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*Research Article*

# VBOC2(α,1−α) ACF Pure Signal Optimization

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This paper examines common optimization algorithms, such as sum and mean square sense applied to the optimization of the autocorrelation functions (ACF) of GPS M-code like signals such as the variable binary offset carrier (VBOC). Before the different versions are examined for efficiency, on GPS M-code like signals such as VBOC2 which vary with the parameter of signal design and optimization alpha, they are checked to make sure that their corresponding ACFs are valid via a set of conditions known as *continuity theorems*. Afterwards, by employing any of the optimization algorithms such as sum and mean square sense on GPS M-code like signals such as VBOC2 we are able to produce ACFs that are one *ninety four percent more efficient* than the corresponding ACFs of the GPS M-code signals.

*Index Terms*—Pulse generation, pulse amplitude modulation, pulse width modulation, multidimensional sequences, signal design, signal analysis, generalized functions, time-frequency analysis, minimization methods, optimization methods.

## 1 Introduction

The main objective of this paper is to introduce $VBOC2(\alpha, 1-\alpha)$ ACF pure signal optimization as a continuation of the discussion on $VBOC2(\alpha, 1-\alpha)$ generalized multidimensional geolocation modulation waveforms or $VBOC2(\alpha, 1-\alpha)$ pure signal design [1]-[7] methodology gaps created over the years as a results of incomplete signal design methodologies.

For the first time I introduced the idea of signal optimization in Chap. 7 of [5] and then for the second time in [2] it was not until I started to design $VBOC2(\alpha, 1-\alpha)$ that it became apparent that $VBOC2(\alpha, 1-\alpha)$ pure signal design [3] and optimization offered unique challenges.

What are the unique and original challenges of $VBOC2(\alpha, 1-\alpha)$ pure signal design [3] and optimization that I believe so far have never been described in any other publication [7]-[27]?

If we take a look at $R_{VBOC2_{(4n,n,\alpha)}}(\tau)$ in Appendix B of Chap. 7 of [5] we notice that $R_{VBOC2_{(4n,n,\alpha)}}(\tau)$ contains only fifteen terms or fourteen nonzero terms and one zero term. Therefore, initially this gave me the impression that the generalized ACF of $VBOC2(\alpha, 1-\alpha)$ or $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ should also contain fifteen terms or fourteen nonzero terms and one zero term. Well coming up with the generalized ACF of $VBOC2(\alpha, 1-\alpha)$ or $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ which contains fifteen terms or fourteen nonzero terms and one zero term was easy; however, *it was IMPOSSIBLE to satisfy the continuity theorems* discussed in this paper.

Therefore, I started to rethink the entire process. If you take a look at the generalized ACF of $VBOC2(\alpha, 1-\alpha)$ or $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ in (38) or (39) of [3] for $p = 2$ we notice that it contains eight non-zero terms; therefore, the ACF of $VBOC2(\alpha, 1-\alpha)$ or $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ in (38) or (39) of [3] for $p = 4$ needed to contain sixteen terms as shown in (38) or (39) of [3]. Hence, $VBOC2(\alpha, 1-\alpha)$ pure signal design [3] and optimization offered unique and original signal design and optimization challenges not encountered anywhere else.

Finally, as a result of this work it is shown that $VBOC2_{(pn,n,\alpha)}$ for $\alpha = 0.4$ offers 94 percent efficiency or for $\alpha = 0.4$ $VBOC2_{(pn,n,\alpha)}$ is 94 percent more efficient than either $BOC_{(pn,n)}/BOC_{(pn/2,n)}(t)$ ($BPSK_0$ being a special case of $BOC_{(pn/2,n)}(t)$) regardless of the optimization criteria in contrast, with the 100 percent efficiency for $VBOC1_{(pn,n,\alpha)}$ we obtained in [2]. In the case of $VBOC1(\alpha)$ we were able to achieve a global optimum for $\alpha = 0.5$; hence, 100 percent efficiency; however, in the case of $VBOC2(\alpha, 1-\alpha)$ we are very close to achieving a global optimum $\alpha = \sim 0.4$ $**$; hence, 94 percent efficiency.

This paper is organized as follows: $VBOC2(\alpha, 1-\alpha)$ pure signal design is discussed next followed by numerical results. Conclusion is given afterwards and followed by a list of references.

## 2 $VBOC2(\alpha, 1-\alpha)$ ACF Pure Signal Optimization

The detailed discussion on $VBOC2(\alpha, 1-\alpha)$ ACF pure signal optimization includes: (1) $VBOC2(\alpha, 1-\alpha)$ ACF continuity theorems; and (2) $VBOC2(\alpha, 1-\alpha)$ ACF optimization theorems.

### 2.1 $VBOC2(\alpha, 1-\alpha)$*ACF continuity theorems.*

*Theorem* 1*: Prove that the ACF* $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$, *given by* (38) and (39) [3], obeys the *continuity theorem of the first kind;* i.e., *it is a continuous function for all values of* $\tau$, $p = \{2,4\}$, *and* $\alpha = [0,1]$.

*Proof of theorem* 1: The *proof of theorem* 1 is straightforward. In order to prove that *the* ACF, $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$, *given by* (38) and (39) [3], obeys the *continuity theorem of the first kind;* i.e., *it is a continuous function for every values of* $\tau$ *and* $p = \{2,4\}$ it suffices to prove that $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$, given by (1) and (2) and $p = \{2,4\}$ obeys the *continuity theorem of the first kind;* i.e., *it is a continuous function for every values of* $\tau$.

We prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the first kind;* i.e., *it is a continuous function for every values of* $\tau$. Because the ACF $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ is an even function of $\tau$ it is sufficient to check the continuity of $R_{VBOC2_{(pn=n,n,\alpha)}}(\tau)$ for values of $\tau$ given by for $0 \le \alpha \le 1/3$

$$\underbrace{\tau_{\mp i}(\alpha)}_{i=\{1,2,\cdots 2p\}} = \underbrace{\left[\underbrace{\begin{matrix} \underbrace{\frac{T_c(1-\alpha)}{2p}\mp}_{\tau_{\mp1}(\alpha)} & \underbrace{\frac{T_c(1+\alpha)}{p}\mp}_{\tau_{\mp5}(\alpha)} \\ \underbrace{\frac{T_c(1+\alpha)}{2p}\mp}_{\tau_{\mp2}(\alpha)} & \underbrace{\frac{T_c(3-\alpha)}{2p}\mp}_{\tau_{\mp6}(\alpha)} \\ \underbrace{\frac{T_c(1-\alpha)}{p}}_{\tau_{\mp3}(\alpha)} & \underbrace{\frac{T_c(3+\alpha)}{2p}\mp}_{\tau_{\mp7}(\alpha)} \\ \underbrace{\frac{T_c}{p}\mp}_{\tau_{\mp4}(\alpha)} & \underbrace{\frac{2T_c}{p}\mp}_{\tau_{\mp8}(\alpha)} \end{matrix}}_{p=2} \begin{matrix} \underbrace{\frac{T_c(5-\alpha)}{2p}\mp}_{\tau_{\mp9}(\alpha)} & \underbrace{\frac{T_c(3+\alpha)}{p}\mp}_{\tau_{\mp13}(\alpha)} \\ \underbrace{\frac{T_c(5+\alpha)}{2p}\mp}_{\tau_{\mp10}(\alpha)} & \underbrace{\frac{T_c(7-\alpha)}{2p}\mp}_{\tau_{\mp14}(\alpha)} \\ \underbrace{\frac{T_c(3-\alpha)}{p}\mp}_{\tau_{\mp11}(\alpha)} & \underbrace{\frac{T_c(7+\alpha)}{2p}\mp}_{\tau_{\mp15}(\alpha)} \\ \underbrace{\frac{3T_c}{p}\mp}_{\tau_{\mp12}(\alpha)} & \underbrace{\frac{4T_c}{p}\mp}_{\tau_{\mp16}(\alpha)} \end{matrix}\right]}_{p=4} \quad (1)$$

and $1/3 \le \alpha \le 1$

$$\underbrace{\tau_{\mp i}(\alpha)}_{i=\{1,2,\cdots 2p\}} = \underbrace{\left[\underbrace{\begin{matrix} \underbrace{\frac{T_c(1-\alpha)}{2p}\mp}_{\tau_{\mp1}(\alpha)} & \underbrace{\frac{T_c(3-\alpha)}{2p}\mp}_{\tau_{\mp5}(\alpha)} \\ \underbrace{\frac{T_c(1-\alpha)}{p}\mp}_{\tau_{\mp2}(\alpha)} & \underbrace{\frac{T_c(1+\alpha)}{p}\mp}_{\tau_{\mp6}(\alpha)} \\ \underbrace{\frac{T_c(1+\alpha)}{2p}}_{\tau_{\mp3}(\alpha)} & \underbrace{\frac{T_c(3+\alpha)}{2p}\mp}_{\tau_{\mp7}(\alpha)} \\ \underbrace{\frac{T_c}{p}\mp}_{\tau_{\mp4}(\alpha)} & \underbrace{\frac{2T_c}{p}\mp}_{\tau_{\mp8}(\alpha)} \end{matrix}}_{p=2} \begin{matrix} \underbrace{\frac{T_c(5-\alpha)}{2p}\mp}_{\tau_{\mp9}(\alpha)} & \underbrace{\frac{T_c(7-\alpha)}{2p}\mp}_{\tau_{\mp13}(\alpha)} \\ \underbrace{\frac{T_c(3-\alpha)}{p}\mp}_{\tau_{\mp10}(\alpha)} & \underbrace{\frac{T_c(3+\alpha)}{p}\mp}_{\tau_{\mp14}(\alpha)} \\ \underbrace{\frac{T_c(5+\alpha)}{2p}\mp}_{\tau_{\mp11}(\alpha)} & \underbrace{\frac{T_c(7+\alpha)}{2p}\mp}_{\tau_{\mp15}(\alpha)} \\ \underbrace{\frac{3T_c}{p}\mp}_{\tau_{\mp12}(\alpha)} & \underbrace{\frac{4T_c}{p}\mp}_{\tau_{\mp16}(\alpha)} \end{matrix}\right]}_{p=4} \quad (2)$$

