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Research Article

VBOC2(α,1−α) Generalized Multidimensional Geolocation Modulation Waveforms

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This paper presents the complete original definition of first generation Variable Binary Offset Carrier $VBOC2(\alpha,1-\alpha)$ generalized multidimensional geolocation modulation waveforms, to improve the standardization of the United States DoD GPS, European Galileo, Russian GLONASS, Chinese Compass, Indian IRNSS in the L-band (1-2 GHz), and the United Nations International Telecommunications Union (ITU) GNSS or geolocation waveforms in the S-band (2-4 GHz) and C-band (4-8 GHz).

In the paper it is argued that the selection of BOC(10,5) (or the military code or M-Code) on both GPS L1 and L2 frequencies is entirely arbitrary because BOC modulation is a special case of $VBOC2(\alpha,1-\alpha)$ for $\alpha=0$ or $\alpha=1$; hence, all the current state-of-the-art GNSS waveforms exhibit sub-optimal signal design performance even at the end-user when generalized global objective functions are applied.

$VBOC2(\alpha,1-\alpha)$ pure signal design or broad definition of generalized autocorrelation function (ACF) and power spectral density (PSD) offers a unique signal design methodology and provides the necessary framework for $VBOC2(\alpha,1-\alpha)$ ACF pure signal optimization to fill in substantial signal design gaps; hence, improving the GNSS signal design and standardization.

Index Terms—Pulse generation, pulse amplitude modulation, pulse width modulation, multidimensional sequences, signal design, signal analysis, generalized functions, time-frequency analysis, minimization methods, optimization methods.

1 Introduction

The main objective of this paper is to introduce the first generation $VBOC2(\alpha,1-\alpha)$ generalized multidimensional geolocation modulation waveforms so as to fill in substantial signal design [1]-[27] methodology gaps created over the years as a results of incomplete signal design methodologies.

The main motivation or the argument of the signal design waveforms has already been established that $VBOC1(\alpha)$ generalized multidimensional geolocation modulation waveforms which is a type of asymmetric non-return-to-zero (or As-NRZ) BOC for $0 \leq \alpha \leq 1$, generalizes the transition from the BPSK waveform $BPSK(t) = VBOC1_{(m,n,\alpha=1)}(t)$ to the current symmetric NRZ (or S-NRZ) BOC modulation, used extensively in GPS L1 and L2 frequencies [5]-[17], because $BOC_{(m,n)}(t) = VBOC1_{(m,n,\alpha=0)}(t)$; however, in the current GNSS standard the choice $\alpha = 0$ in the context of $VBOC1_{(m,n,\alpha)}(t)$ was made entirely arbitrary [1]-[7].

The first argument was made that $VBOC1_{(1,1,\alpha=0.5)}(t)$ is probably better choice on the GPS L1C data signal than $BOC_{(1,1)}(t) = VBOC1_{(1,1,\alpha=0)}(t)$ because the current choice on GPS L1C data signal instead of $BOC_{(1,1)}(t) = VBOC1_{(1,1,\alpha=0)}(t)$ is arbitrary and just as inefficient as the BPSK modulation [1], [2] in terms of out of phase ACF.

The second argument was made that $VBOC1_{(2,1,\alpha=0.5)}(t)$ is also a better choice than $BOC_{(2,1)}(t) = BOC_{(10,5)}(t) = VBOC1_{(2,1,\alpha=0)}(t)$ or the GPS military M-code on both GPS L1 and L2 frequencies [1], [2], because either $VBOC1_{(1,1,\alpha=0.5)}(t)$ or $VBOC1_{(2,1,\alpha=0.5)}(t)$ is 100 percent more efficient than either $BOC_{(1,1)}(t)$ or $BOC_{(2,1)}(t)$ respectively in terms of out of phase ACF [2].

The question now is as follows: can we find a better choice for: (1) $BOC_{(1,1)}(t)$ than $VBOC1_{(1,1,\alpha=0.5)}(t)$? (2) $BOC_{(2,1)}(t) = BOC_{(10,5)}(t)$ than $VBOC1_{(2,1,\alpha=0.5)}(t)$?

For the purposes of this work $VBOC2(\alpha, 1-\alpha)$ really means $VBOC2_{(pn,n,\alpha)}(t)$ for $0 \leq \alpha \leq 1$ which is a special case of the second generation of generalized multidimensional geolocation modulation waveforms $VBOC2(\alpha_1, \alpha_2)$ for $0 \leq \alpha_1, \alpha_2 \leq 1$ and in general $\alpha_1 \neq \alpha_2$.

We notice that $VBOC1(\alpha) = VBOC2(\alpha_1 = \alpha, \alpha_2 = \alpha)$ for $0 \leq \alpha \leq 1$; i.e., $VBOC1(\alpha)$ that is already being discussed in [1], [2] is a special case of $VBOC2(\alpha_1, \alpha_2)$. Since, we have already discussed $VBOC1(\alpha)$ [1], [2] instead of discussing $VBOC2(\alpha_1, \alpha_2)$ for $0 \leq \alpha_1, \alpha_2 \leq 1$ and in general $\alpha_1 \neq \alpha_2$; why not consider $VBOC2(\alpha, 1-\alpha)$ and show if in fact $VBOC2(\alpha, 1-\alpha)$ is better than $VBOC1(\alpha)$ as a first step and if this is the case [3] and then later on consider $VBOC2(\alpha_1, \alpha_2)$ for $0 \leq \alpha_1, \alpha_2 \leq 1$ and in general $\alpha_1 \neq \alpha_2$.

This publication contain substantially original and innovative research, in contrast to my earlier signal design work [7], [21]-[24] and end user work [18]-[20], [25]-[27], in the context of $VBOC2(\alpha, 1-\alpha)$ pure signal design methodology which includes: (1) original and innovative formalized definition and discussion of the $VBOC2(\alpha, 1-\alpha)$ signal waveform; (2) include original and innovative formalized definition and discussion of $VBOC2(\alpha, 1-\alpha)$ the generalized ACF and its complete set of properties; (3) include original and innovative formalized definition and discussion of the $VBOC2(\alpha, 1-\alpha)$ generalized PSD; (4) all (1) through (4) are analyzed as a function of the signal design and optimization parameter, α, and generalized parameter, p which denotes the order of the $VBOC2(\alpha, 1-\alpha)$.

Why is the discussion of all of these so important? First, signal design and optimization parameter, α, exploits a particular asymmetry of the As-NRZ signal coding modulation in order to improve $VBOC2(\alpha, 1-\alpha)$ signal design and optimization. Second, the derivation of generalized ACFs and PSDs as a function of α and p provides for the first time the opportunity to understand the properties of all individual $VBOC2(\alpha, 1-\alpha)$ such as for all positive integer values of p without having to analyze all individual $VBOC2(\alpha, 1-\alpha)$ separately.

This paper is organized as follows: in Sect. 2 $VBOC2(\alpha, 1-\alpha)$ pure signal design is discussed. Sect. 3 contains numerical results; Conclusion is provided in Sect. 4 along with a list of references.

2 $VBOC2(\alpha, 1-\alpha)$ Pure Signal Design

The detailed discussion on $VBOC2(\alpha, 1-\alpha)$ pure signal design starts with the: (1) definition of the $VBOC2_{(m,n,\alpha)}(t)$ waveform; (2) $VBOC2(\alpha, 1-\alpha)$ ACF definition and discussion; and (3) $VBOC2(\alpha, 1-\alpha)$ PSD definition and discussion.

2.1 $VBOC2(\alpha, 1-\alpha)$ Signal Definition and Discussion

Definition 1: $VBOC2_{(m,n,\alpha)}(t)$ waveform is a generalized, periodic function of the $BOC_{(m,n)}(t)$ waveform with period $2T_s$, (T_s : the subcarrier period), for all values of t $-\infty < t < \infty$.

$$VBOC2_{(m,n,\alpha)}(t) = VBOC2_{(m,n,\alpha)}(t \pm 2T_s) \tag{1}$$

and the relation of integers m, n is given by

$$2mT_s = nT_c \tag{2}$$

where T_c is the defined as the chipping period.

Definition 2: $VBOC2_{(m,n,\alpha)}(t)$ waveform is an odd function for all values of $-\infty < t < \infty$.

$$VBOC2_{(m,n,\alpha)}(t) = -VBOC2_{(m,n,\alpha)}(-t) \quad (3)$$

Definition 3: $VBOC2_{(m,n,\alpha)}(t)$ waveform is a periodic function with period T_s for all values of m even and $-\infty < t < \infty$.

$$VBOC2_{(m,n,\alpha)}(t) = VBOC2_{(m,n,\alpha)}(t \pm T_s) \quad (4)$$

Definition 4: The odd/even sub-carrier period of $VBOC2_{(m,n,\alpha)}(t)$ is the superposition of two pulses: a rising pulse and a falling pulse with amplitude/pulse widths, $+1/w_r$ or $+1/w_f$ and $-1/w_f$ or $-1/w_r$ respectively, or a falling pulse and a rising pulse amplitude/pulse widths $-1/w_f$ or $-1/w_r$ and $+1/w_r$ or $+1/w_f$ respectively that satisfy the following

$$w_r = 0.5T_s(1 \pm \alpha) = 0.5\frac{nT_c(1\pm\alpha)}{2m} = \frac{T_c(1\pm\alpha)}{4p} \quad (5)$$

$$w_f = 0.5T_s(1 \mp \alpha) = 0.5\frac{nT_c(1\mp\alpha)}{2m} = \frac{T_c(1\mp\alpha)}{4p}{}^{1} \quad (6)$$

Hence, the waveform $VBOC2_{(m,n,\alpha)}(t)$ can be obtained as

$$VBOC2_{(m,n,\alpha)}(t) = \begin{matrix} \sum_{\substack{q=-\infty \\ q:\,odd}}^{\infty} \left[p_{w_r}(t_1, w_r) - p_{w_f}(t_2, w_f)\right] + \\ \sum_{\substack{q=-\infty \\ q:\,even}}^{\infty} \left[p_{w_f}(t_1, w_f) - p_{w_r}(t_2, w_r)\right] \end{matrix} \quad (7)$$

where

$$t_1 = t - 0.5w_r - qT_s \quad (8)$$

$$t_2 = t - 0.5w_f - w_r - qT_s \quad (9)$$

and $p_w(t - t_0, w)$ is a unit rectangular pulse function with width w centered at t_0 with amplitude $+1$.

