad (9)$$

is the Bessel function of the first kind for $\text{Real}(v) > -\frac{1}{2}$. In order to use (8) into (7) we have to convert $J_v(\beta x)$ into $I_v(\beta x)$ as follows (see [14] pg. 911 ex. 8.406 3.). For v integer

$$I_v(x) = i^{-v} J_v(ix) \leftrightarrow J_v(ix) = i^v I_v(x) \quad (10)$$

Hence, (8) for $\beta = i$ can be written as

$$\int_0^\infty e^{-\alpha x} J_v(ix) x^v dx = \frac{(2i)^v \Gamma(v+\frac{1}{2})}{\sqrt{\pi}(\alpha^2+i^2)^{v+\frac{1}{2}}} = \frac{2^v i^v \Gamma(v+\frac{1}{2})}{\sqrt{\pi}(\alpha^2-1)^{v+\frac{1}{2}}} \quad (11)$$

Or substituting (10) into (11) produces

$$\int_0^\infty e^{-\alpha x} i^v I_v(x) x^v dx = \frac{2^v i^v \Gamma(v+\frac{1}{2})}{\sqrt{\pi}(\alpha^2-1)^{v+\frac{1}{2}}} \quad (12)$$

Or

$$\int_0^\infty e^{-\alpha x} I_v(x) x^v dx = \frac{2^v \Gamma(v+\frac{1}{2})}{\sqrt{\pi}(\alpha^2-1)^{v+\frac{1}{2}}} \quad (13)$$

Using the result of (13), (7) can be written as

$$\frac{\int_0^\infty x^p e^{-x^d} I_p(x) dx}{C_1(p,d)} = \frac{1}{C_1(p,d)} \frac{2^p \Gamma(p_1)}{\sqrt{\pi}(d^2-1)^{p_1}} \quad (14)$$

Finally, substituting the value of $C_1(p,d)$ in (6) into (14) we have

$$\frac{\int_0^\infty x^p e^{-x^d} I_p(x) dx}{C_1(p,d)} = \frac{1}{\frac{2^p \Gamma(p_1)}{\sqrt{\pi}|1-d^2|^{p_1}}} \frac{2^p \Gamma(p_1)}{\sqrt{\pi}(d^2-1)^{p_1}} = 1 \quad (15)$$

The second way to prove it is using direct integration. Substituting (2) into (7) yields

$$\frac{\int_0^\infty x^{2p} \int_{-1}^1 (1-t^2)^{p'} e^{-d'x^t} dt dx'}{C_1(p,d) \sqrt{\pi} 2^p \Gamma(p_1)} = \frac{\int_{-1}^1 (1-t^2)^{p'} \int_0^\infty e^{-d'x^t} x^{2p} dx' dt}{C_1(p,d) \sqrt{\pi} 2^p \Gamma(p_1)} \quad (16)$$

where $p' = p - \frac{1}{2} = p_1 - 1$, $d' = d \mp t$.

Now, from the definition of the gamma function [16] we have

$$\int_0^\infty e^{-d'x^t} x'^{2p} dx' = \frac{\Gamma(2p_1)}{d'^{2p_1}} \quad (17)$$

Substituting, (17) into (16) yields

$$\frac{\int_{-1}^1 (1-t^2)^{p'} \frac{\Gamma(2p_1)}{(d \mp t)^{2p_1}} dt}{C_1(p,d) \sqrt{\pi} 2^p \Gamma(p_1)} = \frac{\Gamma(2p_1) \int_{-1}^1 \frac{(1-t^2)^{p'}}{(d \mp t)^{2p_1}} dt}{C_1(p,d) \sqrt{\pi} 2^p \Gamma(p_1)} \quad (18)$$

In order for (18) to be equal to one then the following should be true

$$\int_{-1}^1 \frac{(1-t^2)^{p_1-1}}{(d \mp t)^{2p_1}} dt = \frac{2^{2p_2} \Gamma(p_1)}{|1-d^2|^{p_1} \Gamma(2p_1)} \quad (19)$$

Utilizing the expression found in (see [14] pg. 322, ex. 3.238 3.) we have

$$\int_a^b \frac{(b-t)^{\mu-1} (t-a)^{\nu-1}}{|t-u|^{\mu+\nu}} dt = \frac{(b-a)^{\mu+\nu-1} [B(\mu, \nu) \equiv \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}]}{|a-u|^\mu |b-u|^\nu} \quad (20)$$

Can we write (19) in the form of (20)? The answer is yes. Hence, the following is obtained

$$\int_{-1}^1 \frac{(1-t^2)^{p_1-1}}{(d \mp t)^{2p_1}} dt = \int_{-1}^1 \frac{(1-t)^{p_1-1} (1+t)^{p_1-1}}{(d \mp t)^{2p_1}} dt \quad (21)$$

By setting the following we have

$$\mu - 1 = p_1 - 1 \quad (22)$$

$$\nu - 1 = p_1 - 1 \quad (23)$$

$$a = -1 \quad (24)$$

$$b = 1 \quad (25)$$

$$d = u \quad (26)$$

Then we have

$$\mu = p_1 \quad (27)$$

$$\nu = p_1 \quad (28)$$

and

$$\mu + \nu = 2p_1 \quad (29)$$

Hence, substituting (24) through (29) into (20) we have

$$\int_{-1}^1 \frac{(1-t)^{p_1-1} (1+t)^{p_1-1}}{|t-d|^{2p_1}} dt = \frac{2^{2p} [B(p_1, p_1) \equiv \frac{\Gamma(p_1)\Gamma(p_1)}{\Gamma(2p_1)}]}{|-1-d|^{p_1} |1-d|^{p_1}} \quad (30)$$

Which is equivalent with

$$\int_{-1}^1 \frac{(1-t)^{p_1-1} (1+t)^{p_1-1}}{|t-d|^{2p_1}} dt = \frac{2^{2p} [B(p_1, p_1) \equiv \frac{\Gamma(p_1)\Gamma(p_1)}{\Gamma(2p_1)}]}{|d^2-1|^{p_1}} \quad (31)$$

Hence, (19) is true which completes the proof.

Substituting, (19) into (18) the following is obtained

$$\frac{\Gamma(2p_1) \int_{-1}^1 \frac{(1-t^2)^{p'}}{(d \mp t)^{2p_1}} dt}{C_1(p,d) \sqrt{\pi} 2^p \Gamma(p_1)} = \frac{\Gamma(2p_1)}{C_1(p,d) \sqrt{\pi} 2^p \Gamma(p_1)} \frac{2^{2p} \Gamma^2(p_1)}{|1-d^2|^{p_1} \Gamma(2p_1)} \quad (32)$$

or

$$\frac{\Gamma(2p_1) \int_{-1}^1 \frac{(1-t^2)^{p'}}{(d \mp t)^{2p_1}} dt}{C_1(p,d) \sqrt{\pi} 2^p \Gamma(p_1)} = \frac{1}{C_1(p,d) \sqrt{\pi} |1-d^2|^{p_1}} = 1 \quad (33)$$

which completes the proof.

Next, we compute the first two moments of Generalized Bessel distribution of the first kind as follows

$$E_{\text{GBessel1}}[X] = \int_0^\infty x f_{\text{GBessel1}}(x; a, c, p) dx \quad (34)$$

It can be shown that from (see [14] pg. 701, ex. 6.623 2.)

$$\int_0^\infty e^{-\alpha x} I_v(x) x^{v+1} dx = \frac{2\alpha^2 v \Gamma(v+\frac{3}{2})}{\sqrt{\pi}(\alpha^2-1)^{v+\frac{3}{2}}} \quad (35)$$

Hence, substituting (35) into (34), we obtain

$$E_{\text{GBessel1}}[X] = \frac{\int_0^\infty x^{p+1} e^{-x^d} I_p(x) dx}{C_1(p,d)} = \frac{a d 2^{p_2} \Gamma(p_3)}{C_1(p,d) \sqrt{\pi} (d^2-1)^{p_3}} \quad (36)$$

Finally, substituting the value of C_1 from (6) we have

$$E_{\text{GBessel1}}[X] = \frac{d 2^{p_2} \Gamma(p_3)}{\frac{2^p \Gamma(p_1)}{\sqrt{\pi}|1-d^2|^{p_1}} \sqrt{\pi} (d^2-1)^{p_3}} = \frac{2p_1 a d}{d^2-1} \quad (37)$$

Next, we compute the second moment as follows:

$$E_{\text{GBessel1}}[X^2] = \int_0^\infty x^2 f_{\text{GBessel1}}(x; a, d, p) dx \quad (38)$$

It can be shown that (see [14] pg. 699, ex. 6.621 1.)

$$\int_0^\infty e^{-\alpha x} I_\nu(x) x^{\nu+2} dx = \frac{\Gamma(2\nu+3) \Gamma(\nu+\frac{3}{2}) \Gamma(\nu+2) \Gamma(\nu+1) \frac{1}{\alpha^2}}{2^\nu \alpha^{2\nu+3} \Gamma(\nu+1)} \quad (39)$$

There are other variation formulas listed in (see [14] pg. 699, ex. 6.621 1.) that can be used to derive identical expression as (39).