First, we substitute values of $\tau = \tau_{\mp1}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) and (39) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-1}(\alpha)] = 1 - \frac{(4p-1)(1-\alpha)}{2p} = \frac{1-2p+(4p-1)\alpha}{2p}}_{\left\{\underbrace{\frac{-3+7\alpha}{4}}_{p=2}, \underbrace{\frac{-7+15\alpha}{8}}_{p=4}\right\}} \quad (3)$$

On the other hand

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+1}(\alpha)]=2\alpha-1+\frac{1-\alpha}{2p}=\frac{1-2p+(4p-1)\alpha}{2p}}_{\left\{\underbrace{\frac{-3+7\alpha}{4}}_{p=2},\underbrace{\frac{-7+15\alpha}{8}}_{p=4}\right\}=R_{VBOC2_{(np,n,\alpha)}}[\tau_{-1}(\alpha)]} \tag{4}$$

Second, we substitute values of $\tau=\tau_{\mp 2}(\alpha)$ in $R_{VBOC2_{(pn=n,n,\alpha)}}(\tau)$ given by (38) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-2}(\alpha)]=2\alpha-1+\frac{1+\alpha}{2p}=\frac{1-2p+(4p+1)\alpha}{2p}}_{\left\{\underbrace{\frac{-3+9\alpha}{4}}_{p=2},\underbrace{\frac{-7+17\alpha}{8}}_{p=4}\right\}} \tag{5}$$

On the other hand

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+2}(\alpha)]=\frac{\underbrace{-6p+4+4\alpha+(4p-3)(1+\alpha)}_{=1-2p+(4p+1)\alpha}}{2p}}_{\left\{\underbrace{\frac{-3+9\alpha}{4}}_{p=2},\underbrace{\frac{-7+17\alpha}{8}}_{p=4}\right\}=R_{VBOC1_{(np,n,\alpha)}}[\tau_{-2}(\alpha)]} \tag{6}$$

Third, we substitute values of $\tau=\tau_{\mp 2}(\alpha)$ in $R_{VBOC2_{(pn=n,n,\alpha)}}(\tau)$ given by (39) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-2}(\alpha)]=2\alpha-1+\frac{1-\alpha}{p}=\frac{1-p+(2p-1)\alpha}{p}}_{\left\{\underbrace{\frac{-1+3\alpha}{2}}_{p=2},\underbrace{\frac{-3+7\alpha}{4}}_{p=4}\right\}} \tag{7}$$

On the other hand

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+2}(\alpha)]=1-\frac{(2p-1)(1-\alpha)}{p}=\frac{1-p+(2p-1)\alpha}{p}}_{\left\{\underbrace{\frac{-1+3\alpha}{2}}_{p=2},\underbrace{\frac{-3+7\alpha}{4}}_{p=4}\right\}=R_{VBOC1_{(np,n,\alpha)}}[\tau_{-2}(\alpha)]} \tag{8}$$

Fourth, we substitute values of $\tau=\tau_{\mp 3}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-3}(\alpha)]=\frac{\underbrace{-3p+2+2\alpha+(4p-3)(1-\alpha)}_{=p-1+(5-4p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-3\alpha}{2}}_{p=2},\underbrace{\frac{3-11\alpha}{4}}_{p=4}\right\}} \tag{9}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+3}(\alpha)]=\frac{\underbrace{2-p-2(p-1)\alpha-(2p-3)(1-\alpha)}_{=p-1+(5-4p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-3\alpha}{2}}_{p=2},\underbrace{\frac{3-11\alpha}{4}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-3}(\alpha)]} \tag{10}$$

Fifth, we substitute values of $\tau=\tau_{\mp 3}(\alpha)$ in $R_{VBOC2_{(pn=n,n,\alpha)}}(\tau)$ given by (39) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-3}(\alpha)]=1-\frac{(2p-1)(1+\alpha)}{2p}=\frac{1-(2p-1)\alpha}{2p}}_{\left\{\underbrace{\frac{1-3\alpha}{4}}_{p=2},\underbrace{\frac{1-7\alpha}{8}}_{p=4}\right\}} \tag{11}$$

On the other hand

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+3}(\alpha)]=\frac{\underbrace{4-2p-4(p-1)\alpha+(2p-3)(1+\alpha)}_{=1-(2p-1)\alpha}}{2p}}_{\left\{\underbrace{\frac{1-3\alpha}{4}}_{p=2},\underbrace{\frac{1-7\alpha}{8}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-3}(\alpha)]} \tag{12}$$

Sixth, we substitute values of $\tau=\tau_{\mp 4}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by either (38) and (39) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-4}(\alpha)]=\frac{\underbrace{2-p-2(p-1)\alpha+2p-3}_{=p-1+(2-2p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-2\alpha}{2}}_{p=2},\underbrace{\frac{3-6\alpha}{4}}_{p=4}\right\}} \tag{13}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+4}(\alpha)]=\frac{\underbrace{(3p-2)-(2p-2)\alpha-2p+1}_{=p-1+(2-2p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-2\alpha}{2}}_{p=2},\underbrace{\frac{3-6\alpha}{4}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-4}(\alpha)]} \tag{14}$$

Seventh, we substitute values of $\tau=\tau_{\mp 5}(\alpha)$ in $R_{VBOC2_{(pn=2n,n,\alpha)}}(\tau)$ given by (38) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-5}(\alpha)]=\frac{\underbrace{3p-2-2(p-1)\alpha-(2p-1)(1+\alpha)}_{=p-1+(3-4p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-5\alpha}{2}}_{p=2},\underbrace{\frac{3-13\alpha}{4}}_{p=4}\right\}} \tag{15}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+5}(\alpha)]=\frac{\underbrace{(5p-6)-2\alpha-(4p-5)(1+\alpha)}_{=p-1+(3-4p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-5\alpha}{2}}_{p=2},\underbrace{\frac{3-13\alpha}{4}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-5}(\alpha)]} \tag{16}$$

Eight, we substitute values of $\tau=\tau_{\mp 5}(\alpha)$ in $R_{VBOC2_{(np=2n,n,\alpha)}}(\tau)$ given by (39) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-5}(\alpha)]=\frac{\underbrace{6p-4-4(p-1)\alpha-(2p-1)(3-\alpha)}_{=-1+(3-2p)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1-\alpha}{4}}_{p=2},\underbrace{\frac{-1-5\alpha}{8}}_{p=4}\right\}} \tag{17}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+5}(\alpha)]=\frac{\underbrace{-6p+8+(2p-3)(3-\alpha)}_{=-1+(3-2p)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1-\alpha}{4}}_{p=2},\underbrace{\frac{-1-5\alpha}{8}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-5}(\alpha)]} \tag{18}$$

Ninth, we substitute values of $\tau=\tau_{\mp 6}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-6}(\alpha)]=\frac{\underbrace{2(5p-6)-4\alpha-(4p-5)(3-\alpha)}_{=3-2p+(4p-9)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1-\alpha}{4}}_{p=2},\underbrace{\frac{-5+7\alpha}{8}}_{p=4}\right\}} \tag{19}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+6}(\alpha)]=\frac{\underbrace{-2p+4(p-2)\alpha+3-\alpha}_{=3-2p+(4p-9)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1-\alpha}{4}}_{p=2},\underbrace{\frac{-5+7\alpha}{8}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-6}(\alpha)]} \tag{20}$$

Tenth, we substitute values of $\tau = \tau_{\mp 6}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (39) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-6}(\alpha)] = \frac{\underbrace{-3p+4+(2p-3)(1+\alpha)}_{=1-p+(2p-3)\alpha}}{p}}_{\left\{\underbrace{\frac{-1+\alpha}{2}}_{p=2},\underbrace{\frac{-3+5\alpha}{4}}_{p=4}\right\}} \tag{21}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{+6}(\alpha)] = \frac{\underbrace{-p+2(p-2)\alpha+1+\alpha}_{=1-p+(2p-3)\alpha}}{p}}_{\left\{\underbrace{\frac{-1+\alpha}{2}}_{p=2},\underbrace{\frac{-3+5\alpha}{4}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-6}(\alpha)]} \tag{22}$$

Eleventh, we substitute values of $\tau = \tau_{\mp 7}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) and (39) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-7}(\alpha)] = \frac{\underbrace{-2p+4(p-2)\alpha+3+\alpha}_{=3-2p+(4p-7)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1+\alpha}{4}}_{p=2},\underbrace{\frac{-5+9\alpha}{8}}_{p=4}\right\}} \tag{23}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{+7}(\alpha)] = \frac{\underbrace{-14p+24+(4p-7)(3+\alpha)}_{=3-2p+(4p-7)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1+\alpha}{4}}_{p=2},\underbrace{\frac{-5+9\alpha}{8}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-7}(\alpha)]} \tag{24}$$

Twelfth, we substitute values of $\tau = \tau_{\mp 8}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) and (39) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-8}(\alpha)] = -\frac{7p-12}{p} + \frac{2(4p-7)}{p} = \frac{p-2}{p}}_{\left\{\underbrace{0}_{p=2},\underbrace{\frac{1}{2}}_{p=4}\right\}} \tag{25}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{+8}(\alpha)] = \frac{9p-20}{p} - \frac{2(4p-9)}{p} = \frac{p-2}{p}}_{\left\{\underbrace{0}_{p=2},\underbrace{\frac{1}{2}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-8}(\alpha)]} \tag{26}$$