It is straightforward to show that $VBOC2_{(m,n,\alpha)}(t)$ obeys *Theorem* 1 of [1]; i.e., for $VBOC2_{(m,n,\alpha)}(t)$ $0 \le \alpha \le 1$.

Corollary 1: $BOC_{(m,n)}(t)$ is simply $VBOC2_{(m,n,\alpha)}(t)$ for $\alpha = 0$.

Corollary 2: $BOC_{(m/2,n)}(t)$ is simply $VBOC2_{(m,n,\alpha)}(t)$ for $\alpha = 1$.

Even though we have defined the $VBOC2_{(m,n,\alpha)}(t)$ signal waveform we want to describe the properties of the signal with itself for values of $0 \le \tau \le T$ in the observation interval. One means of achieving this by correlating the $VBOC2_{(m,n,\alpha)}(t)$ signal waveform with itself; hence, the definition of the ACF, $R_{VBOC2_{(m,n,\alpha)}}(\tau)$.

This concludes $VBOC2(\alpha, 1-\alpha)$ signal definition and discussion; next we continue with $VBOC2(\alpha, 1-\alpha)$ generalized ACF definition and discussion.

2.2 $VBOC1(\alpha)$ *Generalized ACF Definition and Discussion*

Definition 5: The ACF of $VBOC2_{(m,n,\alpha)}(t)$ or $R_{VBOC2_{(m,n,\alpha)}}(\tau, 2T)$ for m odd with the variable coefficient $0 \le \alpha \le 1$ for all values of τ; $t_0 - 2T \le \tau \le t_0 + 2T$ is given by

$$R_{VBOC2_{(m:o^2,n,\alpha)}}(\tau, 2T, t_0) = \frac{v(t)*v(t)}{2T} \quad (10)$$

or in integral form

$$R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T, t_0) = \frac{\int_{t_0}^{t_0+2T} v(t)v(t\pm\tau)dt}{2T} \quad (11)$$

$$v(t) = VBOC2_{(m:o,n,\alpha)}(t) \quad (12)$$

Definition 6: The ACF of $VBOC2_{(m,n,\alpha)}(t)$ or $R_{VBOC2_{(m,n,\alpha)}}(\tau, T)$ for m even with the variable coefficient $0 \le \alpha \le 1$ for all values of τ; $t_0 - T \le \tau \le t_0 + T$ is given by

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T, t_0) = \frac{v(t)*v(t)}{T} \quad (13)$$

or in integral form

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T, t_0) = \frac{\int_{t_0}^{t_0+T} v(t)v(t\pm\tau)dt}{T} \quad (14)$$

$$v(t) = VBOC2_{(m:e,n,\alpha)}(t) \quad (15)$$

There are a couple of things that we should describe about the ACF, $R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T, t_0)$ and $R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T, t_0)$ with respect to $\tau = 0$ and $\tau = \infty$.

It is straight forward to show that the ACF of $VBOC2_{(m,n,\alpha)}(t)$ ($R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T, t_0)$ given by (10) and (11) and $R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T, t_0)$ given by (13) and (14)) obey *theorems* 2 and 3, and *corollaries* 3 and 4 of [1].

Theorem 1 ("*Fundamental theorem of the ACF of the generalized* $VBOC2_{(m,n,\alpha)}(t)$"). Show that if $T = T_s$ starting with:

(1) *Definition* 5 the ACF of $VBOC2_{(m:o,n,\alpha)}(t)$ the following is obtained:

$$R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T_s, t_0) = \frac{\int_{t_0}^{t_0+2T_s} v(t)v(t\pm\tau)dt}{2T_s} \quad (16)$$

(2) *Definition* 6 the ACF of $VBOC2_{(m:e,n,\alpha)}(t)$ the following is obtained:

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T_s, t_0) = \frac{\int_{t_0}^{t_0+T_s} v(t)v(t\pm\tau)dt}{T_s} \quad (17)$$

(3) prove that when $VBOC2_{(m:e,n,\alpha)}(t)$ the following holds

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, 2T_s, t_0) = R_{VBOC2_{(m:e,n,\alpha)}}(\tau, T_s, t_0) \quad (18)$$

[1] For ease of analysis we will assume that the $+$ for w_r and $-$ for w_f.

[2] o: odd and e: even. This short hand notation will be used throughout the paper.

(4) prove that when $VBOC2_{(m:o,n,\alpha)}(t)$ the following holds

$$R_{VBOC2_{(m:o,n,\alpha)}}(\tau, 2T_s, t_0) \neq R_{VBOC2_{(m:o,n,\alpha)}}(\tau, T_s, t_0) \quad (19)$$

Proof of theorem 1 : The *proof of theorem* 1 is straightforward. First, by substituting $T = T_s$ from *definition* 5 (16) is obtained. Second, from *definition* 6 (17) is obtained. Third, now, let us assume that $v(t) = VBOC2_{(m:e,n,\alpha)}(t)$ starting with *definition* 5 or (16) we have:

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, 2T_s, t_0) = \frac{\left| \begin{array}{l} \int_{t_0}^{t_0+2T_s} v(t)v(t\pm\tau)dt = \\ \int_{t_0}^{t_0+T_s} v(t)v(t\pm\tau)dt + \\ \int_{t_0+T_s}^{t_0+2T_s} v(t)v(t\pm\tau)dt \end{array} \right.}{2T_s} \quad (20)$$

By changing the variables $t' = t - T_s$ we notice that

$$\int_{t_0+T_s}^{t_0+2T_s} v(t)v(t \pm \tau)\, dt = \int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau)\, dt' \quad (21)$$

where $t = t' + T_s$.

Based on *definition* 3 (19) becomes:

$$\int_{t_0+T_s}^{t_0+2T_s} v(t)v(t \pm \tau)\, dt = \int_{t_0}^{t_0+T_s} v(t')v(t' \pm \tau)\, dt' \quad (22)$$

or

$$\int_{t_0+T_s}^{t_0+2T_s} v(t)v(t \pm \tau)\, dt = \int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau)\, dt \quad (23)$$

Finally, substituting (23) into (21) the following is obtained:

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, 2T_s, t_0) = \frac{\left| \begin{array}{l} \int_{t_0}^{t_0+2T_s} v(t)v(t\pm\tau)dt = \\ 2\int_{t_0}^{t_0+T_s} v(t)v(t\pm\tau)dt \end{array} \right.}{2T_s} \quad (24)$$

or

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau, 2T_s, t_0) = \frac{\int_{t_0}^{t_0+T_s} v(t)v(t\pm\tau)dt}{T_s} \quad (25)$$

which completed the proof of (18).

Fourth, it is also very straightforward to show that (25) holds also. *Definition* 5 is known as the necessary condition of the ACF of $VBOC2_{(m,n,\alpha)}(t)$ and *definition* 6 as the sufficient condition the ACF of $VBOC2_{(m,n,\alpha)}(t)$.

Next, we will exploit the ACF of $VBOC2_{(m:o,n,\alpha)}(t)$ a bit further; i.e., (21) and (23). Because (21) and (23) are functions of τ

$$\int_{t_0}^{t_0+T_s} v(t')v(t' \pm \tau)\, dt \quad (26)$$

$$\int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau)\, dt \quad (27)$$

for values of $-2T_s \leq \tau \leq 2T_s$ we recognize the following special cases: (a) $0 \leq |\tau| \leq T_s$ and (b) $T_s \leq |\tau| \leq 2T_s$.

First, let us consider (a) $0 \leq |\tau| \leq T_s$ for (26) and (b) $T_s \leq |\tau| \leq 2T_s$ for (26); hence,

$$v(t' \pm \tau)|_{T_s \leq |\tau| \leq 2T_s} = v(t \pm \tau)|_{0 \leq |\tau| \leq T_s} \quad (28)$$

which leads to the following:

$$\underbrace{\int_{t_0}^{t_0+T_s} v(t')v(t' \pm \tau)\, dt}_{T_s \leq |\tau| \leq 2T_s} = \underbrace{\int_{t_0}^{t_0+T_s} v(t')v(t \pm \tau)\, dt}_{0 \leq |\tau| \leq T_s} \quad (29)$$

$$\underbrace{\int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau)\, dt}_{0 \leq |\tau| \leq T_s} = \underbrace{\int_{t_0}^{t_0+T_s} v(t)v(t' \pm \tau)\, dt}_{T_s \leq |\tau| \leq 2T_s} \quad (30)$$

Second, let us consider (b) $T_s \leq |\tau| \leq 2T_s$ for (26) and (a) $0 \leq |\tau| \leq T_s$ for (27); hence,

$$v(t \pm \tau)|_{T_s \leq |\tau| \leq 2T_s} = v(t' \pm \tau)|_{0 \leq |\tau| \leq T_s} \quad (31)$$

which leads to the following:

$$\underbrace{\int_{t_0}^{t_0+T_s} v(t')v(t' \pm \tau)\, dt}_{0 \leq |\tau| \leq T_s} = \underbrace{\int_{t_0}^{t_0+T_s} v(t')v(t \pm \tau)\, dt}_{T_s \leq |\tau| \leq 2T_s} \quad (32)$$

$$\underbrace{\int_{t_0}^{t_0+T_s} v(t)v(t \pm \tau)\, dt}_{T_s \leq |\tau| \leq 2T_s} = \underbrace{\int_{t_0}^{t_0+T_s} v(t)v(t' \pm \tau)\, dt}_{0 \leq |\tau| \leq T_s} \quad (33)$$

Finally, let us compare and contrast (29) with (32) and (30) and (33). Typically, either (29) or (30) is different from (32) or (33) respectively; which leads to the fundamental following *corollary* for the ACF of $VBOC2_{(m,n,\alpha)}(t)$.