Substituting (39) into (38) we obtain:

$$E_{\text{GBessel1}}[X^2] = \frac{a^2 \int_0^\infty x'^p e^{-x'd} I_p(x') dx'}{C_1(p, d)} \quad (40)$$

or

$$E_{\text{GBessel1}}[X^2] = \frac{a^2 \Gamma(2p_3) F(p_3, p_4; p_2; \frac{1}{d^2})}{C_1(p, d) 2^p d^{2p+3} \Gamma(p_2)} \quad (41)$$

where $F(a, b; c; x) \equiv {}_2F_1(a, b; c; x)$ is the Gauss hypergeometric function (see [14] xl index of special functions).

Finally, substituting the value of C_1 from (6) we have

$$E_{\text{GBessel1}}[X^2] = \frac{a^2 \Gamma(2p_3) F(p_3, p_4; p_2; \frac{1}{d^2})}{\frac{2^p \Gamma(p_1)}{\sqrt{\pi} |1-d^2|^{\frac{p_1}{2}}} 2^p d^{2p+3} \Gamma(p_2)} \quad (42)$$

or

$$E_{\text{GBessel1}}[X^2] = \frac{a^2 \sqrt{\pi} |1-d^2|^{\frac{p_1}{2}} \Gamma(2p_3) F(p_3, p_4; p_2; \frac{1}{d^2})}{2^{2p} d^{2p+3} \Gamma(p_1) \Gamma(p_2)} \quad (43)$$

which can be simplified further as

$$E_{\text{GBessel1}}[X^2] = \frac{a^2 |1-d^2|^{\frac{p_1}{2}} 4 p_2 p_1 F(p_3, p_4; p_2; \frac{1}{d^2})}{d^{2p_3}} \quad (44)$$

based on the recursive relation of the gamma function [16].

Next, the variance of the generalized Bessel distribution of the first kind can be computed from

$$\text{Var}_{\text{GBessel1}}[X] = E_{\text{GBessel1}}[X^2] - E_{\text{GBessel1}}^2[X] \quad (45)$$

Or

$$\text{Var}_{\text{GBessel1}}[X] = \frac{4a^2 p_1}{d^2} \left[\frac{p_2 F(p_3, p_4; p_2; \frac{1}{d^2})}{(1-\frac{1}{d^2})^{-p_1}} - \frac{p_1}{(1-\frac{1}{d^2})^2} \right] \quad (46)$$

2.2 Generalized Bessel Distribution of the Second Kind

Next, Dr. Progri, gives for the first time, the correct formulation of the generalized Bessel distribution of the second kind has its pdf given by

$$f_{\text{GBessel2}}(x; a, d, p) = \frac{|x|^p e^{-\frac{d}{a}|x|} K_p(\frac{|x|}{a})}{C_2(p, d) a^{p_2}} \quad (47)$$

for $-\infty < x < \infty$, $a > 0$, $d > 0$, and $p > -1/2$ where (see [14] pg. 917 ex. 8.432 3.) we have

$$K_p(x) = \frac{\sqrt{\pi} x^p}{2^p \Gamma(p_1)} \int_1^\infty (t^2 - 1)^{p_1-1} e^{-xt} dt \quad (48)$$

is the modified Bessel function of the second kind. In contrast to [8]-[11] where $|d| < 1$ here we just let $d \geq 0$. Perhaps some applications of interests would be for $d < 1$ and $a \geq 0$ but in general we assume that $a > 0$ and $d > 0$.

In order for (46) to be a valid pdf then

$$f_{\text{GBessel2}}(x; a, d, p) \geq 0 \quad (49)$$

Or $C_2(p, d) > 0$ and

$$\int_{-\infty}^\infty f_{\text{GBessel2}}(x; a, d, p) dx = \frac{\int_{-\infty}^\infty \frac{|x|^p e^{-\frac{d}{a}|x|} K_p(\frac{|x|}{a}) a^{\frac{x}{a}}}{C_2(p, d)} dx}{C_2(p, d)} = 1 \quad (50)$$

Since, $f_{\text{GBessel2}}(x; a, d, p)$ is an even function; i.e.,

$$f_{\text{GBessel2}}(-x; a, d, p) = f_{\text{GBessel2}}(x; a, d, p) \quad (51)$$

then (150) can be written as

$$\int_{-\infty}^\infty f_{\text{GBessel2}}(x; a, d, p) dx = \frac{2 \int_0^\infty x'^p e^{-x'd} K_p(x') dx'}{C_2(p, d)} = 1 \quad (52)$$

From where we find

$$C_2(p, d) = 2 \int_0^\infty x'^p e^{-x'd} K_p(x') dx' \quad (53)$$

In order to determine $C_2(p, d)$ we consider (see [14] pg. 700 ex. 6.621 3.) we have

$$\int_0^\infty x'^p e^{-x'd} K_p(x') dx' = \frac{\sqrt{\pi} 2^p \Gamma(2p_1) F(2p_1, p_1; p_3; \frac{d-1}{d+1})}{(d+1)^{2p_1} \Gamma(p_3)} \quad (54)$$

Substituting (54) into (53) we have

$$\frac{1}{C_2(p, d)} = \frac{(d+1)^{2p_1} \Gamma(p_3)}{2\sqrt{\pi} 2^p \Gamma(2p_1) F(2p_1, p_1; p_3; \frac{d-1}{d+1})} \quad (55)$$

Or

$$C_2(p, d) = \frac{\sqrt{\pi} 2^{p+2} \Gamma(2p_1) F(2p_1, p_1; p_3; \frac{d-1}{d+1})}{(d+1)^{2p_1} \Gamma(p_3)} \quad (56)$$

Finally, substituting the value of $C_2(p, d)$ into

$$f_{\text{GBessel2}}(x; a, d, p) = \frac{\pi^{-\frac{1}{2}} (d+1)^{2p_1} \Gamma(p_3) |x|^p e^{-\frac{d}{a}|x|} K_p(\frac{|x|}{a})}{(2a)^{p_2} \Gamma(2p_1) F(2p_1, p_1; p_3; \frac{d-1}{d+1})} \quad (57)$$

Another issue is the value of the function for $x = 0$ since $K_p(0) = \infty$. The exact value is known thanks to (Arfken, G.B., and Weber, H.J., 1995, [15], (11.119) pg. 667) ²

¹ This is very different from (1.3) in [8], (2) in [9], and (2.1) in [11]; so, I believe that the result from (1.3) in [8], (2) in [9], and (2.1) in [11] are wrong.

² The same result may be obtained by using the series expansion of the modified Bessel function of the second kind (see [14] pg. 919 ex. 8.446) and

$$\lim_{x \rightarrow 0} x^p K_p(x) = 2^{p-1} \Gamma(p) \quad p > 0 \quad \text{as} \quad \lim_{x \rightarrow 0} x^p K_p(x) = \lim_{p \rightarrow 0} x^p K_p(x) =$$

$$2^{p-1} \Gamma(p) = \infty;$$

$$f_{\text{GBessel2}}(0; a, d, p) = \frac{2^{-2(d+1)^2 p_1 \Gamma(p_3) \Gamma(p)}}{\sqrt{\pi a \Gamma(2p_1) F(2p_1, p_1; p_3; \frac{d-1}{d+1})}} \quad (58)$$

Next, we compute the mean and the variance of the generalized Bessel distribution of the second kind as follows:

$$E_{\text{GBessel2}}[X] = \int_{-\infty}^{\infty} x f_{\text{GBessel2}}(x; a, d, p) dx \quad (59)$$

Or

$$\int_{-\infty}^{\infty} x f_{\text{GBessel2}}(x; a, d, p) dx = 0 \quad (60)$$

Substituting (59) into (58) we obtain:

$$E_{\text{GBessel2}}[X] = 0 \quad (61)$$

Next, we compute the second moment as follows:

$$E_{\text{GBessel2}}[X^2] = \int_{-\infty}^{\infty} x^2 f_{\text{GBessel2}}(x; a, d, p) dx \quad (62)$$

Or

$$\int_{-\infty}^{\infty} x^2 f_{\text{GBessel2}}(x; a, d, p) dx = \frac{2a^2 \int_0^{\infty} x'^{p+2} e^{-x'd} K_p(x') dx'}{C_2(p, d)} \quad (63)$$

Considering the expression found in (see [14] pg. 700 ex. 6.621 3.) we have

$$\int_0^{\infty} x'^{p+2} e^{-x'd} K_p(x') dx' = \frac{2^{p+2} \Gamma(2p_3) F(2p_3, p_1; p_7; \frac{d-1}{d+1})}{\pi^{-\frac{1}{2}} (d+1)^{2p_3} \Gamma(p_7)} \quad (64)$$

Substituting (64) and (56) into (63) we obtain:

$$E_{\text{GBessel2}}[X^2] = \frac{a^2 2 \Gamma(2p_3) \Gamma(p_3) F(2p_3, p_1; p_7; \frac{d-1}{d+1})}{(d+1)^2 \Gamma(p_7) \Gamma(2p_1) F(2p_1, p_1; p_3; \frac{d-1}{d+1})} \quad (65)$$

which can be simplified further as

$$E_{\text{GBessel2}}[X^2] = \frac{a^2 16(p_2)(2p_1) F(2p_3, p_1; p_7; \frac{d-1}{d+1})}{(d+1)^2 (2p_5)(2p_3) F(2p_1, p_1; p_3; \frac{d-1}{d+1})} \quad (66)$$

based on the recursive relation of the gamma function [16].