Thirteenth, we substitute values of $\tau = \tau_{\mp 9}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) and (39) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-9}(\alpha)] = \frac{\underbrace{18p-40-(4p-9)(5-\alpha)}_{=5-2p+(4p-9)\alpha}}{2p}}_{\left\{\underbrace{\frac{-3+7\alpha}{8}}_{p=4}\right\}} \tag{27}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{+9}(\alpha)] = \frac{\underbrace{20-7p+(5p-12)\alpha+(5-\alpha)(p-3)}_{=5-2p+(4p-9)\alpha}}{2p}}_{\left\{\underbrace{\frac{-3+7\alpha}{8}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-9}(\alpha)]} \tag{28}$$

Fourteenth, we substitute values of $\tau = \tau_{\mp 10}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-10}(\alpha)] = \frac{\underbrace{20-7p+(5p-12)\alpha+(p-3)(5+\alpha)}_{=5-2p+(6p-15)\alpha}}{2p}}_{\left\{\underbrace{\frac{-3+9\alpha}{8}}_{p=4}\right\}} \tag{29}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{+10}(\alpha)] = \frac{\underbrace{60-22p+2(p-2)\alpha+(4p-11)(5+\alpha)}_{=5-2p+(6p-15)\alpha}}{2p}}_{\left\{\underbrace{\frac{-3+9\alpha}{8}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-10}(\alpha)]} \tag{30}$$

Fifteenth, we substitute values of $\tau = \tau_{\mp 10}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (39) [3] and we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-10}(\alpha)] = \frac{\underbrace{20-7p+(5p-12)\alpha+2(p-3)(3-\alpha)}_{=2-p+(3p-6)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1+3\alpha}{4}}_{p=4}\right\}} \tag{31}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{+10}(\alpha)] = \frac{\underbrace{11p-28+(4-p)\alpha+2(5-2p)(3-\alpha)}_{=2-p+(3p-6)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1+3\alpha}{4}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-10}(\alpha)]} \tag{32}$$

Sixteenth, we substitute values of $\tau = \tau_{\mp 11}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) [3] we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-11}(\alpha)] = \frac{\underbrace{30-11p+(p-2)\alpha+(4p-11)(3-\alpha)}_{=p-3+(9-3p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-3\alpha}{4}}_{p=4}\right\}} \tag{33}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{+11}(\alpha)] = \frac{\underbrace{6-2p+(6-2p)\alpha+(p-3)(3-\alpha)}_{=p-3+(9-3p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-3\alpha}{4}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-11}(\alpha)]} \tag{34}$$

Seventeenth, we substitute values of $\tau = \tau_{\mp 11}(\alpha)$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (39) [3] we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-11}(\alpha)] = \frac{\underbrace{11p-28+(4-p)\alpha+(5-2p)(5+\alpha)}_{=p-3+(9-3p)\alpha}}{2p}}_{\left\{\underbrace{\frac{1-3\alpha}{8}}_{p=4}\right\}} \tag{35}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{+11}(\alpha)] = \frac{\underbrace{-4(p-3)-4(p-3)\alpha+(p-3)(5+\alpha)}_{=p-3+(9-3p)\alpha}}{2p}}_{\left\{\underbrace{\frac{1-3\alpha}{4}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-11}(\alpha)]} \tag{36}$$

Eighteenth, we substitute values of $\tau = \tau_{\mp 12}$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) and (39) [3] we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-12}(\alpha)]=\frac{\underbrace{-2(p-3)-2(p-3)\alpha+3(p-3)}_{=p-3+(6-2p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-2\alpha}{4}}_{p=4}\right\}} \tag{37}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+12}(\alpha)]=\frac{\underbrace{7p-18-2(p-3)\alpha+3(5-2p)}_{=p-3+(6-2p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-2\alpha}{4}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-12}(\alpha)]} \tag{38}$$

Nineteenth, we substitute values of $\tau=\tau_{\mp 13}$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) [3] we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-13}(\alpha)]=\frac{\underbrace{7p-18-2(p-3)\alpha+(5-2p)(3+\alpha)}_{=p-3+(11-4p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-5\alpha}{4}}_{p=4}\right\}} \tag{39}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+13}(\alpha)]=\frac{\underbrace{13p-42-2\alpha+(13-4p)(3+\alpha)}_{=p-3+(11-4p)\alpha}}{p}}_{\left\{\underbrace{\frac{1-5\alpha}{4}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-13}(\alpha)]} \tag{40}$$

Twentieth, we substitute values of $\tau=\tau_{\mp 13}$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (39) [3] we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-13}(\alpha)]=\frac{\underbrace{14p-36-4(p-3)\alpha+(5-2p)(7-\alpha)}_{=-1+(7-2p)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1-\alpha}{8}}_{p=4}\right\}} \tag{41}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+13}(\alpha)]=\frac{\underbrace{48-14p+(2p-7)(7-\alpha)}_{=-1+(7-2p)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1-\alpha}{8}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-13}(\alpha)]} \tag{42}$$

Twenty-first, we substitute values of $\tau=\tau_{\mp 14}$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) [3] we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-14}(\alpha)]=\frac{\underbrace{26p-84-4\alpha+(13-4p)(7-\alpha)}_{=7-2p+(4p-17)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1-\alpha}{8}}_{p=4}\right\}} \tag{43}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+14}(\alpha)]=\frac{\underbrace{-2p+4(p-4)\alpha+7-\alpha}_{=7-2p+(4p-17)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1-\alpha}{8}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-13}(\alpha)]} \tag{44}$$

Twenty-second, we substitute values of $\tau=\tau_{\mp 14}$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (39) [3] we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-14}(\alpha)]=\frac{\underbrace{24-7p+(2p-7)(3+\alpha)}_{=3-p+(2p-7)\alpha}}{p}}_{\left\{\underbrace{\frac{-1+\alpha}{4}}_{p=4}\right\}} \tag{45}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+14}(\alpha)]=\frac{\underbrace{-p+2(p-4)\alpha+3+\alpha}_{=3-p+(2p-7)\alpha}}{p}}_{\left\{\underbrace{\frac{-1+\alpha}{4}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-14}(\alpha)]} \tag{46}$$

Twenty-third, we substitute values of $\tau=\tau_{\mp 15}$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) and (39) [3] we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-15}(\alpha)]=\frac{\underbrace{-2p+4(p-4)\alpha+7+\alpha}_{=7-2p+(4p-15)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1+\alpha}{8}}_{p=4}\right\}} \tag{47}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+15}(\alpha)]=\frac{\underbrace{102-30p+(4p-15)(7+\alpha)}_{=7-2p+(4p-15)\alpha}}{2p}}_{\left\{\underbrace{\frac{-1+\alpha}{8}}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-15}(\alpha)]} \tag{48}$$

Twenty-fourth and finally, we substitute values of $\tau=\tau_{\mp 16}$ in $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) and (39) [3] we get

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{-16}(\alpha)]=\frac{56-15p}{p}+\frac{4(4p-15)}{p}=\frac{p-4}{p}}_{\left\{\underbrace{0}_{p=4}\right\}} \tag{49}$$

On the other hand we have

$$\underbrace{R_{VBOC2_{(pn,n,\alpha)}}[\tau=\tau_{+16}(\alpha)]=0}_{\left\{\underbrace{0}_{p=4}\right\}=R_{VBOC2_{(pn,n,\alpha)}}[\tau_{-16}(\alpha)]} \tag{50}$$

Equations (1) through (50) complete the proof of *Theorem* 1.

*Theorem 2: Prove that the ACF* $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$*,* given by (38) and (39) [3], obeys the *continuity theorem of the second kind*; i.e., *it is a continuous function for every values of* $\tau$, $p=\{2,4\}$, and $\alpha=\{0,1\}$.

*Proof of theorem* 2: The *proof of theorem* 2 is straightforward. In order to prove that *the* ACF, $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$, *given by* (38) and (39) [3], obeys the *continuity theorem of the second kind*; i.e., *it is a continuous function for all values of* $\tau$, $p=\{2,4\}$ and $\alpha=\{0,1\}$ it suffices to prove $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the second kind*; i.e., *it is a continuous function for all values of* $\tau$.

Since, we already proved in *theorem* 1; i.e., *continuity theorem of the first kind* that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) and (39) [3] *or for all values of* $\tau$, $p = \{2,4\}$ and for $\alpha = [0,1]$ we can employ the results of *theorem* 1 *to prove theorem* 2.

First, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (1) obeys the *continuity theorem of the second kind* for $\{\tau_1(\alpha), \tau_2(\alpha)\}$.

From (3) and (4), and $p = \{2,4\}$ we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_1(0)] = \underbrace{\frac{1-2p+(4p-1)\alpha}{2p} = -1 + \frac{1}{2p}}_{=\left\{\frac{-3}{4},\frac{-7}{8}\right\}} \quad (51)$$

One the other hand, from (5) and (6) and $p = \{2,4\}$ we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_2(0)] = \underbrace{\underbrace{\frac{1-2p+(4p+1)\alpha}{2p} = -1 + \frac{1}{2p}}_{=\left\{\frac{-3}{4},\frac{-7}{8}\right\}}}_{R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_1(0)]} \quad (52)$$

Again, from (3) and (4) and $p = \{2,4\}$ we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_1(1)] = \frac{1-2p+4p-1}{2p} = 1 = R_{BOC_{(n,n)}}[0] \quad (53)$$

Finally, from (5) and (6) and $p = \{2,4\}$ we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_2(1)] = \frac{1-p+2p-1}{p} = 1 = R_{BOC_{(n,n)}}[0] \quad (54)$$

Equations (53) and (54) result from *definition* 2, Appendix B, and Chap. 7 of [5].