Corollary 3: The ACF of $VBOC2_{(m:o,n,\alpha)}(t)$ can either be computed directly as in *definition* 5 in which case requires integration along $2T_s$, as given by (16), or indirectly via four coherent integrations along T_s, as given by (29), (30), (32) and (33); i.e., four independent half ACF (or HACF) periods.

Corollary 4: The ACF of $VBOC2_{(m:e,n,\alpha)}(t)$ is four times more efficient than ACF of $VBOC2_{(m:o,n,\alpha)}(t)$.

Corollary 5: $R_{VBOC2_{(m,n,\alpha)}}(\tau, T, t_0)$ is periodic with period $T = nT_c = 2mT_s$ corresponding to the observation interval $t_0 \leq \tau \leq t_0 + T_c$ and in fact $R_{VBOC2_{(m,n,\alpha)}}(\tau)$ is only one period of ACF $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$ when

$$R_{VBOC2_{(m:o,2n,\alpha)}}(\tau) = \frac{\int_0^{2nT_c} v(t)v(t\pm\tau)dt}{2nT_c} \quad (34)$$

$$R_{VBOC2_{(m:e,n,\alpha)}}(\tau) = \frac{\int_0^{nT_c} v(t)v(t\pm\tau)dt}{nT_c} \quad (35)$$

which is the definition of the ACF whose shape we are most familiar with and it has the largest autocorrelation peak.[3]

Corollary 6: The ACF of $VBOC2_{(m,in,\alpha)}(t)$ or $R_{VBOC2_{(m,in,\alpha)}}(\tau)$ for $i = \{1,2\}$ is in general an even function with respect to τ with the variable coefficient $0 \leq \alpha \leq 1$ for all values of τ; $-\infty < \tau < \infty$; i.e.,

$$R_{VBOC2_{(m,in,\alpha)}}(-\tau) = R_{VBOC2_{(m,in,\alpha)}}(\tau) \text{ for } i = \{1,2\} \quad (36)$$

[3] In general there are an infinite number of shapes of ACFs; however, only one of them has the largest ACF peak.

The closed form expression of the generalized ACF, $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$, with respect to p, is given subject to the following theorem.

Theorem 2: Show that, for values of $p = \{2,4\}$, $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ is the generalized ACF with respect to p; i.e., (1) $R_{VBOC2_{(2\times n,n,\alpha)}}(\tau) = R_{VBOC2_{(pn=2n,n,\alpha)}}(\tau)$; $p = 2$; (2) $R_{VBOC2_{(4\times n,n,\alpha)}}(\tau) = R_{VBOC2_{(pn=4n,n,\alpha)}}(\tau)$; $p = 4$; For $0 \le \alpha \le 1/3$

$$\tau_1^\alpha = \left[\frac{T_c}{2p} = T_s\right](1-\alpha);\ \tau_2^\alpha = \frac{T_c(1+\alpha)}{2p};\ \tau_3^\alpha = \frac{T_c(1-\alpha)}{p}; \quad (37a)$$

$$\tau_4 = \frac{T_c}{p};\ \tau_5^\alpha = \frac{T_c(1+\alpha)}{p};\ \tau_6^\alpha = \frac{T_c(3-\alpha)}{2p}; \quad (37b)$$

$$\tau_7^\alpha = \frac{T_c(3+\alpha)}{2p};\ \tau_8 = \frac{2T_c}{p};\ \tau_9^\alpha = \frac{T_c(5-\alpha)}{2p}; \quad (37c)$$

$$\tau_{10}^\alpha = \frac{T_c(5+\alpha)}{2p};\ \tau_{11}^\alpha = \frac{T_c(3-\alpha)}{p};\ \tau_{12} = \frac{3T_c}{p}; \quad (37d)$$

$$\tau_{13}^\alpha = \frac{T_c(3+\alpha)}{p};\ \tau_{14}^\alpha = \frac{T_c(7-\alpha)}{2p};\ \tau_{15}^\alpha = \frac{T_c(7+\alpha)}{2p};\ \tau_{16} = \frac{4T_c}{p} \quad (37e)$$

and $1/3 \le \alpha \le 1$

$$\tau_1^\alpha = \left[\frac{T_c}{2p} = T_s\right](1-\alpha);\ \tau_2^\alpha = \frac{T_c(1-\alpha)}{p};\ \tau_3^\alpha = \frac{T_c(1+\alpha)}{2p}; \quad (37f)$$

$$\tau_4 = \frac{T_c}{p};\ \tau_5^\alpha = \frac{T_c(3-\alpha)}{2p};\ \tau_6^\alpha = \frac{T_c(1+\alpha)}{p}; \quad (37g)$$

$$\tau_7^\alpha = \frac{T_c(3+\alpha)}{2p};\ \tau_8 = \frac{2T_c}{p};\ \tau_9^\alpha = \frac{T_c(5-\alpha)}{2p}; \quad (37h)$$

$$\tau_{10}^\alpha = \frac{T_c(3-\alpha)}{p};\ \tau_{11}^\alpha = \frac{T_c(5+\alpha)}{2p};\ \tau_{12} = \frac{3T_c}{p}; \quad (37i)$$

$$\tau_{13}^\alpha = \frac{T_c(7-\alpha)}{2p};\ \tau_{14}^\alpha = \frac{T_c(3+\alpha)}{p};\ \tau_{15}^\alpha = \frac{T_c(7+\alpha)}{2p};\ \tau_{16} = \frac{4T_c}{p} \quad (37j)$$

if $R_2(\tau) = R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ is given by the expression below:

$$R_2(\tau) = \left|\begin{array}{l}
\underbrace{\underbrace{1}_{a_{1(p,n,\alpha)}} + \underbrace{\frac{-(4p-1)}{T_c}}_{b_{1(p,n,\alpha)}}}_{r_{1(pn,n,\alpha)};\ p=\{2,4\}}|\tau|, |\tau| \le \tau_1^\alpha \\
\underbrace{-1+2\alpha}_{a_{2(p,n,\alpha)}} + \underbrace{\frac{1}{T_c}}_{b_{2(p,n,\alpha)}} |\tau|, \tau_1^\alpha \le |\tau| \le \tau_2^\alpha \\
\underbrace{\frac{-3p+2+2\alpha}{p}}_{a_{3(p,n,\alpha)}} + \underbrace{\frac{4p-3}{T_c}}_{b_{3(p,n,\alpha)}} |\tau|, \tau_2^\alpha \le |\tau| \le \tau_3^\alpha \\
\underbrace{\frac{2-p-2(p-1)\alpha}{p}}_{a_{4(p,n,\alpha)}} + \underbrace{\frac{2p-3}{T_c}}_{b_{4(p,n,\alpha)}} |\tau|, \tau_3^\alpha \le |\tau| \le \tau_4 \\
\underbrace{\frac{3p-2-2(p-1)\alpha}{p}}_{a_{5(p,n,\alpha)}} + \underbrace{\frac{-(2p-1)}{T_c}}_{b_{5(p,n,\alpha)}} |\tau|,\ \tau_{4(\alpha)} \le |\tau| \le \tau_5^\alpha \\
\underbrace{\frac{(5p-6)-2\alpha}{p}}_{a_{6(p,n,\alpha)}} + \underbrace{\frac{-(4p-5)}{T_c}}_{b_{6(p,n,\alpha)}} |\tau|, \tau_5^\alpha \le |\tau| \le \tau_6^\alpha \\
\underbrace{-1+\frac{2(p-2)\alpha}{p}}_{a_{7(p,n,\alpha)}} + \underbrace{\frac{1}{T_c}}_{b_{7(p,n,\alpha)}} |\tau|, \tau_6^\alpha \le |\tau| \le \tau_7^\alpha \\
\underbrace{\underbrace{-\frac{7p-12}{p}}_{a_{8(p,n,\alpha)}} + \underbrace{\frac{4p-7}{T_c}}_{b_{8(p,n,\alpha)}}}_{r_{8(pn,n,\alpha)};\ p=\{2,4\}} |\tau|,\ \tau_7^\alpha \le |\tau| \le \tau_8 \\
\underbrace{\underbrace{\frac{9p-20}{p}}_{a_{9(p,n,\alpha)}} + \underbrace{\frac{-(4p-9)}{T_c}}_{b_{9(p,n,\alpha)}}}_{r_{9(pn,n,\alpha)};\ p=\{4\}} |\tau|, \tau_8 \le |\tau| \le \tau_9^\alpha \\
\underbrace{\frac{20-7p+(5p-12)\alpha}{2p}}_{a_{10(p,n,\alpha)}} + \underbrace{\frac{p-3}{T_c}}_{b_{10(p,n,\alpha)}} |\tau|, \tau_9^\alpha \le |\tau| \le \tau_{10}^\alpha \\
\underbrace{\frac{30-11p+(p-2)\alpha}{p}}_{a_{11(p,n,\alpha)}} + \underbrace{\frac{4p-11}{T_c}}_{b_{11(p,n,\alpha)}} |\tau|, \tau_{10}^\alpha \le |\tau| \le \tau_{11}^\alpha \\
\underbrace{\frac{-4(p-3)-4(p-3)\alpha}{2p}}_{a_{12(p,n,\alpha)}} + \underbrace{\frac{p-3}{T_c}}_{b_{12(p,n,\alpha)}} |\tau|,\ \tau_{11}^\alpha \le |\tau| \le \tau_{12} \\
\underbrace{\frac{7p-18-2(p-3)\alpha}{p}}_{a_{13(p,n,\alpha)}} + \underbrace{\frac{-(2p-5)}{T_c}}_{b_{13(p,n,\alpha)}} |\tau|, \tau_{12} \le |\tau| \le \tau_{13}^\alpha \\
\underbrace{\frac{13p-42-2\alpha}{p}}_{a_{14(p,n,\alpha)}} + \underbrace{\frac{13-4p}{T_c}}_{b_{14(p,n,\alpha)}} |\tau|, \tau_{13}^\alpha \le |\tau| \le \tau_{14}^\alpha \\
\underbrace{-1+\frac{2(p-4)\alpha}{p}}_{a_{15(p,n,\alpha)}} + \underbrace{\frac{1}{T_c}}_{b_{15(p,n,\alpha)}} |\tau|, \tau_{14}^\alpha \le |\tau| \le \tau_{15}^\alpha \\
\underbrace{\underbrace{\frac{56-15p}{p}}_{a_{16(p,n,\alpha)}} + \underbrace{\frac{4p-15}{T_c}}_{b_{16(p,n,\alpha)}}}_{r_{16(pn,n,\alpha)};\ p=\{4\}} |\tau|, \tau_{15}^\alpha \le |\tau| \le \tau_{16} \\
0,\ |\tau| \ge T_c
\end{array}\right. \quad (38)$$