Next, let us discuss a very special case for $d = 0$. For this very special case only the generalized Bessel distribution of the second kind exists; hence, the value of $C_2(p, d)$ is equal to

$$C_2(p, d) = \frac{\sqrt{\pi} 2^{p_2} \Gamma(2p_1) F(2p_1, p_1; p_3; -1)}{\Gamma(p_3)} \quad (67)$$

From the definition of the hypergeometric function [17] we have

$$F(a, b; 1 + a - b; -1) = \frac{\Gamma(1+a-b) \Gamma(1+\frac{1}{2}a)}{\Gamma(1+a) \Gamma(1+\frac{1}{2}a-b)} \quad (68)$$

Substituting for $a = 2p + 1 = 2p_1$, $b = p + \frac{1}{2} = p_1$, $1 +$

$a - b = 1 + 2p + 1 - p - \frac{1}{2} = p + \frac{3}{2} = p_3$; into (68) we

obtain

$$F(2p_1, p_1; p_3; -1) = \frac{\Gamma(p_3) \Gamma(p_3)}{\Gamma(2p_2) \Gamma(1)} = \frac{\Gamma^2(p_3)}{\Gamma(2p_2)} \quad (69)$$

Substituting, (69) into (67) yields

$$C_2(p, d) = \frac{\sqrt{\pi} 2^{p_2} \Gamma(2p_1) \Gamma(p_3)}{\Gamma(2p_2)} = \sqrt{\pi} 2^p \Gamma(p_1) \quad (70)$$

Hence, the pdf of the generalized Bessel function of the second kind can be written as

$$f_{\text{GBessel2}}(x; a, d = 0, p) = \frac{|x|^p K_p(\frac{|x|}{a})}{\sqrt{\pi} 2^p \Gamma(p_1) a^{p+1}}^3 \quad (71)$$

For this very special case the mean is zero and the second moment is

$$E_{\text{GBessel2}}[X^2] = \frac{16p_2(2p_1) F(2p_3, p_1; p_7; -1)}{(2p_5)(2p_3) F(2p_1, p_1; p_3; -1)} \quad (72)$$

Similarly, we find

$$F(2p_3, p_1; p_7; -1) = \frac{\Gamma(p_7) \Gamma(p_5)}{\Gamma(2p+4)} \quad (73)$$

Substituting, (73) and (69) into (72) produces

$$E_{\text{GBessel2}}[X^2] = \frac{a^2 16p_2(2p_1) \Gamma(p_7) \Gamma(p_5) \Gamma(2p_2)}{(2p_5)(2p_3) \Gamma(p_3) \Gamma(p_3) \Gamma(2p_4)} \quad (74)$$

which can be simplified as

$$E_{\text{GBessel2}}[X^2] = 2a^2 p_1 \quad (75)$$

This concludes the derivations of generalized Bessel distribution. Next, provide the derivations of the series expansion of the generalized Bessel distributions.

3 Series Expansion of the Generalized Bessel Distribution

In this section we provide the closed form expressions of cdfs of the generalized Bessel distribution of the first and second kinds via their series expansion. The series expansion of the generalized Bessel distribution of the first kind is given first followed by the series expansion of the generalized Bessel distribution of the second kind.

3.1 Generalized Bessel Distribution of the First Kind

From the definition of the pdf of the generalized Bessel distribution of the first kind, we define its corresponding cdf as $F_{\text{GBessel1}}(x; a, d, p) = \int_{-\infty}^x f_{\text{GBessel1}}(t; a, d, p) dt$ (76) for $a > 0$, $d > 1$, and $p > 0$, which can also be written as

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{\int_0^x t^p e^{-td} I_p(t) dt}{C_1(p, d)} \quad (77)$$

correct.

by setting the summation index equal to zero.

³ This result is identical to (1) in [7]; so, fortunately I believe that (1) in [7] is

where $C_1(p, d)$ is given by (6).

Using the series expansion of $I_p(t)$ (see [14] pg. 919 ex. 8.445) we have

$$I_p(t) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(p_2 + k)} \frac{t^{p+2k}}{2^{p+2k}} \quad (78)$$

Substituting (78) into (77) we obtain:

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{\int_0^x t^p e^{-td} \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(p_2 + k)} \frac{t^{p+2k}}{2^{p+2k}} dt}{C_1(p, d)} \quad (79)$$

Or by changing the order of summation and integration produces

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{1}{C_1(p, d)} \sum_{k=0}^{\infty} \frac{\int_0^x t^{2p+2k} e^{-td} dt}{k! \Gamma(p_2 + k) 2^{p+2k}} \quad (80)$$

From the definition of the incomplete gamma function [19] we have

$$\int_0^x t^{2p+2k} e^{-td} dt = \frac{\gamma(2p_1+2k, dx)}{d^{2p_1+2k}} \quad (81)$$

Hence, substituting (81) into (80) yields:

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{\sum_{k=0}^{\infty} \frac{\gamma(2p_1+2k, dx)}{k! \Gamma(p_2 + k) 2^{p+2k} d^{2p_1+2k}}}{C_1(p, d)} \quad (82)$$

or

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{\sum_{k=0}^{\infty} \frac{\gamma(2p_1+2k, dx)}{k! \Gamma(p_2 + k) (2d)^{2k}}}{2^p d^{2p+1} C_1(p, d)} \quad (83)$$

Next, we examine the value $C_1(p, d)$ from (6)

$$C_1(p, d) = \frac{2^p \Gamma(p_1)}{\sqrt{\pi} |1-d^2|^{p_1}} = \frac{\Gamma(2p_1)}{2^p |1-d^2|^{p_1} \Gamma(p_2)} \quad (84)$$

Finally, substituting (84) into (83) yields

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{|1-d^2|^{p_1} \Gamma(p_2)}{d^{2p_1} \Gamma(2p_1)} \sum_{k=0}^{\infty} \frac{\gamma(2p_1+2k, dx)}{k! \Gamma(p_2 + k) (2d)^{2k}} \quad (85)$$

We have to find a way to simplify (85). We examine the following recursive relations of the gamma function [16]

$$\Gamma(p_2 + k) = (p + k) \Gamma(p + k) \quad (86)$$

or by applying recursive relation (86) $2k$ times we have

$$\Gamma(p_2 + k) = \Gamma(p_2) [\prod_{n=1}^k (p + n) \equiv (p_2)_k] \quad (87)$$

where $(a)_k$ is the (rising) Pochhammer symbol [20].

Substituting (87) into (85) we obtain:

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{(1-\frac{1}{d^2})^{p_1}}{\Gamma(2p_1)} \sum_{k=0}^{\infty} \frac{\gamma(2p_1+2k, dx)}{k! (p_2)_k (2d)^{2k}} \quad (88)$$

Can we simplify (88) even further?

In order to do that first let us examine the values of cdf for $x = 0$ and $x \rightarrow \infty$. Since, $\gamma(2p_1 + 2k, dx = 0) = 0$ hence,

$$F_{\text{GBessel1}}(0; a, d, p) = 0 \quad (89)$$

Next, since, $\gamma(2p_1 + 2k, dx \rightarrow \infty) \rightarrow \Gamma(2p_1 + 2k)$ we have

$$F_{\text{GBessel1}}(\infty; a, d, p) = \frac{(1-\frac{1}{d^2})^{p_1}}{\Gamma(2p_1)} \sum_{k=0}^{\infty} \frac{\Gamma(2p_1+2k)}{k! (p_2)_k (2d)^{2k}} \quad (90)$$

Equation (90) constitutes a new identity of gamma function series expansion relation for $d > 1$ and $p > 0$:

$$\frac{(1-\frac{1}{d^2})^{p_1}}{\Gamma(2p_1)} \sum_{k=0}^{\infty} \frac{\Gamma(2p_1+2k)}{k! (p_2)_k (2d)^{2k}} \equiv 1 \quad (91)$$

Theorem 1: Prove that (91) is true for $d > 1$ and $p > 0$.