Second, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (1) obeys the *continuity theorem of the second kind* for $\{\tau_3(\alpha), \tau_4(\alpha), \tau_5(\alpha)\}$.

From (9) and (10), for $p = \{2,4\}$, we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_3(0)] = \frac{p-1+(5-4p)\alpha}{p} = 1 - \frac{1}{p} = \left\{\frac{1}{2},\frac{3}{4}\right\} \quad (55)$$

One the other hand, from (13) and (14), for $p = \{2,4\}$, we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_4(0)] = \underbrace{\frac{p-1+2(1-p)\alpha}{p} = 1 - \frac{1}{p} = \left\{\frac{1}{2},\frac{3}{4}\right\}}_{R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_3(0)]} \quad (56)$$

Again from (11) and (12), for $p = \{2,4\}$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_3(1)] = \frac{1-(2p-1)\alpha}{2p} = -1 + \frac{1}{p} = \left\{\frac{-1}{2},\frac{-3}{4}\right\} \quad (57)$$

Again, from (13) and (14), for $p = \{2,4\}$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_4(1)] = \underbrace{\frac{p-1+2(1-p)\alpha}{p} = \frac{1}{p} - 1 = \left\{\frac{-1}{2},\frac{-3}{4}\right\}}_{R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_3(1)]} \quad (58)$$

Finally, from (15) and (16), for $p = \{2,4\}$, we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_5(0)] = \underbrace{\underbrace{\frac{p-1+(3-4p)\alpha}{p} = 1 - \frac{1}{p} = \left\{\frac{1}{2},\frac{3}{4}\right\}}_{R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_3(0)]}}_{R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_4(0)]} \quad (59)$$

Finally, again, from (17) and (18), for $p = \{2,4\}$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_5(1)] = \underbrace{\underbrace{\frac{-1+(3-2p)\alpha}{2p} = \frac{1}{p} - 1 = \left\{\frac{-1}{2},\frac{-3}{4}\right\}}_{R_{VBOC2_{(pn,n,\alpha)}}[\tau_3(1)]}}_{R_{VBOC2_{(pn,n,\alpha)}}[\tau_4(1)]} \quad (60)$$

Third, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the second kind* for $\{\tau_6(\alpha), \tau_7(\alpha)\}$.

From (19), and (20), for $p = \{2,4\}$, we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_6(0)] = \frac{3-2p+(4p-9)\alpha}{2p} = \frac{3}{2p} - 1 = \left\{\frac{-1}{4},\frac{-5}{8}\right\} \quad (61)$$

One the other hand, from (23) and (24), for $p = \{2,4\}$ we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_7(0)] = \underbrace{\frac{3-2p+(4p-7)\alpha}{2p} = \frac{3}{2p} - 1 = \left\{\frac{-1}{4},\frac{-5}{8}\right\}}_{R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_6(0)]} \quad (62)$$

From (21), and (22), for $p = \{2,4\}$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_6(1)] = \frac{1-p+(2p-3)\alpha}{p} = 1 - \frac{2}{p} = \left\{0,\frac{1}{2}\right\} \quad (63)$$

One the other hand, from (23) and (24), for $p = \{2,4\}$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_7(1)] = \underbrace{\frac{3-2p+(4p-7)\alpha}{2p} = 1 - \frac{2}{p} = \left\{0,\frac{1}{2}\right\}}_{R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_6(1)]} \quad (64)$$

Fourth, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the second kind* for $\{\tau_9(\alpha), \tau_{10}(\alpha)\}$.

From (27) and (28), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_9(0)] = \frac{5-2p+(4p-9)\alpha}{2p} = \frac{5}{2p} - 1 = \left\{\frac{-3}{8}\right\} \quad (65)$$

One the other hand, from and (29) and (30), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_{10}(0)] = \underbrace{\frac{5-2p+(6p-15)\alpha}{2p} = \frac{5}{2p} - 1 = \left\{\frac{-3}{8}\right\}}_{R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_9(0)]} \quad (66)$$

Again from (27) and (28) for $p = 4$ we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_9(1)] = \frac{5-2p+(4p-9)\alpha}{2p} = 1 - \frac{2}{p} = \left\{\frac{1}{4}\right\} \quad (67)$$

One the other hand, from (31) and (32), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_{10}(1)] = \underbrace{\frac{2-p+(3p-6)\alpha}{2p} = 1 - \frac{2}{p} = \left\{\frac{1}{2}\right\}}_{R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_9(1)]} \quad (68)$$

Fifth, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the second kind* for $\{\tau_{11}(\alpha), \tau_{12}(\alpha), \tau_{13}(\alpha)\}$.

From (33) and (34), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_{11}(0)] = \frac{p-3+(9-3p)\alpha}{p} = 1 - \frac{3}{p} = \left\{\frac{1}{4}\right\} \quad (69)$$

One the other hand, from and (37) and (38), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_{12}(0)] = \underbrace{\frac{p-3+(6-2p)\alpha}{p} = 1 - \frac{3}{p} = \left\{\frac{1}{4}\right\}}_{R_{VBOC2_{(pn,n,\alpha=0)}}[\tau_{11}(0)]} \quad (70)$$

Again from (35) and (36) for $p = 4$ we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_{11}(1)] = \frac{p-3+(9-3p)\alpha}{2p} = -1 + \frac{3}{p} = \left\{\frac{-1}{4}\right\} \quad (71)$$

One the other hand, from (37) and (38), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_{12}(1)] = \underbrace{\frac{(p-3)-2(p-3)\alpha}{p} = \frac{3}{p} - 1 = \left\{\frac{-1}{4}\right\}}_{R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_{11}(1)]} \quad (72)$$

Again from (39) and (40) for $p = 4$ we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_{13}(0)] = \frac{p-3-(4p-11)\alpha}{p} = 1 - \frac{3}{p} = \left\{\frac{1}{4}\right\} \quad (73)$$

Again from (41) and (42) for $p = 4$ we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_{13}(1)] = \frac{-1-(2p-7)\alpha}{2p} = -1 + \frac{3}{p} = \left\{\frac{-1}{4}\right\} \quad (74)$$

Equations (55) through (71) complete the proof of *theorem* 4.

From (43) and (44), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_{14}(0)] = \frac{7-2p+(4p-17)\alpha}{2p} = \frac{7}{2p} - 1 = \left\{\frac{-1}{8}\right\} \quad (75)$$

From (45) and (46), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_{14}(1)] = \frac{3-p+(2p-7)\alpha}{p} = 1 - \frac{4}{p} = \{0\} \quad (76)$$

From (47) and (48), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_{15}(0)] = \frac{7-2p+(4p-15)\alpha}{2p} = \frac{7}{2p} - 1 = \left\{\frac{-1}{8}\right\} \quad (77)$$

From (47) and (48), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha=1)}}[\tau_{15}(1)] = \frac{7-2p+(4p-15)\alpha}{2p} = 1 - \frac{4}{p} = \{0\} \quad (78)$$

Again from (49) for $p = 4$ we have

$$R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-16}(0)] = \frac{p-4}{p} = 1 - \frac{4}{p} = \{0\} \quad (79)$$

Unfortunately, this is also only one data point.

Finally, from (50), for $p = 4$, we have

$$R_{VBOC2_{(pn,n,\alpha)}}[\tau = \tau_{-16}(1)] = \frac{p-4}{p} = 1 - \frac{4}{p} = \{0\} \quad (80)$$

Unfortunately, this is only one data point.

Equations (51) through (80) complete the proof of *theorem* 2 or *continuity theorem of the second kind of the ACF of* $VBOC2_{(pn,n,\alpha)}$.

*Theorem* 3*: Prove that the ACF* $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$*,* given by (38) and (39) [3], obeys the *continuity theorem of the third kind;* i.e.*, it is a continuous function for all values of* $\tau$ *and* $p = \{2,4\}$ and $\alpha = \{1/3\}$.

The *proof of theorem* 3 is straightforward. In order to prove that *the* ACF, $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$, *given by* (38) and (39) [3], obeys the *continuity theorem of the third kind;* i.e.*, it is a continuous function for every values of* $\tau$*,* $p = \{2,4\}$ and $\alpha = \{1/3\}$ it suffices to prove $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind;* i.e.*, it is a continuous function for every values of* $\tau$.

Since, we already proved in *theorem* 1; i.e., *continuity theorem of the first kind* that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ given by (38) and (39) [3] for *or every values of* $\tau$*,* $p = \{2,4\}$ and for $\alpha = [0,1]$ we can employ the results of *theorem* 1 *to prove theorem* 3.

First, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\left\{\tau_2\left(\alpha = \frac{1}{3}\right), \tau_3\left(\alpha = \frac{1}{3}\right)\right\}$.

From (5) and (6), and $p = \{2,4\}$ we have

$$R_{VBOC2_{(pn,n,\alpha=0)}}\left[\tau_2\left(\frac{1}{3}\right)\right] = \underbrace{\frac{1-2p+(4p+1)\frac{1}{3}}{2p} = -\frac{1}{3} + \frac{2}{3p}}_{=\left\{0,\frac{-1}{3}\right\}} \quad (81)$$

On the other hand, from (11) and (12), and $p = \{2,4\}$ we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_3\left(\frac{1}{3}\right)\right] = \frac{1-(2p-1)\frac{1}{3}}{2p} = \frac{2}{3p} - \frac{1}{3} = \left\{0,\frac{-1}{3}\right\} \quad (82)$$

Again, from (10) and (11), and for $p = \{2,4\}$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_3\left(\frac{1}{3}\right)\right] = \frac{p-1+(5-4p)\frac{1}{3}}{p} = \frac{2}{3p} - \frac{1}{3} = \left\{0,\frac{-1}{3}\right\} \quad (83)$$

Finally, from (7) and (8), and $p = \{2,4\}$ we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_2\left(\frac{1}{3}\right)\right] = \frac{1-p+(2p-1)\frac{1}{3}}{p} = \frac{2}{3p} - \frac{1}{3} = \left\{0,\frac{-1}{3}\right\} \quad (84)$$

Second, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [[3]] obeys the *continuity theorem of the third kind* for $\left\{\tau_5\left(\alpha = \frac{1}{3}\right), \tau_6\left(\alpha = \frac{1}{3}\right)\right\}$.