$$R_2(\tau) = \begin{cases} \underbrace{\underbrace{1}_{a_{1(p,n,\alpha)}} + \underbrace{\frac{-(4p-1)}{T_c}}_{b_{1(p,n,\alpha)}}|\tau|}_{r_{1(pn,n,\alpha)};\ p=\{2,4\}}, |\tau| \le \tau_1^\alpha \\ \underbrace{-1+2\alpha}_{a_{2(p,n,\alpha)}} + \underbrace{\frac{1}{T_c}}_{b_{2(p,n,\alpha)}}|\tau|, \tau_1^\alpha \le |\tau| \le \tau_2^\alpha \\ \underbrace{1}_{a_{3(p,n,\alpha)}} + \underbrace{\frac{-(2p-1)}{T_c}}_{b_{3(p,n,\alpha)}}|\tau|, \tau_2^\alpha \le |\tau| \le \tau_3^\alpha \\ \underbrace{\frac{2-p-2(p-1)\alpha}{p}}_{a_{4(p,n,\alpha)}} + \underbrace{\frac{2p-3}{T_c}}_{b_{4(p,n,\alpha)}}|\tau|, \tau_3^\alpha \le |\tau| \le \tau_4 \\ \underbrace{\frac{3p-2-2(p-1)\alpha}{p}}_{a_{5(p,n,\alpha)}} + \underbrace{\frac{-(2p-1)}{T_c}}_{b_{5(p,n,\alpha)}}|\tau|,\ \tau_{4(\alpha)} \le |\tau| \le \tau_5^\alpha \\ \underbrace{-\frac{3p-4}{p}}_{a_{6(p,n,\alpha)}} + \underbrace{\frac{2p-3}{T_c}}_{b_{6(p,n,\alpha)}}|\tau|, \tau_5^\alpha \le |\tau| \le \tau_6^\alpha \\ \underbrace{-1+\frac{2(p-2)\alpha}{p}}_{a_{7(p,n,\alpha)}} + \underbrace{\frac{1}{T_c}}_{b_{7(p,n,\alpha)}}|\tau|, \tau_6^\alpha \le |\tau| \le \tau_7^\alpha \\ \underbrace{\underbrace{-\frac{7p-12}{p}}_{a_{8(p,n,\alpha)}} + \underbrace{\frac{4p-7}{T_c}}_{b_{8(p,n,\alpha)}}}_{r_{8(pn,n,\alpha)};\ p=\{2,4\}}|\tau|,\ \tau_7^\alpha \le |\tau| \le \tau_8 \\ \underbrace{\underbrace{\frac{9p-20}{p}}_{a_{9(p,n,\alpha)}} + \underbrace{\frac{-(4p-9)}{T_c}}_{b_{9(p,n,\alpha)}}}_{r_{9(pn,n,\alpha)};\ p=\{4\}}|\tau|, \tau_8 \le |\tau| \le \tau_9^\alpha \\ \underbrace{\frac{20-7p+(5p-12)\alpha}{2p}}_{a_{10(p,n,\alpha)}} + \underbrace{\frac{p-3}{T_c}}_{b_{10(p,n,\alpha)}}|\tau|, \tau_9^\alpha \le |\tau| \le \tau_{10}^\alpha \\ \underbrace{\frac{11p-28+(4-p)\alpha}{2p}}_{a_{11(p,n,\alpha)}} + \underbrace{\frac{-(2p-5)}{T_c}}_{b_{11(p,n,\alpha)}}|\tau|, \tau_{10}^\alpha \le |\tau| \le \tau_{11}^\alpha \\ \underbrace{\frac{-4(p-3)-4(p-3)\alpha}{2p}}_{a_{12(p,n,\alpha)}} + \underbrace{\frac{p-3}{T_c}}_{b_{12(p,n,\alpha)}}|\tau|,\ \tau_{11}^\alpha \le |\tau| \le \tau_{12} \\ \underbrace{\frac{7p-18-2(p-3)\alpha}{p}}_{a_{13(p,n,\alpha)}} + \underbrace{\frac{-(2p-5)}{T_c}}_{b_{13(p,n,\alpha)}}|\tau|, \tau_{12} \le |\tau| \le \tau_{13}^\alpha \\ \underbrace{\frac{24-7p}{p}}_{a_{14(p,n,\alpha)}} + \underbrace{\frac{2p-7}{T_c}}_{b_{14(p,n,\alpha)}}|\tau|, \tau_{13}^\alpha \le |\tau| \le \tau_{14}^\alpha \\ \underbrace{-1+\frac{2(p-4)\alpha}{p}}_{a_{15(p,n,\alpha)}} + \underbrace{\frac{1}{T_c}}_{b_{15(p,n,\alpha)}}|\tau|, \tau_{14}^\alpha \le |\tau| \le \tau_{15}^\alpha \\ \underbrace{\underbrace{\frac{56-15p}{p}}_{a_{16(p,n,\alpha)}} + \underbrace{\frac{4p-15}{T_c}}_{b_{16(p,n,\alpha)}}}_{r_{16(pn,n,\alpha)};\ p=\{4\}}|\tau|, \tau_{15}^\alpha \le |\tau| \le \tau_{16} \\ 0,\ |\tau| \ge T_c \end{cases} \quad (39)$$

$$T = nT_c = 2mT_s, \text{ and } m,\ n \text{ integers} \quad (39a)$$

For $0 \le \alpha \le 1/3$ and $1/3 \le \alpha \le 1$ the ACF $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ is given by (38) and (39) respectively.

The *proof of theorem* 2 is straightforward.

First, we recognize that substituting $p = 2$ in (38) we get

$$R_1(\tau) = \begin{cases} 1 - \frac{(4p-1)|\tau|}{T_c}, & |\tau| \le \tau_1^\alpha \\ -1+2\alpha - \frac{|\tau|}{T_c}, & \tau_1^\alpha \le |\tau| \le \tau_2^\alpha \\ \frac{-3p+2+2\alpha}{p} + \frac{(4p-3)|\tau|}{T_c}, & \tau_2^\alpha \le |\tau| \le \tau_3^\alpha \\ \frac{2-p-2(p-1)\alpha}{p} - \frac{(2p-3)|\tau|}{T_c}, & \tau_3^\alpha \le |\tau| \le \tau_4 \\ \frac{(5p-6)-(2p-2)\alpha}{p} - \frac{(2p-1)|\tau|}{T_c}, & \tau_4 \le |\tau| \le \tau_5^\alpha \\ -\frac{7p-12}{p} + \frac{(4p-7)|\tau|}{T_c}, & \tau_5^\alpha \le |\tau| \le \tau_6 \\ 0, & |\tau| \ge T_c \end{cases} \quad (40)$$

and $p = 2$ and $1/3 \le \alpha \le 1$ in (39) we get

$$R_1(\tau) = \begin{cases} 1 - \frac{(4p-1)|\tau|}{T_c}, & |\tau| \le \tau_1^\alpha \\ -1+2\alpha - \frac{|\tau|}{T_c}, & \tau_1^\alpha \le |\tau| \le \tau_2^\alpha \\ 1 - \frac{(2p-1)|\tau|}{T_c}, & \tau_2^\alpha \le |\tau| \le \tau_3^\alpha \\ \frac{2-p-2(p-1)\alpha}{p} + \frac{(2p-3)|\tau|}{T_c}, & \tau_3^\alpha \le |\tau| \le \tau_4 \\ \frac{3p-2-2(p-1)\alpha}{p} - \frac{(2p-1)|\tau|}{T_c}, & \tau_4 \le |\tau| \le \tau_5^\alpha \\ -\frac{7p-12}{p} + \frac{(4p-7)|\tau|}{T_c}, & \tau_5^\alpha \le |\tau| \le \tau_6 \\ 0, & |\tau| \ge T_c \end{cases} \quad (41)$$

Equations (40) and (41), for $p = 2$, are the same as in Chap. 7 of [5]. Also, (38) and (39), for $p = 4$, are the same as Chap. 7 of [5]. This completed the *proof of theorem* 2.

Although we did not show the enormous convolution computational process for computing the generalized ACF of $VBOC2_{(m,n,\alpha)}(t)$ or $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ with respect to the signal design and optimization parameter α and generalized parameter p there is no way of knowing that by simple inspection this expression is correct or not especially considering the fact that $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ takes an entire page.