Proof of theorem 1: In order to prove that (91) is true, first we examine the following recursive relation of the gamma function [16]

$$\Gamma(2p_1 + 2k) = (2p + 2k) \Gamma(2p + 2k) \quad (92)$$

or by applying recursive relation (92) $2k$ times we have

$$\Gamma(2p_1 + 2k) = \Gamma(2p_1) (2p_1)_{2k} \quad (93)$$

Substituting, (93) and (87) into summation of either (90) or (91) we obtain:

$$\sum_{k=0}^{\infty} \frac{\Gamma(2p_1+2k)}{k! (p_2)_k (2d)^{2k}} \equiv \Gamma(2p_1) \sum_{k=0}^{\infty} \frac{(2p_1)_{2k}}{k! (p_2)_k (2d)^{2k}} \quad (94)$$

Finally, substituting (94) into either (90) or (91) we obtain

$$\left(1 - \frac{1}{d^2}\right)^{p_1} \sum_{k=0}^{\infty} \frac{(2p_1)_{2k}}{k! (p_2)_k (2d)^{2k}} \equiv 1 \quad (95)$$

Or

$$\left(1 - \frac{1}{d^2}\right)^{-p_1} \equiv \sum_{k=0}^{\infty} \frac{(2p_1)_{2k}}{k! (p_2)_k (2d)^{2k}} \quad (96)$$

In order to prove that (96) is true then we examine its binomial expansion (see Arfken, G.B., and Weber, H.J., 1995, [15],

(5.99) pg. 317) by setting $x = -\frac{1}{d^2}$ and $m = -p - \frac{1}{2}$ we have

$$\left(1 - \frac{1}{d^2}\right)^{-p_1} \equiv \sum_{k=0}^{\infty} \frac{(-p-\frac{1}{2})!}{k! (-p-\frac{1}{2}-k)!} \left(-\frac{1}{d^2}\right)^k \quad (97)$$

which is equivalent with

$$\left(1 - \frac{1}{d^2}\right)^{-p_1} \equiv \sum_{k=0}^{\infty} \binom{k-p-\frac{1}{2}}{k} (-1)^k \quad (98)$$

To prove that (98) is identical to (96) it remains to prove that

$$\binom{-p_1}{k} (-1)^k \equiv \frac{\prod_{n=1}^{2k} (p_n)}{\prod_{n=1}^k (p_{2n})} \quad (99)$$

For all values of $p > 0$ and integer values of $k \geq 1$. Next, we recognize that

$$\prod_{n=1}^{2k} (p_n) \equiv \prod_{n=1, odd}^{2k} (p_n) \prod_{n=1, even}^{2k} (p_n) \quad (100)$$

Next, we examine

$$\prod_{n=1, even}^{2k} (p_n) = \prod_{m=1}^k (p_{2m}) \quad (101)$$

Substituting (101) into (99) yields

$$\binom{-p_1}{k} (-1)^k \equiv \prod_{n=1, odd}^{2k} (p_n) \quad (102)$$

which leads to the binomial coefficient identity [18]

$$\binom{-p_1}{k}(-1)^k \equiv \binom{k+p-\frac{1}{2}}{k} \equiv \prod_{m=1}^k \left(p_{2m} - \frac{1}{2}\right) \quad (103)$$

which is exactly the value of the binomial coefficient which completes the proof.

Or using the definition of the (rising) Pochhammer symbol [20] (96) can be written as

$$\left(1 - \frac{1}{d^2}\right)^{-p_1} \equiv \sum_{k=0}^{\infty} \frac{(p_2)_k (p_1)_k}{k! (p_2)_k d^{2k}} \quad (104)$$

or with the help of the hyper geometric function [17]

$$(1 - d^{-2})^{-p_1} \equiv \sum_{k=0}^{\infty} \frac{(p_1)_k (1)_k}{(1)_k} \frac{(d^{-2})^k}{k!} \equiv F(p_1, 1; 1; d^{-2}) \quad (105)$$

which is identical to

$$(1 - d^{-2})^{-p_1} \equiv \sum_{k=0}^{\infty} \frac{(p_1)_k (1)_k}{(1)_k} \frac{(d^{-2})^k}{k!} \equiv F(1, p_1; 1; d^{-2}) \quad (106)$$

Now that (90) and (81) have been proven. Let us try to simplify (88). Indeed, from the definition of the incomplete gamma function [19] we have

$$\gamma(\alpha + 1, dx) = \alpha \gamma(\alpha, dx) - \frac{(dx)^\alpha}{e^{dx}}; \quad \alpha = 2p + 2k \quad (107)$$

or by applying recursive relation (107) $2k$ times [21] we have

$$\gamma(\alpha + 1, dx) = (2p_1)_{2k} \gamma(2p_1, dx) - g(x; p, k) \quad (108)$$

where $g(x; p, k)$ is given by (see Thompson 2013 [21])

$$g(x; p, k) = (2p_1)_{2k} (dx)^{2p} e^{-dx} \sum_{m=1}^{2k} \frac{(dx)^m}{(2p+1)_m} \quad (109)$$

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{(x^2)^{p_1} e^{-dx} \left[\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(p_1)_k (1)_m}{(p_3)_{m+k} (p_2)_{m+k}} \frac{(x_1)^k (x_2)^m}{k! m!} + \frac{dx}{2p_2} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(p_1)_k (1)_m}{(p_4)_{m+k} (p_3)_{m+k}} \frac{(x_1)^k (x_2)^m}{k! m!} \right]}{(d^2-1)^{-p_1} \Gamma(2p_2)} \quad (116)$$

where

$$x_1 = \frac{x^2}{4}; \quad x_2 = \frac{x^2 d^2}{4} \quad (116a)$$

Finally, Dr. Progi has successfully derived for the first time

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{(d^2-1)^{p_1} e^{-dx} x^{2p_1} \left(F_{2;0;0}^{0;1;1} \left[\begin{matrix} -; p_1; 1; \\ p_3, p_2; -; -; \end{matrix} \middle| x_1, x_2 \right] + \frac{dx}{2p_2} F_{2;0;0}^{0;1;1} \left[\begin{matrix} -; p_1; 1; \\ p_4, p_3; -; -; \end{matrix} \middle| x_1, x_2 \right] \right)}{\Gamma(2p_2)} \quad (117)$$

or

$$F_{\text{GBessel1}}(x; a, d, p) = \left[\frac{F_{2;0;0}^{0;1;1} \left[\begin{matrix} -; p_1; 1; \\ p_3, p_2; -; -; \end{matrix} \middle| x_1, x_2 \right]}{C_{11}^{-1}(p, d)} + \frac{x F_{2;0;0}^{0;1;1} \left[\begin{matrix} -; p_1; 1; \\ p_4, p_3; -; -; \end{matrix} \middle| x_1, x_2 \right]}{C_{12}^{-1}(p, d)} \right] x^{2p_1} e^{-dx}; \quad C_{11}(p, d) = \frac{(d^2-1)^{p_1}}{\Gamma(2p_2)}; \quad C_{12}(p, d) = \frac{C_{11}(p, d)d}{2p_2} \quad (118)$$

This concludes the derivations of the closed form expression of the cdf of the generalized Bessel distribution of the first kind via its series expansion of the modified Bessel function of the first kind.

Substituting (108) into (88) produces

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{\sum_{k=0}^{\infty} \frac{\gamma(2p_1, dx) (2p_1)_{2k}}{k! (p_2)_k (2d)^{2k}}}{|1-d^{-2}|^{-p_1} \Gamma(2p_1)} - G_{\text{GBessel1}}(x; a, d, p) \quad (110)$$

Or

$$F_{\text{GBessel1}}(x; a, d, p) = \frac{\gamma(2p_1, dx)}{\Gamma(2p_1)} - G_{\text{GBessel1}}(x; a, d, p) \quad (111)$$

where

$$G_{\text{GBessel1}}(x; a, d, p) = \frac{(dx)^{2p} e^{-dx} \sum_{k=0}^{\infty} \frac{(p_1)_k \sum_{m=1}^{2k} \frac{(dx)^m}{(2p_1)_m}}{k! (d)^{2k}}}{(1-d^{-2})^{p_1} \Gamma(2p_1)} \quad (112)$$

or with the help of the regularized incomplete Gamma function (112) can be written as

$$F_{\text{GBessel1}}(x; a, d, p) = P(2p_1, x) - G_{\text{GBessel1}}(x; a, d, p) \quad (113)$$

If there is another way to find the close form expression of (113) then

$$G_{\text{GBessel1}}(x; a, d, p) = P(2p_1, x) - F_{\text{GBessel1}}(x; a, d, p) \quad (114)$$

Let us rewrite (88) in a different form using the series expansion of the incomplete gamma function based on (see [14] pg. 899 ex. 8.351 2.) we have

$$\gamma(\alpha', dx) = \frac{(dx)^{\alpha'} e^{-dx} \Phi(1, 1 + \alpha'; dx)}{\alpha'} = \frac{(dx)^{\alpha'} e^{-dx} \sum_{n=0}^{\infty} \frac{(1)_n}{(1+\alpha')_n} \frac{(dx)^n}{n!}}{\alpha'} \quad (115)$$

where $\Phi(a, b; x)$ is the confluent hypergeometric function (see [14] pg. 1023 ex. 9.21 2.).

Substituting (115) into (88) we obtain for $\alpha' = \alpha + 1$

the cdf of the Bessel function distribution of the first kind which can be written with the help of the Kampé de Fériet function [22]-[25] as follows

3.2 Generalized Bessel Distribution of the Second Kind

From the definition of the pdf of the Generalized Bessel distribution of the second kind, we define its corresponding cdf as

$$F_{\text{GBessel2}}(x; a, d, p) = \int_{-\infty}^x f_{\text{GBessel2}}(t; a, d, p) dt \quad (119)$$

for $a > 0$, $d > 0$, and $p > 1$, which can also be written as

$$F_{\text{GBessel2}}(x; a, d, p) = \frac{\int_{-\infty}^x |t|^p e^{-|t|d} K_p(|t|) dt}{C_2(p, d)} \quad (120)$$

where $C_2(p, d)$ is given by (56).