From (15) and (16), and for $p = \{2,4\}$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_5\left(\frac{1}{3}\right)\right] = \frac{p-1+(3-4p)\frac{1}{3}}{p} = \frac{-p}{3p} = \left\{\frac{-1}{3},\frac{-1}{3}\right\} \quad (85)$$

On the other hand, from (21) and (22), and for $p = \{2,4\}$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_6\left(\frac{1}{3}\right)\right] = \frac{3-2p+(4p-9)\frac{1}{3}}{2p} = \frac{-p}{3p} = \left\{\frac{-1}{3},\frac{-1}{3}\right\} \quad (86)$$

Again, from (17) and (18), and for $p = \{2,4\}$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_5\left(\frac{1}{3}\right)\right] = \frac{-1+(3-2p)\frac{1}{3}}{2p} = \frac{-p}{3p} = \left\{\frac{-1}{3},\frac{-1}{3}\right\} \quad (87)$$

On the other hand, from (21), and (22), and for $p = \{2,4\}$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_6\left(\frac{1}{3}\right)\right] = \frac{3-2p+(4p-9)\frac{1}{3}}{2p} = \frac{-p}{3p} = \left\{\frac{-1}{3},\frac{-1}{3}\right\} \quad (88)$$

Third, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\left\{\tau_{10}\left(\alpha = \frac{1}{3}\right), \tau_{11}\left(\alpha = \frac{1}{3}\right)\right\}$.

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_{10}\left(\frac{1}{3}\right)\right]=\frac{5-2p+(6p-15)\frac{1}{3}}{2p}=0=\{0\} \quad (89)$$

From and (29) and (30), for $p=4$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_{11}\left(\frac{1}{3}\right)\right]=\frac{p-3+(9-3p)\frac{1}{3}}{p}=0=\{0\} \quad (90)$$

Again, from (33) and (34), for $p=4$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_{11}\left(\frac{1}{3}\right)\right]=\frac{p-3+(9-3p)\frac{1}{3}}{p}=0=\{0\} \quad (91)$$

On the other hand, from (35) and (36) for $p=4$ we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_{10}\left(\frac{1}{3}\right)\right]=\frac{2-p+(3p-6)\frac{1}{3}}{2p}=0=\{0\} \quad (92)$$

Fourth, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\left\{\tau_{12}\left(\alpha=\frac{1}{3}\right)\right\}$.

From and (37) and (38), for $p=4$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_{12}\left(\frac{1}{3}\right)\right]=\frac{p-3+(6-2p)\frac{1}{3}}{p}=\frac{p-3}{3p}=\left\{\frac{1}{12}\right\} \quad (93)$$

Fifth, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\left\{\tau_{13}\left(\alpha=\frac{1}{3}\right)\right\}$.

From (39) and (40), for $p=4$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_{13}\left(\frac{1}{3}\right)\right]=\frac{(p-3)-(4p-11)\frac{1}{3}}{p}=\frac{2-p}{3p}=\left\{\frac{-1}{6}\right\} \quad (94)$$

On the other hand, from (41) and (42), for $p=4$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_{13}\left(\frac{1}{3}\right)\right]=\frac{-1-(2p-7)\frac{1}{3}}{2p}=\frac{2-p}{3p}=\left\{\frac{-1}{6}\right\} \quad (95)$$

Sixth, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\left\{\tau_{14}\left(\alpha=\frac{1}{3}\right)\right\}$.

From (43) and (44) for $p=4$ we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_{14}\left(\frac{1}{3}\right)\right]=\frac{7-2p+(4p-17)\frac{1}{3}}{2p}=\frac{2}{3p}-\frac{1}{3}=\left\{\frac{-1}{6}\right\} \quad (96)$$

One the other hand, from (45) and (46), for $p=4$, we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau_{14}\left(\frac{1}{3}\right)\right]=\frac{3-p+(2p-7)\frac{1}{3}}{p}=\frac{-1}{3}+\frac{2}{3p}=\left\{\frac{-1}{6}\right\} \quad (97)$$

Seventh, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [[3]] obeys the *continuity theorem of the third kind* for $\left\{\tau_{15}\left(\alpha=\frac{1}{3}\right)\right\}$.

From (47) and (48) for $p=4$ we have

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau=\tau_{-15}\left(\frac{1}{3}\right)\right]=\underbrace{\frac{7-2p+(4p-15)\frac{1}{3}}{2p}=\frac{1}{p}-\frac{1}{3}}_{=\left\{\frac{-1}{12}\right\}} \quad (98)$$

Eight and finally, we prove that $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, given by (38) and (39) [3] obeys the *continuity theorem of the third kind* for $\left\{\tau_{16}\left(\alpha=\frac{1}{3}\right)\right\}$.

Finally, from (49) for $p=4$

$$R_{VBOC2_{\left(pn,n,\alpha=\frac{1}{3}\right)}}\left[\tau=\tau_{-16}\left(\frac{1}{3}\right)\right]=\frac{p-4}{p}=1-\frac{4}{p}=\{0\} \quad (99)$$

Equations (81) through (99) complete the proof of *theorem* 3 or *continuity theorem of the second kind of the ACF of* $VBOC2_{(pn,n,\alpha)}$.

*Corollary* 1: Based on *theorems* 2 [1], 2 and 3 we conclude that $R_{VBOC1_{(pn,n,\alpha)}}(0)=1=\max$ regardless of $p$ and/or $\alpha$; i.e., $R_{VBOC1_{(pn,n,\alpha)}}[\tau_i(\alpha)]\le 1$ regardless of $p$ and/or $\alpha$.

### *2.2 $VBOC1(\alpha)$ ACF Optimization Theorems*

*Theorem* 4*: Prove that the ACF $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$, given by* (38) and (39) [3], obeys the *optimization theorem of the first kind;* i.e., *find the values of* $\alpha$ that minimizes the out of phase ACF,

$$\underbrace{\left|\begin{matrix} R_{VBOC2_{(pn,n,\alpha)}}[\tau^{1}(\alpha)] \\ \vdots \\ R_{VBOC2_{(pn,n,\alpha)}}[\tau^{4p-2}(\alpha)] \end{matrix}\right.}_{\min\limits_{\alpha}} \rightarrow \underbrace{\sum_{i=1}^{2p-1} R_{VBOC2_{(pn,n,\alpha)}}\left[\tau^{i}(\alpha)\right]}_{\min\limits_{\alpha}} \quad (100)$$

$$\underbrace{\tau^{i}(\alpha)}_{i=\{1,2,\cdots 2p\}}=\underbrace{\underbrace{\begin{bmatrix} \underbrace{\frac{T_c(1-\alpha)}{2p}}_{\tau^1(\alpha)} & \underbrace{\frac{T_c(1+\alpha)}{p}}_{\tau^5(\alpha)} & \underbrace{\frac{T_c(5-\alpha)}{2p}}_{\tau^8(\alpha)} & \underbrace{\frac{T_c(3+\alpha)}{p}}_{\tau^{12}(\alpha)} \\ \underbrace{\frac{T_c(1+\alpha)}{2p}}_{\tau^2(\alpha)} & \underbrace{\frac{T_c(3-\alpha)}{2p}}_{\tau^6(\alpha)} & \underbrace{\frac{T_c(5+\alpha)}{2p}}_{\tau^9(\alpha)} & \underbrace{\frac{T_c(7-\alpha)}{2p}}_{\tau^{13}(\alpha)} \\ \underbrace{\frac{T_c(1-\alpha)}{p}}_{\tau^3(\alpha)} & \underbrace{\frac{T_c(3+\alpha)}{2p}}_{\tau^7(\alpha)} & \underbrace{\frac{T_c(3-\alpha)}{p}}_{\tau^{10}(\alpha)} & \underbrace{\frac{T_c(7+\alpha)}{2p}}_{\tau^{14}(\alpha)} \\ \underbrace{\frac{T_c}{p}}_{\tau^4(\alpha)} & & \underbrace{\frac{3T_c}{p}}_{\tau^{11}(\alpha)} & \end{bmatrix}}_{p=2}}_{p=4} \quad (101)$$

$$\underbrace{\tau^{i}(\alpha)}_{i=\{1,2,\cdots 2p\}}=\underbrace{\underbrace{\begin{bmatrix} \underbrace{\frac{T_c(1-\alpha)}{2p}}_{\tau^1(\alpha)} & \underbrace{\frac{T_c(3-\alpha)}{2p}}_{\tau^5(\alpha)} & \underbrace{\frac{T_c(5-\alpha)}{2p}}_{\tau^8(\alpha)} & \underbrace{\frac{T_c(7-\alpha)}{2p}}_{\tau^{12}(\alpha)} \\ \underbrace{\frac{T_c(1-\alpha)}{p}}_{\tau^2(\alpha)} & \underbrace{\frac{T_c(1+\alpha)}{p}}_{\tau^6(\alpha)} & \underbrace{\frac{T_c(3-\alpha)}{p}}_{\tau^9(\alpha)} & \underbrace{\frac{T_c(3+\alpha)}{p}}_{\tau^{13}(\alpha)} \\ \underbrace{\frac{T_c(1+\alpha)}{2p}}_{\tau^3(\alpha)} & \underbrace{\frac{T_c(3+\alpha)}{2p}}_{\tau^7(\alpha)} & \underbrace{\frac{T_c(5+\alpha)}{2p}}_{\tau^{10}(\alpha)} & \underbrace{\frac{T_c(7+\alpha)}{2p}}_{\tau^{14}(\alpha)} \\ \underbrace{\frac{T_c}{p}}_{\tau^4(\alpha)} & & \underbrace{\frac{3T_c}{p}}_{\tau^{11}(\alpha)} & \end{bmatrix}}_{p=2}}_{p=4} \quad (102)$$