However, the following material which discusses the broad properties of generalized ACF of $VBOC2_{(m,n,\alpha)}(t)$ or $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ and the material discussed in [3] which discusses in great detail the complete set of the continuity and optimization theorems guarantees that generalized ACF of $VBOC2_{(m,n,\alpha)}(t)$ or $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ is the correct expression.

Corollary 7: From *Corollary* 1 $VBOC2_{(m,n,\alpha=0)}(t) = BOC_{(m,n)}(t)$ and from *Theorem* 2, the ACF $R_{VBOC2_{(m,n,\alpha=0)}}(\tau) = R_{BOC_{(m,n)}}(\tau)$ hence, prove that the generalized ACF with respect to p; i.e., is

(1) $R_{BOC_{(2n,n)}}(\tau) = R_{VBOC2_{(pn=2n,n,\alpha=0)}}(\tau)$; $p = 2$;

(2) $R_{BOC_{(4n,n)}}(\tau) = R_{VBOC2_{(pn=4n,n,\alpha=0)}}(\tau)$; $p = 4$;

$$R_{BOC_{(pn,n)}}(\tau)=\begin{cases}\underbrace{\underbrace{1}_{a_{1(p,n)}}+\underbrace{\frac{-(4p-1)}{T_c}}_{b_{1(p,n)}}|\tau|}_{r_{1(pn,n)};\ p=\{1,2,3,4\}}, & |\tau|\le\underbrace{\frac{T_c}{2p}=T_s}_{\tau_1}\\ \underbrace{-\underbrace{\frac{3p-2}{p}}_{a_{3(p,n)}}+\underbrace{\frac{4p-3}{T_c}}_{b_{3(p,n)}}|\tau|}_{r_{3(pn,n)};\ p=\{1,2,3,4\}}, & \frac{T_c}{2p}\le|\tau|\le\underbrace{\frac{T_c}{p}}_{\tau_3}\\ \underbrace{\underbrace{\frac{5p-6}{p}}_{a_{4(p,n)}}+\underbrace{\frac{-(4p-5)}{T_c}}_{b_{4(p,n)}}|\tau|}_{r_{4(pn,n)};\ p=\{2,3,4\}}, & \frac{T_c}{p}\le|\tau|\le\underbrace{\frac{3T_c}{2p}}_{\tau_4}\\ \underbrace{-\underbrace{\frac{7p-12}{p}}_{a_{6(p,n)}}+\underbrace{\frac{(4p-7)}{T_c}}_{b_{6(p,n)}}|\tau|}_{r_{6(pn,n)};\ p=\{2,3,4\}}, & \frac{3T_c}{2p}\le|\tau|\le\underbrace{\frac{2T_c}{p}}_{\tau_6}\\ \underbrace{\underbrace{\frac{9p-20}{p}}_{a_{7(p,n)}}+\underbrace{\frac{-(4p-9)}{T_c}}_{b_{7(p,n)}}|\tau|}_{r_{7(pn,n)};\ p=\{3,4\}}, & \frac{2T_c}{p}\le|\tau|\le\underbrace{\frac{5T_c}{2p}}_{\tau_7}\\ \underbrace{-\underbrace{\frac{(11p-30)}{p}}_{a_{9(p,n)}}+\underbrace{\frac{(4p-11)}{T_c}}_{b_{9(p,n)}}|\tau|}_{r_{9(pn,n)};\ p=\{3,4\}}, & \frac{5T_c}{2p}\le|\tau|\le\underbrace{\frac{3T_c}{p}}_{\tau_9}\\ \underbrace{\underbrace{\frac{13p-42}{p}}_{a_{10(p,n)}}+\underbrace{\frac{-(4p-13)}{T_c}}_{b_{10(p,n)}}|\tau|}_{r_{10(pn,n)};\ p=\{4\}}, & \frac{3T_c}{p}\le|\tau|\le\underbrace{\frac{7T_c}{2p}}_{\tau_{10}}\\ \underbrace{-\underbrace{\frac{(15p-56)}{p}}_{a_{12(p,n)}}+\underbrace{\frac{(4p-15)}{T_c}}_{b_{12(p,n)}}|\tau|}_{r_{12(pn,n)}}, & \frac{7T_c}{2p}\le|\tau|\le\underbrace{\frac{4T_c}{p}}_{\tau_{12}}\\ 0, & |\tau|\ge T_c\end{cases}\tag{42}$$

$$T=nT_c=2mT_s,\ \text{and}\ m,\ n\ \text{integers}\tag{42a}$$

Proof of corollary 7: The *proof of corollary* 7 is straightforward.

First, we recognize that substituting $p=2$ in (38) we get

$$R_{BOC_{(pn,n)}}(\tau)=\begin{cases}1-\frac{(4p-1)|\tau|}{T_c}, & |\tau|\le\frac{T_c}{2p}=T_s\\ -\frac{3p-2}{p}+\frac{(4p-3)|\tau|}{T_c}, & \frac{T_c}{2p}\le|\tau|\le\frac{T_c}{p}\\ \frac{5p-6}{p}-\frac{(4p-5)|\tau|}{T_c}, & \frac{T_c}{p}\le|\tau|\le\frac{3T_c}{2p}\\ -\frac{7p-12}{p}+\frac{(4p-7)|\tau|}{T_c}, & \frac{3T_c}{2p}\le|\tau|\le\frac{2T_c}{p}\\ 0, & |\tau|\ge T_c\end{cases}\tag{43}$$

$$T=nT_c=2mT_s,\ \text{and}\ m=np,\ n\ \text{integers}\tag{43a}$$

Equation (43), for $p=2$, is the same as in Chap. 7 [5] and (82) in [6].

Equation (42), for $p=4$, is the same as in Chap. 7 [5]. This completed the *proof of corollary* 7.

Corollary 8: From *corollary* 2 $VBOC2_{(m,n,\alpha=1)}(t)=BOC_{(m/2,n)}(t)$ and from *theorem* 2, the ACF $R_{VBOC2_{(m,n,\alpha=1)}}(\tau)=R_{BOC_{(m/2,n)}}(\tau)$ hence, prove that the generalized ACF with respect to p; i.e., is

(1) $R_{BOC_{(n,n)}}(\tau)=R_{VBOC2_{(np=2n,n,\alpha=1)}}(\tau);\ p=2;$

(2) $R_{BOC_{(2n,n)}}(\tau)=R_{VBOC2_{(np=4n,n,\alpha=1)}}(\tau);\ p=4;$

$$R_{BOC_{(pn/2,n)}}(\tau)=\begin{cases}\underbrace{\underbrace{1}_{a_{3(p,n)}}-\underbrace{\frac{2p-1}{T_c}}_{b_{3(p,n)}}\frac{|\tau|}{1}}_{r_{3(pn,n)};\ p=\{2,4\}}, & 0\le|\tau|\le\underbrace{\frac{T_c}{p}}_{\tau_3}\\ \underbrace{\underbrace{\frac{4-3p}{p}}_{a_{6(p,n)}}+\underbrace{\frac{2p-3}{T_c}}_{b_{6(p,n)}}\frac{|\tau|}{1}}_{r_{6(pn,n)};\ p=\{2,4\}}, & \frac{T_c}{p}\le|\tau|\le\underbrace{\frac{2T_c}{p}}_{\tau_6}\\ \underbrace{\frac{5p-12}{p}}_{a_{11(p,n)}}+\underbrace{\frac{5-2p}{T_c}}_{b_{11(p,n)}}\frac{|\tau|}{1}, & \frac{2T_c}{p}\le|\tau|\le\underbrace{\frac{3T_c}{2p}}_{\tau_{11}}\\ \underbrace{\underbrace{\frac{24-7p}{p}}_{a_{14(p,n)}}+\underbrace{\frac{2p-7}{T_c}}_{b_{14(p,n)}}\frac{|\tau|}{1}}_{r_{14(pn,n)};\ p=\{4\}}, & \frac{3T_c}{2p}\le|\tau|\le\underbrace{\frac{4T_c}{p}}_{\tau_{14}}\\ 0, & |\tau|\ge T_c\end{cases}\tag{44}$$

$$T=nT_c=2mT_s,\ \text{and}\ m,\ n\ \text{integers}\tag{44a}$$

Proof of corollary 8*:* The *proof of corollary* 8 is straightforward. First, we recognize that that for up to values of $p=2$ in (44) or for $\alpha=1$ at (41) we get

$$R_{BOC_{(pn/2,n)}}(\tau)=\begin{cases}\underbrace{\underbrace{1}_{a_{3(p,n)}}-\underbrace{\frac{2p-1}{T_c}}_{b_{3(p,n)}}|\tau|}_{r_{3(pn,n)};\ p=\{2\}}, & 0\le|\tau|\le\underbrace{\frac{T_c}{p}}_{\tau_3}\\ \underbrace{\underbrace{\frac{4-3p}{p}}_{a_{6(p,n)}}+\underbrace{\frac{2p-3}{T_c}}_{b_{6(p,n)}}|\tau|}_{r_{6(pn,n)};\ p=\{2\}}, & \frac{T_c}{p}\le|\tau|\le\underbrace{\frac{2T_c}{p}}_{\tau_6}\\ 0, & |\tau|\ge T_c\end{cases}\tag{45}$$

Equation (45), for $p=2$, is the same as in Chap. 7 [5] and (100) in [6].

Equation (45), for $p=4$, is the same as in Chap. 7 [5]. This completed the *proof of corollary* 8.

This completes the first discussion on $VBOC2_{(m,n,\alpha)}(t)$ and $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$; further material with regards to $VBOC2_{(m,n,\alpha)}(t)$ pure signal optimization $R_{VBOC2_{(m=pn,n,\alpha)}}(\tau)$ continues in [3].