Let us define

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\int_0^x t^p e^{-td} K_p(t) dt}{C_2(p, d)} \quad (121)$$

then

$$F_{\text{GBessel2}}(x; a, d, p) = \frac{1-2\mathcal{F}_{\text{GBessel2}}(-x; a, d, p) \quad x < 0}{1+2\mathcal{F}_{\text{GBessel2}}(x; a, d, p) \quad x \geq 0} \quad (122)$$

Hence, if we compute the series expansion of (121); then, we have computed the series expansion of (120) via (122).

Using the relation between the modified Bessel function of the first and second kind from (see Arfken, G.B., and Weber, H.J., 1995, [15], (11.118) pg. 667 or [14] pg. 928 ex. 8.485) for real (not an integer) values of p

$$K_p(t) = \frac{\pi}{2} \frac{I_{-p}(t) - I_p(t)}{\sin \pi p} \quad (123)$$

Substituting (123) into (121) yields

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\pi}{2} \frac{\int_0^x t^p e^{-td} [I_{-p}(t) - I_p(t)] dt}{C_2(p, d) \sin \pi p} \quad (124)$$

or equivalently,

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\pi}{2} \frac{\int_0^x t^p e^{-td} I_{-p}(t) dt - \int_0^x t^p e^{-td} I_p(t) dt}{C_2(p, d) \sin \pi p} \quad (125)$$

or

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\int_0^x t^p e^{-td} I_{-p}(t) dt - \int_0^x t^p e^{-td} I_p(t) dt}{\tilde{C}_2(p, d)} \quad (126)$$

where

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\frac{2^p}{d} \sum_{k=0}^{\infty} \frac{\gamma(2k+1, dx)}{k! \Gamma(-p+k+1) (2d)^{2k}} \left[\frac{{}_F^{0:1;1}_{2:0;0} \left[\begin{matrix} -p_1; 1; \\ p_3, p_2; -; -; \end{matrix} x_1, x_2 \right]}{C_{11}^{-1}(p, d)} + \frac{{}_x F^{0:1;1}_{2:0;0} \left[\begin{matrix} -p_1; 1; \\ p_4, p_3; -; -; \end{matrix} x_1, x_2 \right]}{C_{12}^{-1}(p, d)} \right] x^{2p_1} e^{-dx}}{\tilde{C}_2(p, d)} \quad (135)$$

Next, we must show that (135) (via (122)) is indeed a valid cdf. Hence, we must first show that $\mathcal{F}_{\text{GBessel2}}(0; a, d, p) = 0$, $\mathcal{F}_{\text{GBessel2}}(\infty; a, d, p) = 0.5$; and the function is monotonically increasing.

Showing that $\mathcal{F}_{\text{GBessel2}}(0; a, d, p) = 0$ is straightforward. Showing that $\mathcal{F}_{\text{GBessel2}}(\infty; a, d, p) = 0.5$ is more complicated. From

$$\sum_{k=0}^{\infty} \frac{\Gamma(2k+1)}{k! \Gamma(-p+k+1) (2d)^{2k}} \equiv \frac{2F(2p_1, p_1; p_3; \frac{d-1}{d+1}) \sin \pi p}{d^{-1} \Gamma^{-1}(2p_1) \sqrt{\pi} (d+1)^{2p_1} \Gamma(p_3)} + \frac{d \Gamma(2p_1)}{2^{2p} |1-d^2|^{p_1} \Gamma(p_2)} \quad (138)$$

$$\tilde{C}_2(p, d) = \frac{2C_2(p, d) \sin \pi p}{\pi} = \frac{2^{p_4} \Gamma(2p_1) F(2p_1, p_1; p_3; \frac{d-1}{d+1}) \sin \pi p}{\sqrt{\pi} (d+1)^{2p_1} \Gamma(p_3)} \quad (127)$$

Since, the second integral is already known

$$\int_0^x \frac{t^p I_p(t) dt}{e^{td}} = \frac{\left[\frac{{}_F^{0:1;1}_{2:0;0} \left[\begin{matrix} -p_1; 1; \\ p_3, p_2; -; -; \end{matrix} x_1, x_2 \right]}{C_{11}^{-1}(p, d)} + \frac{{}_x F^{0:1;1}_{2:0;0} \left[\begin{matrix} -p_1; 1; \\ p_4, p_3; -; -; \end{matrix} x_1, x_2 \right]}{C_{12}^{-1}(p, d)} \right] x^{2p_1}}{C_1^{-1}(p, d)} e^{dx} \quad (128)$$

It only remains to evaluate the first integral.

Using the series expansion of $I_{-p}(t)$ (see [14] pg. 919 ex. 8.445) we have

$$I_{-p}(t) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(-p+k+1)} \frac{t^{-p+2k}}{2^{-p+2k}} \quad (129)$$

Substituting (129) into (125) we obtain:

$$\int_0^x t^p e^{-td} [I_{-p}(t) \equiv \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(-p+k+1)} \frac{t^{-p+2k}}{2^{-p+2k}}] dt \quad (130)$$

Or by changing the order of summation and integration produces

$$\int_0^x t^p e^{-td} I_{-p}(t) dt = \sum_{k=0}^{\infty} \frac{\int_0^x t^{2k} e^{-td} dt}{k! \Gamma(-p+k+1) 2^{-p+2k}} \quad (131)$$

From the definition of the incomplete gamma function [19] we have

$$\int_0^x t^{2k} e^{-td} dt = \frac{\gamma(2k+1, dx)}{d^{2k+1}} \quad (132)$$

Hence, substituting (132) into (131) yields:

$$\int_0^x t^p e^{-td} I_{-p}(t) dt = \sum_{k=0}^{\infty} \frac{\gamma(2k+1, dx)}{k! \Gamma(-p+k+1) 2^{-p+2k} d^{2k+1}} \quad (133)$$

or

$$\int_0^x t^p e^{-td} I_{-p}(t) dt = \frac{2^p}{d} \sum_{k=0}^{\infty} \frac{\gamma(2k+1, dx)}{k! \Gamma(-p+k+1) (2d)^{2k}} \quad (134)$$

Finally, substituting (134) and (128) into (126) we obtain:

$$\mathcal{F}_{\text{GBessel2}}(\infty; a, d, p) = \frac{\frac{2^p}{d} \sum_{k=0}^{\infty} \frac{\Gamma(2k+1)}{k! \Gamma(-p+k+1) (2d)^{2k}} C_1(p, d)}{\tilde{C}_2(p, d)} \quad (136)$$

Let us assume that this is true. Then we would have shown that

$$\frac{2^{p_2}}{d} \sum_{k=0}^{\infty} \frac{\Gamma(2k+1)}{k! \Gamma(-p+k+1) (2d)^{2k}} \equiv \tilde{C}_2(p, d) + 2C_1(p, d) \quad (137)$$

or

Or following the derivations that we completed earlier we have

$$\gamma(2k+1, dx) = \frac{(dx)^{2k+1} e^{-dx}}{2k+1} \left[\sum_{m=0}^{\infty} \frac{x_2^m}{\left(k+\frac{3}{2}\right)_m (k+1)_m} + \frac{dx}{2(k+1)} \sum_{m=0}^{\infty} \frac{x_2^m}{(k+2)_m \left(k+\frac{3}{2}\right)_m} \right] \text{ii} \quad (139)$$

Substituting (139) into the summation of (135) yields

$$\sum_{k=0}^{\infty} \frac{\gamma(2k+1, dx)}{k! \Gamma(-p+k+1) (2d)^{2k}} = \frac{x d e^{-dx} \left[\sum_{k,m=0}^{\infty} \frac{(1)_k \left(\frac{1}{2}\right)_k (1)_m}{\left(\frac{3}{2}\right)_{k+m} (1)_{k+m} (1-p)_k} \frac{x_1^k x_2^m}{k! m!} + \frac{dx}{2} \sum_{k,m=0}^{\infty} \frac{(1)_k \left(\frac{1}{2}\right)_k (1)_m}{(2)_{k+m} \left(\frac{3}{2}\right)_{k+m} (1-p)_k} \frac{x_1^k x_2^m}{k! m!} \right]}{\Gamma(-p+1)} \text{iii} \quad (140)$$

which can be written with the help of the Kampé de Fériet function [22]-[25] as follows

$$\sum_{k=0}^{\infty} \frac{\gamma(2k+1, dx)}{k! \Gamma(-p+k+1) (2d)^{2k}} = \frac{x d e^{-dx} \left[F_{2;1;0}^{0;2;1} \left[\begin{matrix} -1, \frac{1}{2}; 1; \\ \frac{3}{2}; 1-p; -; \end{matrix} ; x_1, x_2 \right] + \frac{dx}{2} F_{2;1;0}^{0;2;1} \left[\begin{matrix} -1, \frac{1}{2}; 1; \\ \frac{3}{2}; 1-p; -; \end{matrix} ; x_1, x_2 \right] \right]}{\Gamma(1-p)} \text{iv} \quad (141)$$