*Proof of theorem* 4: The *proof of theorem* 4 is also straightforward. Because the min-max of the ACF of $VBOC2_{(pn,n,\alpha)}(t)$, or $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, that depends $\alpha$ on for values of $p = \{2,4\}$ and $\underbrace{\tau^i(\alpha)}_{i=\{1,2,\cdots 2p\}}$ given by (101) and (102); hence, it suffices to show that:

For $p = \{2,4\}$ find the value of $\alpha$ that minimizes $\tau^1(\alpha)$, $\cdots$, $\tau^{13}(\alpha)$

$$\underbrace{\left|\begin{array}{l} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^{14}(\alpha)] \end{array}\right.}_{\min\limits_{\alpha}} = \underbrace{\frac{\begin{bmatrix} 1-2p+(4p-1)\alpha \\ 1-2p+(4p+1)\alpha \\ 2p-2+(10-8p)\alpha \\ 2p-2+(4-4p)\alpha \\ 2p-2+(6-8p)\alpha \\ 3-2p+(4p-9)\alpha \\ 3-2p+(4p-7)\alpha \\ 5-2p+(4p-9)\alpha \\ 5-2p+(6p-15)\alpha \\ 2p-6+(18-6p)\alpha \\ 2p-6+(12-4p)\alpha \\ 2p-6+(22-8p)\alpha \\ 7-2p+(4p-17)\alpha \\ 7-2p+(4p-15)\alpha \end{bmatrix} \cup \begin{bmatrix} 1-2p+(4p-1)\alpha \\ 2-2p+(4p-2)\alpha \\ 1+(1-2p)\alpha \\ 2p-2+(4-4p)\alpha \\ -1+(3-2p)\alpha \\ 2-2p+(4p-6)\alpha \\ 3-2p+(4p-7)\alpha \\ 5-2p+(4p-9)\alpha \\ 2-p+(3p-6)\alpha \\ p-3+(9-3p)\alpha \\ 2p-6+(12-4p)\alpha \\ -1+(7-2p)\alpha \\ 6-2p+(4p-14)\alpha \\ 7-2p+(4p-15)\alpha \end{bmatrix}}{2p}}_{\min\limits_{\alpha}} \tag{103}$$

*Optimization theorem of the first kind* says that proving (103) is equivalent with proving the following

$$\min_{\alpha} \underbrace{\left\{\sum_{i=1}^{4p-2=14} R_{VBOC2_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right]\right\}}_{\frac{8-4p+(0-4p)\alpha}{2p} = \underset{0\leq\alpha\leq 1/3}{\alpha=\frac{8-4p}{4p}}} \tag{104}$$

$$\min_{\alpha} \underbrace{\left\{\sum_{i=1}^{4p-2=14} R_{VBOC2_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right]\right\}}_{\frac{15-8p-(21-8p)\alpha}{2p} = \underset{1/3\leq\alpha\leq 1}{\alpha=\frac{15-8p}{21-8p}}} \tag{105}$$

The first solution is $0 \leq \alpha \leq 1/3$

$$2 - p - p\alpha = 0 \text{ or } \alpha = \frac{2-\mathrm{p}}{\mathrm{p}} = \left\{\frac{-1}{2} = -0.5\right\} \tag{106}$$

And the second solution is

$$15 - 8p - (21 - 8p)\alpha = 0 \text{ or } \alpha = \frac{8p-15}{8p-21} = \left\{\frac{17}{9} > 1\right\} \tag{107}$$

Both the first solution for $\alpha = -0.5$ and the second solution for $\alpha = \frac{17}{9} > 1$ are unacceptable. In order to fix the optimization problem we consider the following:

$$\underbrace{\left|\begin{array}{l} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^2(\alpha)] \\ -R_{VBOC1_{(pn,n,\alpha)}}[\tau^3(\alpha)] \\ -R_{VBOC1_{(pn,n,\alpha)}}[\tau^4(\alpha)] \\ -R_{VBOC1_{(pn,n,\alpha)}}[\tau^5(\alpha)] \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^6(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^9(\alpha)] \\ -R_{VBOC1_{(pn,n,\alpha)}}[\tau^{10}(\alpha)] \\ -R_{VBOC1_{(pn,n,\alpha)}}[\tau^{11}(\alpha)] \\ -R_{VBOC1_{(pn,n,\alpha)}}[\tau^{12}(\alpha)] \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^{13}(\alpha)] \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^{14}(\alpha)] \end{array}\right. = \frac{\begin{array}{c} 1-2p+(4p-1)\alpha \\ 1-2p+(4p+1)\alpha \\ 2-2p+(8p-10)\alpha \\ 2-2p+(4p-4)\alpha \\ 2-2p+(8p-6)\alpha \\ 3-2p+(4p-9)\alpha \\ 3-2p+(4p-7)\alpha \\ 5-2p+(4p-9)\alpha \\ 5-2p+(6p-15)\alpha \\ 6-2p+(6p-18)\alpha \\ 6-2p+(4p-12)\alpha \\ 6-2p+(8p-22)\alpha \\ 7-2p+(4p-17)\alpha \\ 7-2p+(4p-15)\alpha \end{array} \cup \begin{array}{c} 1-2p+(4p-1)\alpha \\ 2-2p+(4p-2)\alpha \\ -1+(2p-1)\alpha \\ 2-2p+(4p-4)\alpha \\ 1+(2p-3)\alpha \\ 2-2p+(4p-6)\alpha \\ 3-2p+(4p-7)\alpha \\ 5-2p+(4p-9)\alpha \\ 2-p+(3p-6)\alpha \\ 3-p+(3p-9)\alpha \\ 6-2p+(4p-12)\alpha \\ 1+(2p-7)\alpha \\ 6-2p+(4p-14)\alpha \\ 7-2p+(4p-15)\alpha \end{array}}{2p}}_{\min\limits_{\alpha}} \tag{108}$$

*Optimization theorem of the first kind* says that proving (108) is equivalent with proving the following

$$\min_{\alpha} \underbrace{\left|\begin{array}{l} \sum_{i=1,\neq 3,4,5,10,11,12}^{4p-2=14} R_{VBOC2_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right] \\ \quad -\sum_{i=3,\neq 6,7,7,9}^{12} R_{VBOC2_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right] \end{array}\right.}_{\frac{56-28p-(144-72p)\alpha}{2p} = \underset{0\leq\alpha\leq 1/3}{\alpha=\frac{28}{72}\times\frac{2-p}{2-p}=\frac{7}{18}=0.3888889}} \tag{109}$$

$$\min_{\alpha} \underbrace{\left|\begin{array}{l} \sum_{i=1,\neq 3,4,5,10,11,12}^{4p-2=14} R_{VBOC2_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right] \\ \quad -\sum_{i=3,\neq 6,7,7,9}^{12} R_{VBOC2_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right] \end{array}\right.}_{\frac{40-20p-(96-48p)\alpha}{2p} = \underset{1/3\leq\alpha\leq 1}{\alpha=\frac{20p-40}{48p-96}}} \tag{110}$$

The first solution in $0 \leq \alpha \leq 1/3$ is

$$56 - 28p - (144 - 72p)\alpha = 0 \text{ or } \alpha = \frac{28}{72} \times \frac{2-\mathrm{p}}{2-\mathrm{p}} = \frac{7}{18} \tag{111}$$

And the second solution in $1/3 \leq \alpha \leq 1$ is

$$40 - 20p - (96 - 48p)\alpha = 0 \text{ or } \alpha = \underbrace{\frac{20\mathrm{p}-40}{48\mathrm{p}-96} = \frac{20(\mathrm{p}-2)}{48(\mathrm{p}-2)}}_{=\frac{5}{12}=0.4166666667} \tag{112}$$

The first solution in $0 \leq \alpha \leq 1/3$ is $\alpha = \frac{28}{72} \times \frac{2-p}{2-p} = \frac{7}{18} = 0.3888889$ and it is independent of $p$. However, since this is outside of the interval then the optimum solution for $\alpha = 1/3$ and it is independent of $p$.

The second solution for in $1/3 \leq \alpha \leq 1$ is independent of $p$ and it is equal to $\alpha = \frac{5}{12} = 0.416666667$; hence, the optimum $\alpha$ is in fact independent of $p$ and it is given by

$$\alpha_{\mathrm{opt}} = \frac{5}{12} = 0.416666667 \tag{113}$$

Hence, the *optimization theorem of the first kind* proves that $\alpha = 0.416666667$, independent of $p$, minimizes the out-of-phase autocorrelation peaks of the ACF of $VBOC2_{(pn,n,\alpha)}(t)$, or $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$, for all values of $p$ and this completes the proof of *theorem* 4.