Next, we continue with $VBOC2(\alpha,1-\alpha)$ PSD definition and discussion.

2.3 $VBOC2(\alpha,1-\alpha)$ *Generalized PSD Definition and Discussion*

Definition 7: The PSD of $VBOC2_{(m,n,\alpha)}(t)$ or $G_{VBOC2_{(m,n,\alpha)}}(f)$ with the variable coefficient $0\le\alpha\le1$ for all values of f; $-\infty<f<\infty$ is given by

$$G_{VBOC2_{(m,n,\alpha)}}(f,T)=\underbrace{\mathcal{F}\left\{R_{VBOC2_{(m,n,\alpha)}}(\tau,T,t_0)\right\}}_{\frac{V(f)V(f)}{T}=\frac{V^2(f)}{T}}\tag{46}$$

where $\mathcal{F}\{v(t)\}$ denotes the Fourier Transform (FT) of $v(t)$; i.e.,

$$V(f)=\mathcal{F}\{VBOC2_{(m,n,\alpha)}(t)\}=\mathcal{F}\{v(t)\}\tag{47}$$

Now that we have defined the PSD of $VBOC2_{(m,n,\alpha)}(t)$, it is straightforward to show that $G_{VBOC2_{(m,n,\alpha)}}(f,T)$ obeys *theorem* 11 in [1] so we leave it as an exercise to the reader.

Corollary 9: The PSD of $VBOC2_{(m,n,\alpha)}(t)$, $G_{VBOC2_{(m,n,\alpha)}}(f,T)$, is in general NOT periodic; simply, because $V(f)$ is in general not periodic, even if $v(t)$ is periodic.

Corollary 10: The PSD of $VBOC2_{(m,n,\alpha)}(t)$; i.e., $G_{VBOC2_{(m,n,\alpha)}}(f,T)$ is always an even function of $-\infty < f < \infty$; i.e., from (46) the following holds

$$G_{VBOC2_{(m,n,\alpha)}}(-f) = G_{VBOC2_{(m,n,\alpha)}}(f) \tag{48}$$

Theorem 3: Show that, for values of $p = \{2,4\}$, $G_{VBOC2_{(m=np,n,\alpha)}}(f)$ is the generalized ACF with respect to p; i.e.,

(1) $G_{VBOC2_{(2n,n,\alpha)}}(f) = G_{VBOC2_{(pn=2n,n,\alpha)}}(f)$; $p = 2$;

(2) $G_{VBOC2_{(4n,n,\alpha)}}(f) = G_{VBOC2_{(pn=4n,n,\alpha)}}(f)$; $p = 4$.

Find the generalized PSD of $G_{VBOC1_{(m=np,n,\alpha)}}(f)$ given by

$$G_{VBOC2_{(pn,n,\alpha)}}(f) = \left| \begin{array}{l} \mathcal{F}\left\{R_{VBOC2_{(pn,n,\alpha)}}(\tau)\right\} \\ = \frac{V(f)V(f)}{T} = \frac{V^2(f)}{T} \end{array} \right. \tag{49}$$

or

$$G_{VBOC2_{(pn,n,\alpha)}}(\omega) = 2\int_0^{\infty} R_{VBOC2_{(pn,n,\alpha)}}(\tau)\cos(\omega\tau)d\tau \tag{50}$$

where $R_{VBOC2_{(pn,n,\alpha)}}(\tau)$ is given by (38) and (39), $T = nT_c = 2mT_s$, and $\omega = 2\pi f$.

The *proof of theorem* 3 is straightforward and is very similar to the proof of *Theorems* 12 and 4 in [1].

Hence, we compute the generalized PSD of $VBOC2_{(pn,n,\alpha)}$, or $G_{VBOC2_{(pn,n,\alpha)}}(\omega)$, as a Fourier transform of the ACF such as

$$G_{VBOC2_{(pn,n,\alpha)}}(\omega) = 2\underbrace{\int_0^{\infty} R_{VBOC2_{(pn,n,\alpha)}}(\tau)\cos(\omega\tau)d\tau}_{\sum_{i=1}^{4p}\underbrace{\int_{\tau_{i-1}}^{\tau_i} r_{i_{(pn,n,\alpha)}}(\tau)\cos(\omega\tau)d\tau}_{g_{i_{(np,n,\alpha)}}(\omega)}} \tag{51}$$

where t_i and $r_{i_{(pn,n,\alpha)}}(\tau)$ are given below in the recall of $R_{VBOC2_{(np,n,\alpha)}}(\tau)$ given by (38) and (39).

As we can see from (51) and (38) and (39) the computation of $G_{VBOC2_{(np,n,\alpha)}}(\omega)$ is in general a laborious process because it involves the computation of twelve integrals to obtain the generalized expression of $G_{VBOC2_{(np,n,\alpha)}}(\omega)$.

Let us provide the details for obtaining the twelve integrals that lead to the generalized expression of $G_{VBOC2_{(np,n,\alpha)}}(\omega)$ given by (51) by considering the integral of the general expression as follows:

$$g_{i_{(pn,n,\alpha)}}(\omega) = \int_{\tau_{i-1}}^{\tau_i} r_{i_{(pn,n,\alpha)}}(\tau)\cos(\omega\tau)d\tau \tag{52}$$

where $r_{i_{(pn,n,\alpha)}}(\tau)$ is given by

$$r_{i_{(pn,n,\alpha)}}(\tau) = a_{i_{(p,n,\alpha)}} + \tau b_{i_{(p,n,\alpha)}} \tag{53}$$

where $a_{i_{(p,n,\alpha)}}$ and $b_{i_{(p,n,\alpha)}}$ are independent of τ given in (38) and (39).

Substituting (53) into (52) the following is obtained:

$$g_{i_{(pn,n,\alpha)}}(\omega) = \left| \begin{array}{l} a_{i_{(p,n,\alpha)}} \underbrace{\int_{\tau_{i-1}}^{\tau_i}\cos(\omega\tau)d\tau}_{q_a(\tau_i,\tau_{i-1},\omega)} + \\ b_{i_{(p,n,\alpha)}} \underbrace{\int_{\tau_{i-1}}^{\tau_i}\tau\cos(\omega\tau)d\tau}_{q_b(\tau_i,\tau_{i-1},\omega)} \end{array} \right. \tag{54}$$

In order to compute (54) we compute two integrals $q_a(\tau_i,\tau_{i-1},\omega)$ as a direct integral as follows:

$$q_a(\tau_i,\tau_{i-1},\omega) = \int_{\tau_{i-1}}^{\tau_i}\cos(\omega\tau)d\tau = \frac{\sin(\omega\tau_i)-\sin(\omega\tau_{i-1})}{\omega} \tag{55}$$

and $q_b(\tau_i,\tau_{i-1},\omega)$ using integration by parts

$$q_b(t_i,t_{i-1},\omega) = \left| \begin{array}{l} \int_{\tau_{i-1}}^{\tau_i}\tau\cos(\omega\tau)d\tau = \\ \frac{\tau_i\sin(\omega t_i)-\tau_{i-1}\sin(\omega\tau_{i-1})}{\omega} + \\ \frac{\cos(\omega\tau_i)-\cos(\omega\tau_{i-1})}{\omega^2} \end{array} \right. \tag{56}$$

Substituting (56) and (55) into (54) we obtain the detailed expression of $g_{i_{(pn,n,\alpha)}}(\omega)$ as follows:

$$g_{i_{(pn,n,\alpha)}}(\omega) = \left| \begin{array}{l} a_{i_{(p,n,\alpha)}} \frac{\sin(\omega\tau_i)-\sin(\omega\tau_{i-1})}{\omega} + \\ b_{i_{(p,n,\alpha)}} \left| \begin{array}{l} \frac{\tau_i\sin(\omega\tau_i)-\tau_{i-1}\sin(\omega\tau_{i-1})}{\omega} + \\ \frac{\cos(\omega\tau_i)-\cos(\omega\tau_{i-1})}{\omega^2} \end{array} \right. \end{array} \right. \tag{57}$$

or

$$g_{i_{(pn,n,\alpha)}}(\omega) = \left| \begin{array}{l} \frac{\left[a_{i_{(p,n,\alpha)}}+b_{i_{(p,n,\alpha)}}\tau_i\right]\sin(\omega\tau_i)}{\omega} - \\ \frac{\left[a_{i_{(p,n,\alpha)}}+b_{i_{(p,n,\alpha)}}\tau_{i-1}\right]\sin(\omega\tau_{i-1})}{\omega} + \\ b_{i_{(p,n,\alpha)}} \frac{\cos(\omega\tau_i)-\cos(\omega\tau_{i-1})}{\omega^2} \end{array} \right. \tag{58}$$

Finally, substituting (58) into (51) we obtain the final expression for PSD of the generalized $VBOC2_{(pn,n,\alpha)}$ or $G_{VBOC2_{(pn,n,\alpha)}}(\omega)$ as follows

$$G_{VBOC2_{(pn,n,\alpha)}}(\omega) = 2\left| \begin{array}{l} \underbrace{\sum_{i=1}^{2p} g_{i_{(np,n,\alpha)}}(\omega)}_{G_1;\, p=\{2,4\}} \\ + \underbrace{\sum_{i=2p+1}^{4p} g_{i_{(np,n,\alpha)}}(\omega)}_{G_2;\, p=\{4\}} \end{array} \right. \tag{59}$$