Finally, Dr. Proghi has successfully derived for the first time the cdf of the generalized Bessel function distribution of the second kind by substituting (141) into (135) yields:

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = x e^{-dx} \left[\frac{F_{2;1;0}^{0;2;1} \left[\begin{matrix} -1, \frac{1}{2}; 1; \\ \frac{3}{2}; 1-p; -; \end{matrix} ; x_1, x_2 \right]}{C_{21}^{-1}(p, d)} + \frac{x F_{2;1;0}^{0;2;1} \left[\begin{matrix} -1, \frac{1}{2}; 1; \\ \frac{3}{2}; 1-p; -; \end{matrix} ; x_1, x_2 \right]}{C_{22}^{-1}(p, d)} + \frac{x^2 p F_{2;0;0}^{0;1;1} \left[\begin{matrix} -; p_1; 1; \\ p_3, p_2; -; -; \end{matrix} ; x_1, x_2 \right]}{C_{23}^{-1}(p, d)} + \frac{x^2 p_1 F_{2;0;0}^{0;1;1} \left[\begin{matrix} -p_1; 1; \\ p_4, p_3; -; -; \end{matrix} ; x_1, x_2 \right]}{C_{24}^{-1}(p, d)} \right] \text{v} \quad (142)$$

where $C_{2i}(p, d)$ for $i = \{1, 2, 3, 4\}$ are given by

$$C_{21}(p, d) = \frac{2^p}{\Gamma(1-p) \tilde{C}_2(p, d)}; C_{22}(p, d) = \frac{2^{p-1} d}{\Gamma(1-p) \tilde{C}_2(p, d)} \quad (142a)$$

$$C_{23}(p, d) = \frac{-C_{11}(p, d) C_1}{\tilde{C}_2(p, d)}; C_{24}(p, d) = \frac{-C_{12}(p, d) C_1}{\tilde{C}_2(p, d)} \quad (142b)$$

where $\tilde{C}_2(p, d)$ is given by (127).

$$K_p(t) = \frac{1}{2} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \left(\frac{t}{2}\right)^{2k-p} + (-1)^{p+1} \sum_{k=0}^{\infty} \frac{\ln \frac{t}{2} \frac{\psi(k+1) + \psi(p+k+1)}{2}}{k! (p+k)!} \left(\frac{t}{2}\right)^{p+2k} \quad (143)$$

Substituting (143) into (121) yields

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\int_0^x t^p e^{-td} \left[\frac{\sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \left(\frac{t}{2}\right)^{2k-p} + (-1)^{p+1} \sum_{k=0}^{\infty} \frac{\ln \frac{t}{2} \frac{\psi(k+1) + \psi(p+k+1)}{2}}{2^{p+2k} k! (p+k)!} \right]}{C_2(p, d)} dt \quad (144)$$

Or by changing the order of summation and integration we obtain:

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\frac{1}{2} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \int_0^x t^p t^{2k-p} e^{-td} dt + (-1)^{p+1} \sum_{k=0}^{\infty} \int_0^x t^p e^{-td} t^{p+2k} \left[\ln \frac{t}{2} \frac{\psi(k+1) + \psi(p+k+1)}{2} \right]}{C_2(p, d)} dt \quad (145)$$

Or which can be written as

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{2^{p-1} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \int_0^x t^p t^{2k-p} e^{-td} dt + (-1)^{p+1} \sum_{k=0}^{\infty} \int_0^x t^p e^{-td} \ln \frac{t}{2} dt \left[\frac{\psi(k+1) + \psi(p+k+1)}{2} \right] \int_0^x t^p e^{-td} dt}{C_2(p, d)} \quad (146)$$

In order to evaluate (146) we need to compute three integrals. (132); Hence, (146) can be written as
The third integral is exactly (81) and the first integral is exactly

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\frac{2^{p-1}}{d} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)!}{k!} \gamma(2k+1, dx) + \frac{(-1)^{p+1}}{2^p} \sum_{k=0}^{\infty} \int_0^x t^p e^{-td} \ln \frac{t}{2} dt \left[\frac{\psi(k+1) + \psi(p+k+1)}{2} \right] \frac{\gamma(\alpha+1, dx)}{d^{\alpha+1}}}{C_2(p, d)} \quad (147)$$

It remains to evaluate the second integral of (146) or the remaining integral of (147). This integral can be evaluated by using the method of partial derivatives as follows:

$$\int_0^x t^\alpha e^{-td} dt = \int_0^x 2^\alpha \left(\frac{t}{2}\right)^\alpha e^{-td} dt = \frac{\gamma(\alpha+1, dx)}{d^{\alpha+1}} \quad (148)$$

Taking the partial derivative of (148) with respect to α can be written as

$$\frac{\partial}{\partial \alpha} \int_0^x 2^\alpha \left(\frac{t}{2}\right)^\alpha e^{-td} dt = \frac{\partial}{\partial \alpha} \frac{\gamma(\alpha+1, dx)}{d^{\alpha+1}} \quad (149)$$

$$\frac{\partial \int_0^x 2^\alpha \left(\frac{t}{2}\right)^\alpha e^{-td} dt}{\partial \alpha} = \int_0^x \frac{\partial 2^\alpha}{\partial \alpha} \left(\frac{t}{2}\right)^\alpha e^{-td} dt + \int_0^x 2^\alpha \frac{\partial \left(\frac{t}{2}\right)^\alpha}{\partial \alpha} e^{-td} dt \quad (151)$$

Evaluating the first integral of (151) yields

$$\int_0^x \frac{\partial 2^\alpha}{\partial \alpha} \left(\frac{t}{2}\right)^\alpha e^{-td} dt = \ln(2) \left[\int_0^x t^\alpha e^{-td} dt \equiv \frac{\gamma(\alpha+1, dx)}{d^{\alpha+1}} \right] \quad (152)$$

The second integral of (151) is obtained from:

$$\int_0^x 2^\alpha \frac{\partial \left(\frac{t}{2}\right)^\alpha}{\partial \alpha} e^{-td} dt = \int_0^x t^\alpha \ln\left(\frac{t}{2}\right) e^{-td} dt \quad (153)$$

Equation (153) is in fact our desired integral.

Next it remains to evaluate the right-hand side of (149) as follows:

$$\frac{\partial}{\partial \alpha} \frac{\gamma(\alpha+1, dx)}{d^{\alpha+1}} = \frac{d^{\alpha+1} \frac{\partial}{\partial \alpha} \gamma(\alpha+1, dx) - \gamma(\alpha+1, dx) \frac{\partial d^{\alpha+1}}{\partial \alpha}}{d^{2\alpha+2}} \quad (154)$$

$$\frac{\partial}{\partial \alpha} \gamma(\alpha+1, dx) = \Gamma(\alpha+1) \left\{ \psi(\alpha+1) \left[1 - e^{-dx} \sum_{m=0}^{\infty} \frac{(dx)^m}{m!} \right] - e^{-dx} \frac{\partial}{\partial \alpha} \left[\sum_{m=0}^{\infty} \frac{(dx)^m}{m!} \right] \right\} \quad (158)$$

Substituting (147) and (81) into (146) we have a very

$$\mathcal{F}_{\text{GBessel2}}(x; a, d, p) = \frac{\frac{2^{p-1}}{d} \sum_{k=0}^{p-1} \frac{(-1)^k (p-k-1)! \gamma(2k+1, dx)}{k! (4d^2)^k} + \frac{(-1)^{p+1}}{2^p} \sum_{k=0}^{\infty} \frac{\frac{\partial}{\partial \alpha} \frac{\gamma(\alpha+1, dx)}{d^{\alpha+1}} - [\ln(d) + \ln(2)] \frac{\gamma(\alpha+1, dx)}{d^{\alpha+1}} \frac{[\psi(k+1) + \psi(p+k+1)] \gamma(\alpha+1, dx)}{2}}{2^{2k} k! (p+k)!}}{C_2(p, d)} \quad \text{vii} \quad (159)$$

This concluded the derivations of the closed form expression of the generalized Bessel distribution of the second kind.

4 Numerical, Theoretical Results

In this section we provide numerical theoretical results on the exact and approximated generalized Bessel function distributions of the first and second kinds based analytical derivations presented in the previous section.

As far as I can tell, Arnaut 2003 [6] presented a solid methodology for showing both the pdfs and cdfs; however, only for x positive and only for a special class of Bessel function distributions.