*Theorem 5: Prove that the ACF* $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$, *given by* (38) and (39) [3], obeys the *optimization theorem of the fourth kind*[1] *(or in the minimum mean square error (MMSE) sense* [28]*);* i.e., *find the values of* $\alpha$ that minimizes the out-of-phase ACF,

$$\min_{\alpha}\left|\begin{matrix} R_{VBOC2_{(pn,n,\alpha)}}[\tau^{1}(\alpha)] \\ \vdots \\ R_{VBOC2_{(pn,n,\alpha)}}[\tau^{2p-2}(\alpha)] \end{matrix}\right| \rightarrow 0 \tag{114}$$

which is equivalent with

$$\alpha_{min} = \frac{\sum_{i=1}^{2p-2} a_i(p) b_i(p)}{\sum_{i=1}^{2p-2}[a_i(p)]^2} = \frac{\boldsymbol{a}(p)^{\boldsymbol{T}}\boldsymbol{b}(p)}{\boldsymbol{a}(p)^{\boldsymbol{T}}\boldsymbol{a}(p)} = \frac{\boldsymbol{a}(p)^{\boldsymbol{T}}\boldsymbol{b}(p)}{\|\boldsymbol{a}(p)\|^2} \tag{115}$$

$\tau^i(\alpha)$ for $i = \{1,2,\cdots 2p-2\}$ given by (101) and (102) in *Theorem 4* and $p = \{2,4\}$ and $a_i(p)$ and $b_i(p)$ are given by

$$\min_{\alpha}\begin{bmatrix} R_{VBOC2_{(pn,n,\alpha)}}[\tau^{1}(\alpha)] \\ \vdots \\ R_{VBOC2_{(pn,n,\alpha)}}[\tau^{14}(\alpha)] \end{bmatrix} = \underbrace{\frac{\begin{bmatrix} 1-2p \\ 1-2p \\ 2p-2 \\ 2p-2 \\ 2p-2 \\ 3-2p \\ \underbrace{3-2p}_{p=\{2,4\}} \\ 5-2p \\ 5-2p \\ 2p-6 \\ 2p-6 \\ 2p-6 \\ 7-2p \\ \underbrace{7-2p}_{p=\{4\}} \end{bmatrix}}{2p}}_{\boldsymbol{b}(p)} - \underbrace{\frac{\begin{bmatrix} 1-4p \\ -1-4p \\ 8p-10 \\ 4p-4 \\ 8p-6 \\ 9-4p \\ \underbrace{7-4p}_{p=\{2,4\}} \\ 9-4p \\ 15-6p \\ 6p-18 \\ 4p-12 \\ 8p-22 \\ 17-4p \\ \underbrace{15-4p}_{p=\{4\}} \end{bmatrix}}{2p}}_{\boldsymbol{a}(p)}\alpha \tag{116}$$

$$\min_{\alpha}\begin{bmatrix} R_{VBOC2_{(pn,n,\alpha)}}[\tau^{1}(\alpha)] \\ \vdots \\ R_{VBOC2_{(pn,n,\alpha)}}[\tau^{14}(\alpha)] \end{bmatrix} = \underbrace{\frac{\begin{bmatrix} 1-2p \\ 2-2p \\ 1 \\ 2p-2 \\ -1 \\ 2-2p \\ \underbrace{3-2p}_{p=\{2,4\}} \\ 5-2p \\ 2-p \\ p-3 \\ 2p-6 \\ -1 \\ 6-2p \\ \underbrace{7-2p}_{p=\{4\}} \end{bmatrix}}{2p}}_{\boldsymbol{b}(p)} - \underbrace{\frac{\begin{bmatrix} 1-4p \\ 2-4p \\ 2p-1 \\ 4p-4 \\ 2p-3 \\ 6-4p \\ \underbrace{7-4p}_{p=\{2,4\}} \\ 9-4p \\ 6-3p \\ 3p-9 \\ 4p-12 \\ 2p-7 \\ 14-4p \\ \underbrace{15-4p}_{p=\{4\}} \end{bmatrix}}{2p}}_{\boldsymbol{a}(p)}\alpha \tag{117}$$

*Proof of theorem* 5 : The *proof of theorem* 5 is also straight forward.

For $p = 4$ from the *optimization theorem of the second kind (or in the MMSE sense);* i.e., *find the values of* $\alpha$ that minimizes the out-of-phase ACF, based on (116),

$$\alpha_{\min} = \frac{\boldsymbol{a}(p)^{\boldsymbol{T}}\boldsymbol{b}(p)}{\|\boldsymbol{a}(p)\|^2} = \frac{744-574p+144p^2}{2056-1588p+408p^2} = \frac{372-287p+72p^2}{1028-794p+204p^2} \tag{118}$$

or

$$\alpha_{\min} = \frac{\boldsymbol{a}(p)^{\boldsymbol{T}}\boldsymbol{b}(p)}{\|\boldsymbol{a}(p)\|^2} = \frac{376}{1116} \cong 0.3369 \quad \text{for } 0 \leq \alpha \leq 1/3 \tag{119}$$

and based on (117) we have

$$\alpha_{\min} = \frac{\boldsymbol{a}(p)^{\boldsymbol{T}}\boldsymbol{b}(p)}{\|\boldsymbol{a}(p)\|^2} = \frac{400-308p+78p^2}{928-694p+174p^2} \tag{120}$$

or

$$\alpha_{\min} = \frac{\boldsymbol{a}(p)^{\boldsymbol{T}}\boldsymbol{b}(p)}{\|\boldsymbol{a}(p)\|^2} = \frac{416}{936} = 0.4444 \quad \text{for } 1/3 \leq \alpha \leq 1 \tag{121}$$

In order to find that the optimum $\alpha$ we consider only the $p^2$ coefficients from (118) and (120) as given by

$$\alpha_{\text{opt}} = \frac{72}{204} = 0.352941176470588; \text{ for } 0 \leq \alpha \leq 1/3 \tag{122}$$

or

$$\alpha_{\text{opt}} = \frac{78}{174} = 0.448275862068966 \text{ ; for } 1/3 \leq \alpha \leq 1 \tag{123}$$

Since, $\alpha_{\text{opt}} = 0.352941176470588 > 1/3$ outside the range; hence, $\alpha_{\text{opt}} = 0.448275862068966$ is the optimum solution in terms of mean square sense. This completes the proof of *theorem 5*.

*Corollary* 2: $\alpha = 0.448275862068966$ is the asymptotic value for which the MMSE optimum $\alpha$s from the *optimization theorem of the second kind* of the ACF of the generalized $VBOC2_{(pn,n,\alpha)}(t)$ approach the asymptotic optimum $\alpha$ as indicated in (123).

*Corollary* 3: or *performance enhancement or efficiency* both in terms of optimization theorems of the *first kind* and *fourth kind*.

*Performance enhancement or efficiency* based on the *optimization theorem of the first kind* or $\Upsilon_{2(p,\alpha)}$ is defined as the ratio or $\Upsilon_{2(p,\alpha)}$ the sum of all out-of-phase autocorrelation peaks that depend on $\alpha$ by their number as follows:

$$\Upsilon_{2(p,\alpha)} = \frac{\left|\underbrace{\sum_{i=1}^{14} R_{VBOC1_{(pn,n,\alpha)}}[\tau^{i}(\alpha)]}_{0\leq\alpha\leq1/3}\right|}{14\times 2p} = \frac{|56-28p-(144-72p)\alpha|}{14\times 2p} \tag{124}$$

and

$$\Upsilon_{2(p,\alpha)} = \frac{\left|\underbrace{\sum_{i=1}^{14} R_{VBOC1_{(pn,n,\alpha)}}[\tau^{i}(\alpha)]}_{0\leq\alpha\leq1/3}\right|}{14\times 2p} = \frac{|56-28p-(144-72p)\alpha|}{14\times 2p} \tag{125}$$

or

$$\Upsilon_{2(p,0\leq\alpha\leq1/3)} = \frac{|56-28p-(144-72p)\alpha|}{14\times 2p} = \frac{\left|2-p-\left(\frac{36}{7}-\frac{18}{7}p\right)\alpha\right|}{p} \tag{126}$$

and

$$\Upsilon_{2(p,1/3\leq\alpha\leq1)} = \frac{|40-20p-(96-48p)\alpha|}{14\times 2p} = \frac{\left|\frac{10}{7}-\frac{5}{7}p-\left(\frac{24}{7}-\frac{12}{7}p\right)\alpha\right|}{p} \tag{127}$$

[1] *Optimization theorems of the second and third kind* are not considered for $VBOC2_{(pn,n,\alpha)}(t)$ only for $VBOC1_{(pn,n,\alpha)}(t)$ [2].

As we take the limit as $p \to \infty$ we find efficiency only as function of $\alpha$ as follows:

$$\Upsilon_{2(0\le\alpha\le1/3)} = \lim_{p\to\infty} \Upsilon_{2(p,0\le\alpha\le1/3)} = \left|\frac{18}{7}\alpha - 1\right| \tag{128}$$

and

$$\Upsilon_{2(1/3\le\alpha\le1)} = \lim_{p\to\infty} \Upsilon_{2(p,1/3\le\alpha\le1)} = \left|\frac{12}{7}\alpha - \frac{5}{7}\right| \tag{129}$$

or $\frac{12}{7}\alpha - \frac{5}{7} = 0$ or $\alpha = \frac{5}{12} = 0.416666666666667$ gives efficiency equal to zero (or infinite, or 100 percent) improvements of the *performance enhancement*.