Leaving the computation of G_1 and G_2 to the reader we obtain

$$G_{VBOC2_{(pn,n,\alpha)}}(\omega) = 2(G_s + G_C) \tag{60}$$

where

$$G_s = \sum_{i=1}^{2} G_{is} \text{ and } G_C = \sum_{i=1}^{2} G_{iC} \tag{61}$$

or

$$G_S = \frac{\left| \begin{array}{l} \underbrace{\frac{p-2}{p}\sin\left(\frac{\omega 2T_c}{p}\right)}_{p=\{2,4\}} \\ \underbrace{-\frac{p-2}{p}\sin\left(\frac{\omega 2T_c}{p}\right)+\frac{p-4}{p}\sin\left(\frac{\omega 4T_c}{p}\right)}_{p=\{4\}} \end{array} \right.}{\omega} = 0 \tag{62}$$

and

$$G_C = \frac{\underbrace{\left|\begin{array}{c} -4p\cos\left[\frac{\omega T_C(1-\alpha)}{2p}\right]+4(1-p)\cos\left[\frac{\omega T_C(1+\alpha)}{2p}\right] \\ +2p\cos\left[\frac{\omega T_C(1-\alpha)}{p}\right]+4(p-1)\cos\left(\frac{\omega T_C}{p}\right) \\ +2(p-2)\cos\left[\frac{\omega T_C(1+\alpha)}{p}\right]+4(1-p)\cos\left[\frac{\omega T_C(3-\alpha)}{2p}\right] \\ +4(2-p)\cos\left[\frac{\omega T_C(3+\alpha)}{2p}\right]+(4p-7)\cos\left(\frac{\omega 2T_C}{p}\right)+4p-1 \end{array}\right.}_{p=\{2,4\}} \underbrace{\left|\begin{array}{c} (12-5p)\cos\left[\frac{\omega T_C(5-\alpha)}{2p}\right]+(8-3p)\cos\left[\frac{\omega T_C(5+\alpha)}{2p}\right] \\ +(3p-8)\cos\left[\frac{\omega T_C(3-\alpha)}{p}\right]+(3p-8)\cos\left(\frac{\omega 3T_C}{p}\right) \\ +(2p-8)\cos\left[\frac{\omega T_C(3+\alpha)}{p}\right]+(12-4p)\cos\left[\frac{\omega T_C(7-\alpha)}{2p}\right] \\ +(16-4p)\cos\left[\frac{\omega T_C(7+\alpha)}{2p}\right] \\ +(4p-15)\cos\left(\frac{\omega 4T_C}{p}\right)+(4p-9)\cos\left(\frac{\omega 2T_C}{p}\right) \end{array}\right.}_{p=\{4\}}}{\omega^2 T_c} \tag{63}$$

Since, $G_S = 0$ for any values of p; hence, (60) becomes

$$G_{VBOC2_{(pn,n,\alpha)}}(\omega) = 2G_C \tag{64}$$

Now, let us compute the individual expressions of $G_{VBOC2_{(pn,n,\alpha)}}(\omega)$ for values of $p = \{2,4\}$ based on (63) or (64).

First, for $p = \{2\}$ from (63) or (64) we obtain

$$G_{VBOC2_{(2n,n,\alpha)}}(\omega) = \frac{2\left|\begin{array}{c} -8\cos\left[\frac{\omega T_C(1-\alpha)}{4}\right]-4\cos\left[\frac{\omega T_C(1+\alpha)}{4}\right] \\ +4\cos\left[\frac{\omega T_C(1-\alpha)}{2}\right]+4\cos\left(\frac{\omega T_C}{2}\right) \\ -4\cos\left[\frac{\omega T_C(3-\alpha)}{4}\right]+\cos\left(\frac{\omega T_C}{4}\right)+7 \end{array}\right.}{\omega^2 T_c} \tag{65}$$

Equation (65) is identical in Appendix A of Chap. 7 of [5].

Second and finally, for $p = \{4\}$ from (63) or (64) we obtain

$$G_{VBOC2_{(4n,n,\alpha)}}(\omega) = \frac{2\left|\begin{array}{c} -16\cos\left[\frac{\omega T_C(1-\alpha)}{8}\right]-12\cos\left[\frac{\omega T_C(1+\alpha)}{8}\right] \\ +8\cos\left[\frac{\omega T_C(1-\alpha)}{4}\right]+12\cos\left(\frac{\omega T_C}{4}\right) \\ +4\cos\left[\frac{\omega T_C(1+\alpha)}{4}\right]-12\cos\left[\frac{\omega T_C(3-\alpha)}{8}\right] \\ -8\cos\left[\frac{\omega T_C(3+\alpha)}{8}\right]+9\cos\left(\frac{\omega T_C}{2}\right)+15 \\ -8\cos\left[\frac{\omega T_C(5-\alpha)}{8}\right]-4\cos\left[\frac{\omega T_C(5+\alpha)}{8}\right] \\ +4\cos\left[\frac{\omega T_C(3-\alpha)}{4}\right]+4\cos\left(\frac{\omega 3T_C}{4}\right) \\ -4\cos\left[\frac{\omega T_C(7-\alpha)}{8}\right] \\ +\cos(\omega T_c)+7\cos\left(\frac{\omega T_C}{2}\right) \end{array}\right.}{\omega^2 T_c} \tag{66}$$

or

$$G_{VBOC2_{(4n,n,\alpha)}}(\omega) = \frac{2\left|\begin{array}{c} -16\cos\left[\frac{\omega T_C(1-\alpha)}{8}\right]-12\cos\left[\frac{\omega T_C(1+\alpha)}{8}\right] \\ +8\cos\left[\frac{\omega T_C(1-\alpha)}{4}\right]+12\cos\left(\frac{\omega T_C}{4}\right) \\ +4\cos\left[\frac{\omega T_C(1+\alpha)}{4}\right]-12\cos\left[\frac{\omega T_C(3-\alpha)}{8}\right] \\ -8\cos\left[\frac{\omega T_C(3+\alpha)}{8}\right]+16\cos\left(\frac{\omega T_C}{2}\right)+15 \\ -8\cos\left[\frac{\omega T_C(5-\alpha)}{8}\right]-4\cos\left[\frac{\omega T_C(5+\alpha)}{8}\right] \\ +4\cos\left[\frac{\omega T_C(3-\alpha)}{4}\right]+4\cos\left(\frac{\omega 3T_C}{4}\right) \\ -4\cos\left[\frac{\omega T_C(7-\alpha)}{8}\right]+\cos(\omega T_c) \end{array}\right.}{\omega^2 T_c} \tag{67}$$

Equation (67) is identical in Appendix A of Chap. 7 of [5].

Equations (51) through (67) complete the proof of *theorem* 3.

Definition 7, *theorem* 3 and *corollaries* 9 and 10 are the most important *definition*, *theorem*, and *corollaries* of PSDs of the generalized, $VBOC2_{(m,n,\alpha)}(t)$, or $G_{VBOC2_{(m,n,\alpha)}}(f)$.

This concludes $VBOC2(\alpha, 1-\alpha)$ generalized PSD definition and discussion; next, we continue with $VBOC2(\alpha, 1-\alpha)$ numerical results and one example as it pertains to the GPS M-code [3]-[5], [8]-[10].

3 Numerical, Theoretical Results

Next, let us consider a $VBOC2(\alpha, 1-\alpha)$ example. In coming up with an example it is important to know that $VBOC2_{(1,1,\alpha)}(t) = VBOC1_{(1,1,\alpha)}(t)$; hence, this is already discussed in [1], [2]; i.e., $VBOC2_{(1,1,\alpha=0.5)}(t)$ is just as good as $VBOC1_{(1,1,\alpha=0.5)}(t)$ instead of $BOC_{(1,1)}(t)$ which is currently proposed in GPS III L1C or L2C signals [11]-[15].

3.1 *$VBOC2(\alpha, 1-\alpha)$ Examples*

Now that we have developed this intellectual apparatus on the profound meaning of ACF $R_{VBOC2_{(m,n,\alpha)}}(\tau)$ and PSD $G_{VBOC2_{(m,n,\alpha)}}(f)$ functions let us consider a more specific example of $VBOC2_{(2,1,\alpha)}(t)$ and its corresponding ACF for $VBOC2_{(2,1,\alpha)}(t)$ or $R_{VBOC2_{(2,1,\alpha)}}(\tau)$.

Corollary 11: From *definition* 1, $VBOC2_{(2,1,\alpha)}(t)$ is simply

$$VBOC2_{(m=2,n=1,\alpha)}(t) = VBOC2_{(m=2,n=1,\alpha)}(t \pm 2T_s) \tag{68}$$

and the relation between T_s and T_c is given by

$$4T_s = T_c \tag{68a}$$

hence,

$$VBOC2_{(m=2,n=1,\alpha)}(t) = VBOC2_{(m=2,n=1,\alpha)}(t \pm 0.5T_c) \tag{69}$$

Corollary 12: From *definition* 2 $VBOC2_{(m=2,n=1,\alpha)}(t)$ waveform is an odd function for all values of $-\infty < t < \infty$.

$$VBOC2_{(m=2,n=1,\alpha)}(t) = -VBOC2_{(m=2,n=1,\alpha)}(-t) \tag{70}$$

Corollary 13: From *definition* 3 $VBOC2_{(m=2,n=1,\alpha)}(t)$ waveform is a periodic function with period T_s, for all values of m even and t $-\infty < t < \infty$.

$$VBOC2_{(m=2,n=1,\alpha)}(t) = VBOC2_{(m=2,n=1,\alpha)}(t \pm T_s) \tag{71}$$

Corollary 14: From *definition* 4: The odd/even sub-carrier period of $VBOC2_{(m=2,n=1,\alpha)}(t)$ is the superposition of two pulses: a rising pulse and a falling pulse with amplitude/pulse widths, $+1/w_r$ or $+1/w_f$ and $-1/w_f$ or $-1/w_r$ respectively, or a falling pulse and a rising pulse amplitude/pulse widths $-1/w_f$ or $-1/w_r$ and $+1/w_r$ or $+1/w_f$ respectively that satisfy the following

$$w_r = 0.5T_s(1 \pm \alpha) = 0.5\frac{T_c(1 \pm \alpha)}{4}, \tag{72}$$

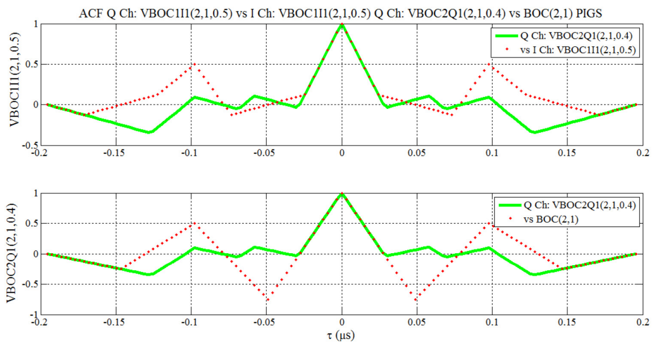

FIGURE 1: ACF of $VBOC2_{(2n,n,0.4)}(t)$ on Q channel vs. $VBOC1_{(2n,n,0.5)}(t)$ on I channel and $VBOC2_{(2n,n,0.4)}(t)$ vs. $BOC_{(2n,n)}(t)$ on Q channel.