4.1 Generalized Bessel Function Distribution of the First and Second Kinds

In this subsection we present, numerical results of the

The left-hand side of (149) can be computed by the help of Leibnitz integral rule [43] or differentiation under the integration sign rule [44] or Reynold's transport theorem [45] as follows

$$\frac{\partial}{\partial \alpha} \int_0^x 2^\alpha \left(\frac{t}{2}\right)^\alpha e^{-td} dt = \int_0^x \frac{\partial}{\partial \alpha} \left[2^\alpha \left(\frac{t}{2}\right)^\alpha \right] e^{-td} dt \quad (150)$$

Equation (150) can be further written as

The second part of (154) can be easily computed as

$$\frac{\partial d^{\alpha+1}}{\partial \alpha} = d^{\alpha+1} \ln(d) \quad (155)$$

Substituting (155) into (164) produces

$$\frac{\partial}{\partial \alpha} \frac{\gamma(\alpha+1, dx)}{d^{\alpha+1}} = \frac{\frac{\partial}{\partial \alpha} \gamma(\alpha+1, dx) - \gamma(\alpha+1, dx) \ln(d)}{d^{\alpha+1}} \quad (156)$$

The first derivative of (156) can be obtained from (see Gradshteyn, Ryzhik [14] pg. 899 ex. 8.352 1.) as follows

$$\frac{\partial}{\partial \alpha} \gamma(\alpha+1, dx) = \frac{\partial}{\partial \alpha} \Gamma(\alpha+1) \left[1 - e^{-dx} \sum_{m=0}^{\infty} \frac{(dx)^m}{m!} \right] \quad (157)$$

Which is equivalent with

complicated expression of $\mathcal{F}_{\text{GBessel2}}(x; a, d, p)$ of integer p

generalized Bessel function of the first and second kinds using numerical integration.

Figure 1 shows the plots of (top) the generalized Bessel of the first kind pdf and cdf for $a = 1$, $d = 2$, and $p = 2$; (bottom) original generalized Bessel of the second kind pdf and cdf [26] for $a = 0.85$, $d = 0.5$, and $p = 2$ for three different scenarios. Scenario (a) is the original pdf and cdf based on (113). Scenario (b) is the exponentiated pdf and cdf for $\alpha = 0.5$. And scenario (c) is the convolution pdf and cdf. Details on how the pdfs and cdfs are produced are given in [26]. The important thing is that in order to produce scenario (c) the stats of the generalized Bessel are needed which are analytically computed in (37) and (46).

In all three scenarios generalized Bessel of the second kind pdf and cdf (based on the derivations that follow next) are only used by virtue of comparison or contrast to show that in all three cases that both the generalized Bessel of the first kind pdf and

cdf for $a = 1$, $d = 2$, and $p = 2$ and the generalized Bessel of the second kind pdf and cdf for $a = 0.85$, $d = 2$, and $p = 2$ are in fact valid pdfs and cdfs.

Figure 2 shows exactly the same results as the ones in Fig. 1 with the only difference that $p = 2.5$ in Fig. 2 instead of $p = 2$ in Fig. 1.

The important results here is that even for non-integer values of p the generalized Bessel of the first and second kinds pdfs and cdfs are valid.

It is important to show that pdf calculations are based on the closed form expression and for the cdf are based on the numerical approximation. In a separate paper we will produce the exact algorithms and MATLAB functions for computing the cdfs from their closed form expressions.

4.2 Approximations of the Generalized Bessel Function Distribution of the First and Second Kinds

Because the numerical computation of the generalized Bessel function distribution cdf of the first and second kinds is laborious based on their closed form expression as it requires the numerical evaluation of the Kampé de Fériet function [22]-[25], for most practical applications the generalized Bessel function distribution cdf of the first and second kinds can be approximated from the family of well-known distribution functions to within 5 % absolute error accuracy.

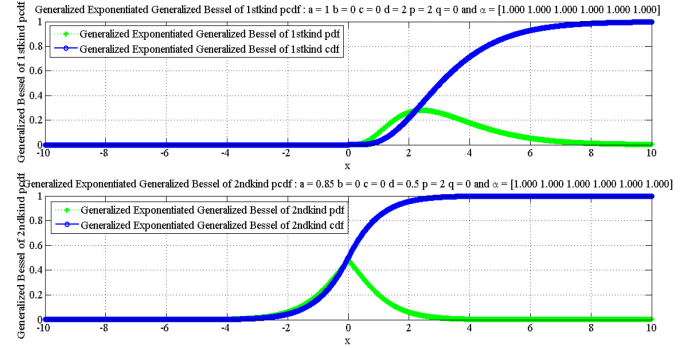
This is exactly what is shown in Fig. 3.

Figure 3 (a) shows numerical calculations of the generalized Bessel of the first kind pdf and cdf for $a = 1$, $d = 1.6$, and $p = 3$ approximates Gamma pdf and cdf with for $a = 1$ and $c = 2p + 1$.

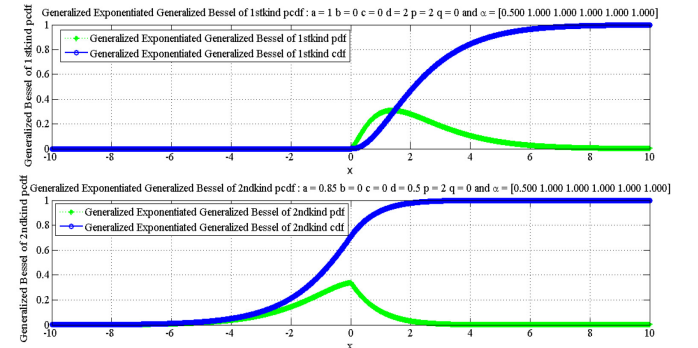
Figure 3 (b) shows numerical calculations of the half-generalized Bessel of the second kind pdf and cdf for $a = 1$, $d = 0$, $p = 3$ approximates half normal pdf and cdf with for $a = \sqrt{2p}$. Figure 3 (c) shows that the absolute error of the (top) first approximation is $\pm 2\%$ for the pdf and $\pm 4\%$ for the cdf and (bottom) second approximation is $\pm 2\%$ for the pdf and $\pm 2.5\%$ for the cdf.

5 Conclusions

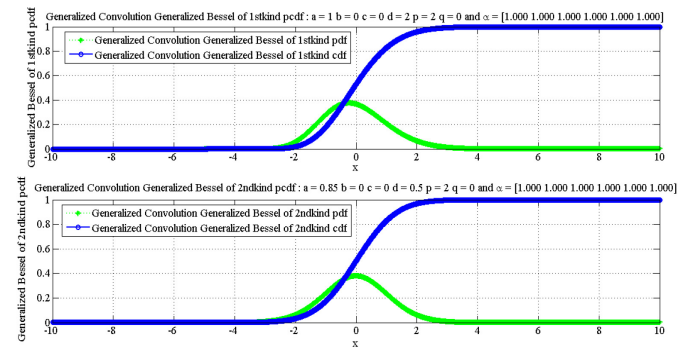
We have been able to correct the major errors reported in the literature regarding the generalized Bessel of the first and second kind pdf and cdf. All the derivations provided in this paper are entirely original^{viii}.



(a) (top) original generalized Bessel of the first kind pdf and cdf for $a = 1$, $d = 2$, and $p = 1$; (bottom) original generalized Bessel of the second kind pdf and cdf for $a = 0.84$, $d = 0.5$, and $p = 2$.

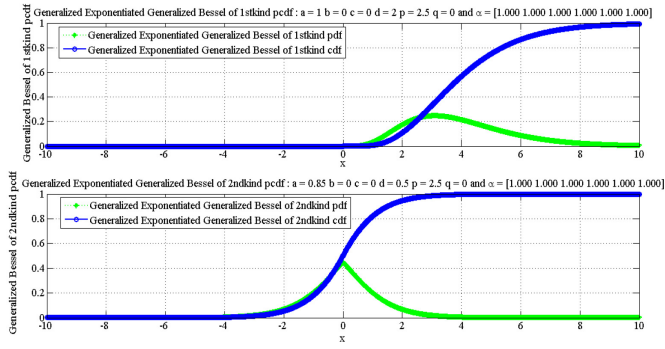


(b) exponentiated (top) generalized Bessel of the first kind pdf and cdf for $a = 1$, $d = 2$, $p = 2$; (bottom) generalized Bessel of the second kind pdf and cdf for $a = 0.85$, $d = 0.5$, $p = 2$ and $\alpha = 0.5$

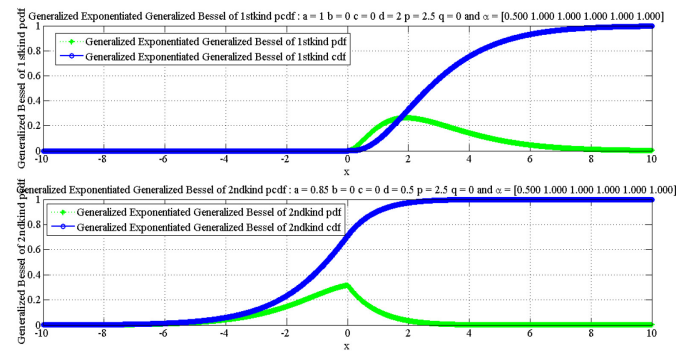


(c) generalized convolution (top) generalized Bessel of the first kind pdf and cdf for $a = 1$, $d = 2$, $p = 2$; (bottom) generalized Bessel of the second kind pdf and cdf for $a = 0.85$, $d = 0.5$, and $p = 2$.