*Performance enhancement or efficiency* based on the *optimization theorem of the second kind* is defined as the ratio of the sum of all out-of-phase autocorrelation peaks squared by their number and denoted by $\Upsilon_{2\mathrm{MS}(p,\alpha)}$ as follows:

$$\Upsilon_{2MS(p,0\le\alpha\le1/3)} = \frac{\underbrace{\sum_{i=1}^{14}\left(R_{VBOC1_{(pn,n,\alpha)}}[\tau_i(\alpha)]\right)^2}_{0\le\alpha\le1/3}}{14} \tag{130}$$

and

$$\Upsilon_{2MS(p,1/3\le\alpha\le1)} = \frac{\underbrace{\sum_{i=1}^{14}\left(R_{VBOC1_{(pn,n,\alpha)}}[\tau_i(\alpha)]\right)^2}_{1/3\le\alpha\le1}}{14} \tag{131}$$

or

$$\Upsilon_{2MS(p,0\le\alpha\le1/3)} = \frac{\underbrace{[\boldsymbol{b}(p)-\boldsymbol{a}(p)\alpha]^T[\boldsymbol{b}(p)-\boldsymbol{a}(p)\alpha]}_{0\le\alpha\le1/3}}{14} \tag{132}$$

and

$$\Upsilon_{2MS(p,1/3\le\alpha\le1)} = \frac{\underbrace{[\boldsymbol{b}(p)-\boldsymbol{a}(p)\alpha]^T[\boldsymbol{b}(p)-\boldsymbol{a}(p)\alpha]}_{1/3\le\alpha\le1}}{16} \tag{133}$$

or

$$\Upsilon_{2MS(p,\alpha)} = \frac{\underbrace{\boldsymbol{b}(p)^T\boldsymbol{b}(p)-[\boldsymbol{b}(p)^T\boldsymbol{a}(p)+\boldsymbol{a}(p)^T\boldsymbol{b}(p)]\alpha+\boldsymbol{a}(p)^T\boldsymbol{a}(p)\alpha^2}_{0\le\alpha\le1/3}}{14} \tag{134}$$

and

$$\Upsilon_{2MS(p,\alpha)} = \frac{\underbrace{\boldsymbol{b}(p)^T\boldsymbol{b}(p)-[\boldsymbol{b}(p)^T\boldsymbol{a}(p)+\boldsymbol{a}(p)^T\boldsymbol{b}(p)]\alpha+\boldsymbol{a}(p)^T\boldsymbol{a}(p)\alpha^2}_{1/3\le\alpha\le1}}{14} \tag{135}$$

or

$$\Upsilon_{2,1MS(p,0\le\alpha\le1/3)} = \frac{\begin{matrix}(288-224p+56p^2)\\ -(1288-1016p+288p^2)\alpha\\ +(1968-1460p+364p^2)\alpha^2\end{matrix}}{56p^2} \tag{136}$$

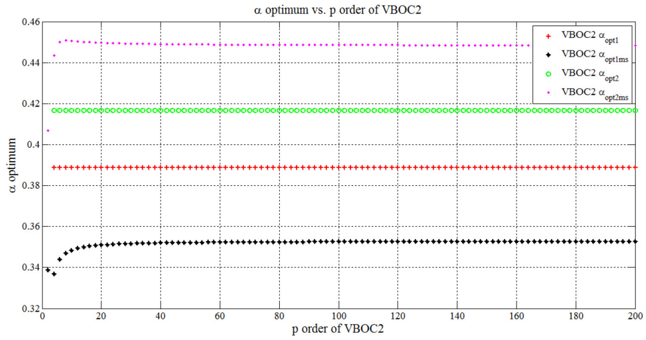
and

$$\Upsilon_{2MS(p,1/3\le\alpha\le1)} = \frac{\begin{matrix}(184-146p+38p^2)\\ -(703-558p+156p^2)\alpha\\ +(928-694p+174p^2)\alpha^2\end{matrix}}{56p^2} \tag{137}$$

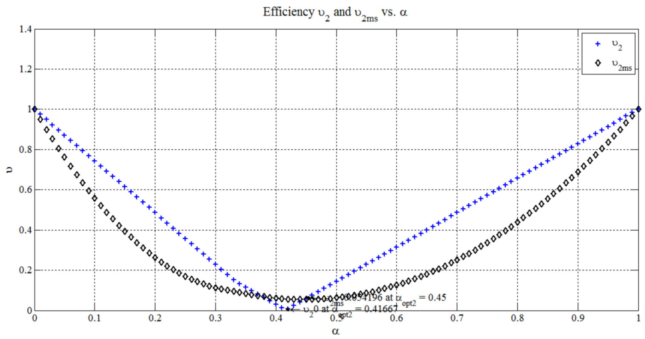
As we take the limit as $p \to \infty$ we find efficiency only as function of $\alpha$ as follows:

$$\Upsilon_{2MS(0\le\alpha\le1/3)} = \lim_{p\to\infty} \Upsilon_{2MS(p,\alpha)} = 1 - \frac{36}{7}\alpha + \frac{93}{14}\alpha^2 \tag{138}$$

and




(a) $\alpha$ optimum vs. order $p$ of $VBOC2_{(pn,n,\alpha)}(t)$




(b) Efficiency $\Upsilon_{1(\alpha)}$ (+) and $\Upsilon_{1\mathrm{MS}(\alpha)}$ (◊) vs $\alpha$

FIGURE 1: Illustration of $VBOC2_{(pn,n,\alpha)}(t)$ ACF optimization theorems.

$$\Upsilon_{2MS(1/3\le\alpha\le1)} = \lim_{p\to\infty} \Upsilon_{2MS(p,\alpha)} = \frac{19}{28} - \frac{78}{28}\alpha + \frac{87}{28}\alpha^2 \tag{139}$$

This concluded the discussion on the $VBOC2(\alpha, 1-\alpha)$ ACF optimization theorems.

## 3 Numerical, Theoretical Results

Now that we have produced a closed form expression of the efficiency $\Upsilon_{2\mathrm{MS}(p,\alpha)}$ and $\Upsilon_{2\mathrm{MS}(\alpha)}$ as a function of the signal design and optimization parameter $\alpha$ and the generalized parameter $p$ (or subcarrier frequency) let us consider two examples which illustrate that: (1) $\Upsilon_{2\mathrm{MS}(p,\alpha)}$ is in fact independent of $p$; and (2) $\Upsilon_{2\mathrm{MS}(\alpha)}$ has a minimum for $\alpha = 0.4$.

Figure 1(a) presents the $\alpha$ optimum vs. order $p$ of $VBOC2_{(pn,n,\alpha)}(t)$. There are four curves shown in Figure 1(a) correspond to $VBOC2_{(pn,n,\alpha)}(t)$ optimization theorems 4 and 5 however for different range of $\alpha$s: (1)-(2) first and second for $0 \le \alpha \le 1/3$; and (3)-(4) third and fourth $1/3 \le \alpha \le 1$. The fifth and sixth curves which corresponds to $VBOC2_{(pn,n,\alpha)}(t)$ shows that as $p$ changes from two to one hundred $\alpha_{\mathrm{opt}}$ is either equal to or converges to 0.416666666666667 or

$0.448275862068966$ to regardless of generalized optimization parameter $p$ (or of sub-carrier frequency) and regardless of the optimization theorem(s).

The argument can be made that the choice we make for $\alpha$ will equally affect the standardization of the waveforms regardless of sub-carrier frequency; i.e., even though the choice on $BOC_{(2,1)}(t) = BOC_{(10,5)}(t) = VBOC2_{(2,1,\alpha=0)}(t)$ or the GPS military M-code [11], [18] on both GPS L1 and L2 frequencies is arbitrary it only happened by accident that they have the same efficiency as $BOC_{(1,1)}(t) = VBOC2_{(1,1,\alpha=0)}(t)$ [1], [2]. Next, we consider how efficient are these choices?

Figure 1(b) depicts the efficiency $\Upsilon_{2\mathrm{MS}(\alpha)}$ corresponding to $VBOC2_{(pn,n,\alpha)}(t)$ respectively vs. signal design and optimization parameter $\alpha$. As shown in Figure 1(b) efficiency for $\Upsilon_{2\mathrm{MS}(\alpha)}$ does have a minimum exactly at $\alpha = 13/29$ equal to $\Upsilon_{2\mathrm{MS}(\alpha=0.4)} = 0.054187192118227$ in corollary 3.

For $\alpha = \{0,1\}$ we have $\Upsilon_{2\mathrm{MS}(\alpha=\{0,1\})} = 1$ as opposed to for $\alpha = 13/29$ we have $\Upsilon_{2\mathrm{MS}(\alpha=13/29)} = 0.054187192118227$. Hence, $\alpha = 13/29$ offers 94 percent efficiency or for $\alpha = 13/29\, VBOC2_{(pn,n,\alpha)}$ is 94 percent more efficient than either $BOC_{(pn,n)}/BOC_{(pn/2,n)}(t)$ ($BPSK_0$ being a special case of $BOC_{(pn/2,n)}(t)$) in terms of mean square sense.

## 4 Conclusions

This journal paper is the first complete (i.e., in its entirety) discussion on ACF pure signal optimization for $VBOC2(\alpha, 1-\alpha)$.

ACF pure signal optimization as a function of a signal design and optimization parameter $\alpha$ is very laborious and creative work; however, the results are fantastic: it is shown that $VBOC2_{(pn,n,\alpha)}$ for $\alpha = 13/29$ offers 94 percent efficiency or for $\alpha = 0.44$ $VBOC2_{(pn,n,\alpha)}$ is 94 percent more efficient than either $BOC_{(pn,n)}/BOC_{(pn/2,n)}(t)$ ($BPSK_0$ being a special case of $BOC_{(pn/2,n)}(t)$) in terms of MMSE sense and $p \geq 2$; i.e., regardless of the subcarrier frequency.

*Future work*: Is it possible that from optimization of the system (transmitter, channel, and receiver) one can obtain a different optimum $\alpha$ than $\alpha = 0.44$ as a result of the optimization of the received out-of-phase ACF? The results of this laborious investigation will be presented in the future.

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