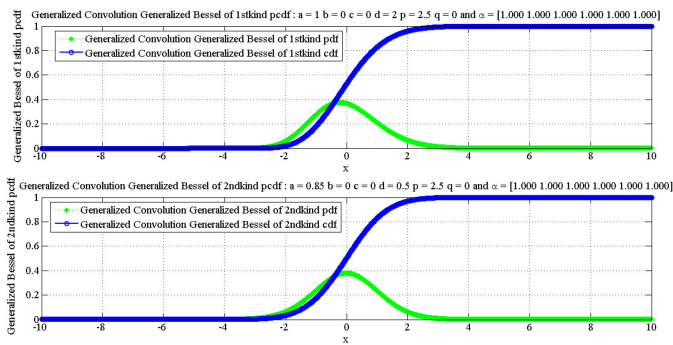
FIGURE 1: Numerical calculations of the generalized Bessel of the first kind pdf and cdf vs. Generalized Bessel of the second kind: (a) original pdf and cdf; (b) exponentiated pdf and cdf for $\alpha = 0.5$; (c) generalized convolution.



(a) (top) original generalized Bessel of the first kind pdf and cdf for $a = 1$, $d = 2$, and $p = 2.5$; (bottom) original generalized Bessel of the second kind pdf and cdf for $a = 0.85$, $d = 0.5$, and $p = 2.5$.

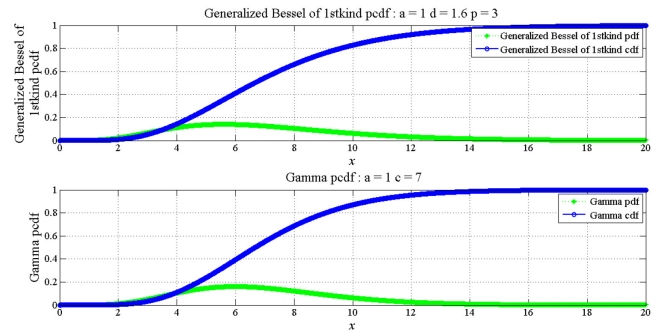


(b) exponentiated (top) generalized Bessel of the first kind pdf and cdf for $a = 1$, $d = 2$, $p = 2.5$; (bottom) generalized Beta of the first kind pdf and cdf for $a = 0.85$, $d = 0.5$, and $p = 2.5$ and $\alpha = 0.5$

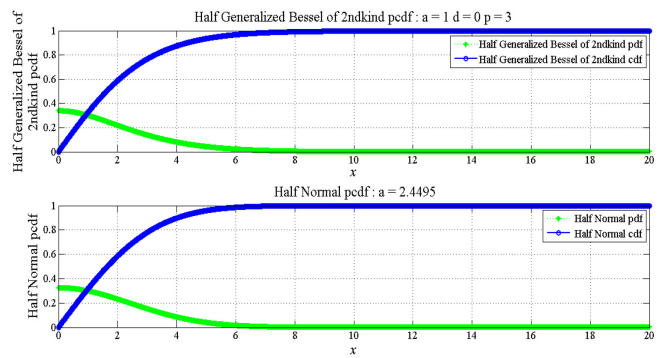


(c) generalized convolution (top) generalized Bessel of the first kind pdf and cdf for $a = 1$, $d = 2$, $p = 2.5$; (bottom) generalized Bessel of the second kind pdf and cdf for $a = 0.85$, $d = 0.5$, and $p = 2.5$.

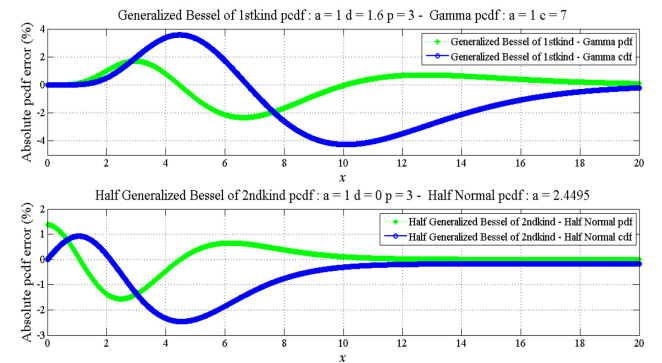
FIGURE 2: Numerical calculations of the generalized Bessel of the first kind pdf and cdf vs. Generalized Bessel of the second kind: (a) original pdf and cdf; (b) exponentiated pdf and cdf for $\alpha = 0.5$; (c) generalized convolution.



(a) (top) original generalized Bessel of the first kind pdf and cdf for $a = 1$, $d = 1.6$, and $p = 3$; (bottom) Gamma pdf and cdf for $a = 1$ and $c = 7 = 2p + 1$.



(b) (top) generalized Bessel of the first second kind pdf and cdf for $a = 1$, $d = 0$, $p = 3$; (bottom) half normal pdf and cdf for $a = \sqrt{2p} = 6$.



(c) absolute error from the generalized Bessel of the (top) first second kind pdf and cdf for $a = 1$, $d = 0$, $p = 3$; (bottom) half normal pdf and cdf for $a = \sqrt{2p} = 6$.

FIGURE 3: Numerical calculations of the generalized Bessel of the first and second kind pdf and cdf vs. approximated Bessel of the first and second kind pdf and cdf.

For each generalized Bessel distribution model, the closed form expression of the cdf is given by means of series expansion of the modified Bessel functions which leads to incomplete gamma functions, hypergeometric series, and the

Kampé de Fériet function. Numerical results are derived for each case to validate the theoretical models presented in the paper.

It is also shown that most practical applications the generalized Bessel function distribution cdf of the first and second kinds can be approximated from the family of well-known distribution functions to within five percent or less absolute error accuracy.

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ⁱ The details of these derivations, the numerical efficient algorithm, and MATLAB functions will be published in a separate paper in 2018.

ⁱⁱ Ditto.

ⁱⁱⁱ Ditto.

^{iv} Ditto.

^v Ditto.

^{vi} If we assume that $x \rightarrow \infty$ then we get

$$\frac{\partial}{\partial \alpha} \int_0^\infty 2^\alpha \left(\frac{t}{2}\right)^\alpha e^{-td} dt = \frac{\partial}{\partial \alpha} \frac{\Gamma(\alpha+1)}{d^{\alpha+1}} \quad (1)$$

Taking the derivative of the left-hand side yields

$$\frac{\partial \int_0^\infty 2^\alpha \left(\frac{t}{2}\right)^\alpha e^{-td} dt}{\partial \alpha} = \int_0^\infty \frac{\partial 2^\alpha}{\partial \alpha} \left(\frac{t}{2}\right)^\alpha e^{-td} dt + \int_0^\infty 2^\alpha \frac{\partial \left(\frac{t}{2}\right)^\alpha}{\partial \alpha} e^{-td} dt \quad (2)$$

Evaluating the first integral of (2) yields

$$\int_0^\infty \frac{\partial 2^\alpha}{\partial \alpha} \left(\frac{t}{2}\right)^\alpha e^{-td} dt = \ln(2) \left[\int_0^\infty t^\alpha e^{-td} dt \right] \equiv \frac{\Gamma(\alpha+1)}{d^{\alpha+1}} \quad (3)$$

The second integral of (2) is obtained from:

$$\int_0^\infty 2^\alpha \frac{\partial \left(\frac{t}{2}\right)^\alpha}{\partial \alpha} e^{-td} dt = \int_0^\infty t^\alpha \ln\left(\frac{t}{2}\right) e^{-td} dt \quad (4)$$

https://en.wikipedia.org/wiki/Reynolds_transport_theorem

Equation (4) is in fact our desired integral. Next it remains to evaluate the right-hand side of (1) as follows:

$$\frac{\partial}{\partial \alpha} \frac{\Gamma(\alpha+1)}{d^{\alpha+1}} = \frac{d^{\alpha+1} \frac{\partial}{\partial \alpha} \Gamma(\alpha+1) - \Gamma(\alpha+1) \frac{\partial d^{\alpha+1}}{\partial \alpha}}{d^{2\alpha+2}} \equiv \frac{d^{\alpha+1} \psi(\alpha+1) \Gamma(\alpha+1) - \Gamma(\alpha+1) d^{\alpha+1} \ln(d)}{d^{2\alpha+2}} \quad (5)$$

Equation (5) can be further simplified as

$$\frac{\partial}{\partial \alpha} \frac{\Gamma(\alpha+1)}{d^{\alpha+1}} = \frac{\Gamma(\alpha+1)}{d^{\alpha+1}} [\psi(\alpha+1) - \ln(d)] \quad (6)$$

Finally substituting (3), (4), and (6) into (1) we obtain the desired result

$$\int_0^\infty t^\alpha \ln\left(\frac{t}{2}\right) e^{-td} dt = \frac{\Gamma(\alpha+1)}{d^{\alpha+1}} \{\psi(\alpha+1) - [\ln(d) + \ln(2)]\} \quad (7)$$

Is this expansion correct? In Gradshteyn, Ryzhik [14] pg. 573 ex. 4.352 1.

We have a similar answer. So, I believe that (7) is correct.

^{vii} In this case a lot more work is required to obtain the closed form expression. The details of these derivations, the numerical efficient algorithm, and MATLAB functions will be published in a separate paper soon.

^{viii} Reformatted in 2021 to improve the quality of the formatting of the equations.