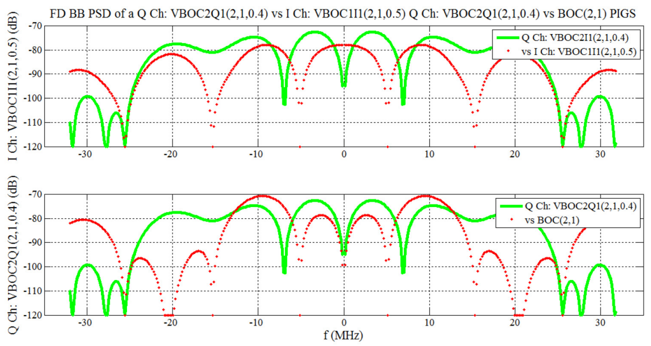

FIGURE 2: PSD of $VBOC2_{(2n,n,0.4)}(t)$ on Q channel vs. $VBOC1_{(2n,n,0.5)}(t)$ on I channel and $VBOC2_{(2n,n,0.4)}(t)$ vs. $BOC_{(2n,n)}(t)$ on Q channel.

$$w_f = 0.5T_s(1 \mp \alpha) = 0.5\frac{T_c(1\mp\alpha)}{4} \text{ [4]} \tag{73}$$

$$0 \le w_r \le 0.5T_c \text{ or } 0.5T_c \le w_f = T_c;\ 0 \le \alpha \le 1 \tag{73a}$$

Hence, the waveform $VBOC2_{(m=2,n=1,\alpha)}(t)$ can be obtained as

$$VBOC2_{(2,1,\alpha)}(t) = \begin{matrix} \sum_{\substack{q=-\infty \\ q:\,odd}}^{\infty} \left[p_{w_r}(t_1, w_r) - p_{w_f}(t_2, w_f)\right] \\ + \sum_{\substack{q=-\infty \\ q:\,even}}^{\infty} \left[p_{w_f}(t_1, w_f) - p_{w_r}(t_2, w_r)\right] \end{matrix} \tag{74}$$

where

$$t_1 = t - 0.5w_r - qT_s \tag{75}$$

$$t_2 = t - 0.5w_f - w_r - qT_s \tag{76}$$

and $p_w(t - t_0, w)$ is a unit rectangular pulse function with width w centered at t_0 with amplitude $+1$.

It is straightforward to show that $VBOC2_{(m=2,n=1,\alpha)}(t)$ obeys *theorem* 1 of [1]; i.e., for $VBOC2_{(m=2,n=1,\alpha)}(t)$ $0 \le \alpha \le 1$.

The ACF for $VBOC2_{(2,1,\alpha)}(t)$ is defined as $R_{VBOC2_{(2,1,\alpha)}}(\tau)$ given by *definition* 25: in Chap. 7 in [5] (full convolution for m even), which, for $0 \le \alpha \le 1/3$, is given by

$$R_{VBOC2_{(2,1,\alpha)}}(\tau) = \left| \begin{array}{ll} 1 - \frac{7|\tau|}{T_c}, & 0 \le |\tau| \le \frac{T_c(1-\alpha)}{4} \\ 2\alpha - 1 + \frac{|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{4} \le |\tau| \le \frac{T_c(1+\alpha)}{4} \\ \alpha - 2 + \frac{5|\tau|}{T_c}, & \frac{T_c(1+\alpha)}{4} \le |\tau| \le \frac{T_c(1-\alpha)}{2} \\ -\alpha + \frac{|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{2} \le |\tau| \le \frac{T_c}{2} \\ 2 - \alpha - \frac{3|\tau|}{T_c}, & \frac{T_c}{2} \le |\tau| \le \frac{T_c(3-\alpha)}{4} \\ \frac{|\tau|}{T_c} - 1, & \frac{T_c(3-\alpha)}{4} \le |\tau| \le T_c \\ 0, & |\tau| \ge T_c = 4T_s \end{array} \right. \tag{77}$$

and for $1/3 \le \alpha \le 1$

$$R_{VBOC2_{(2,1,\alpha)}}(\tau) = \left| \begin{array}{ll} 1 - \frac{7|\tau|}{T_c}, & 0 \le |\tau| \le \frac{T_c(1-\alpha)}{4} \\ 2\alpha - 1 + \frac{|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{4} \le |\tau| \le \frac{T_c(1-\alpha)}{2} \\ 1 - \frac{3|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{2} \le |\tau| \le \frac{T_c(1+\alpha)}{4} \\ -\alpha + \frac{|\tau|}{T_c}, & \frac{T_c(1+\alpha)}{4} \le |\tau| \le \frac{T_c}{2} \\ 2 - \alpha - \frac{3|\tau|}{T_c}, & \frac{T_c}{2} \le |\tau| \le \frac{T_c(3-\alpha)}{4} \\ \frac{|\tau|}{T_c} - 1, & \frac{T_c(3-\alpha)}{4} \le |\tau| \le T_c \\ 0, & |\tau| \ge T_c = 4T_s \end{array} \right. \tag{78}$$

Corollary 15: Prove that the ACFs of $VBOC2_{(2,1,\alpha)}(t)$ given by (77) and (78) will result in the $BOC_{(2,1)}(t)$ for $\alpha = 0$ as indicated in *corollary* 1 and $BOC_{(1,1)}(t)$ for $\alpha = 1$ as indicated in *corollary* 2.

Proof of corollary 15: The *proof of corollary* 15 is straightforward. First, we substitute values of $\alpha = 0$ in (77) and we get $R_{VBOC2_{(2n,n,\alpha=0)}}(\tau) = R_{BOC_{(2n,n)}}(\tau)$ as indicated in *corollary* 1 of $VBOC2_{(2,1,\alpha)}(t)$.

$$R_{VBOC2_{(2n,n,\alpha=0)}}(\tau) = \underbrace{\left\| \begin{array}{ll} 1 - \frac{7|\tau|}{T_c}, & |\tau| \le \frac{T_c}{4} = T_s \\ -2 + \frac{5|\tau|}{T_c}, & \frac{T_c}{4} \le |\tau| \le \frac{T_c}{2} \\ 2 - \frac{3|\tau|}{T_c}, & \frac{T_c}{2} \le |\tau| \le \frac{3T_c}{4} \\ \frac{|\tau|}{T_c} - 1, & \frac{3T_c}{4} \le |\tau| \le T_c \\ 0, & |\tau| \ge T_c \end{array} \right.}_{R_{BOC_{(2n,n)}}(\tau)} \tag{79}$$

Second, we substitute values of $\alpha = 1$ in (78) and we get $R_{VBOC2_{(2n,n,\alpha=1)}}(\tau) = R_{BOC_{(n,n)}}(\tau)$ as indicated in *corollary* 2 of $VBOC2_{(2,1,\alpha)}(t)$.

$$R_{VBOC2_{(2n,n,\alpha=1)}}(\tau) = \underbrace{\left\| \begin{array}{ll} 1 - \frac{3|\tau|}{T_c}, & |\tau| \le \frac{T_c}{2} \\ -1 + \frac{|\tau|}{T_c}, & \frac{T_c}{2} \le |\tau| \le T_c \\ 0, & |\tau| \ge T_c \end{array} \right.}_{R_{BOC_{(n,n)}}(\tau)} \tag{80}$$

Equations (79) and (80) complete the *proof of corollary* 15.

[4] For ease of analysis we will assume that the $+$ for w_r and $-$ for w_f.

Corollary 16: Finally, the PSD of $VBOC2_{(2,1,\alpha)}(t)$ or $G_{VBOC2_{(2,1,\alpha)}}(\omega)$ is given by

$$G_{VBOC2_{(2,1,\alpha)}}(\omega)=\frac{\left|\begin{array}{c}4-16\cos\left[\frac{\omega T_c(1-\alpha)}{4}\right]-8\cos\left[\frac{\omega T_c(1+\alpha)}{4}\right]\\+8\cos\left[\frac{\omega T_c(1-\alpha)}{2}\right]+8\cos\left(\frac{\omega T_c}{2}\right)\\-8\cos\left[\frac{\omega T_c(3-\alpha)}{4}\right]+2\cos(\omega T_c)\end{array}\right.}{\omega^2 T_c} \tag{81}$$

After some simplifications it can be shown that

$$G_{VBOC2_{(2,1,\alpha=0)}}(\omega)=\left|\begin{array}{c}\frac{4}{\omega^2 T_c}\tan^2\left(\frac{\omega T_c}{8}\right)\sin^2\left(\frac{\omega T_c}{2}\right)=\\\frac{1}{\omega^2 T_s}\tan^2\left(\frac{\omega T_s}{2}\right)\sin^2(2\omega T_s)\end{array}\right. \tag{82}$$

as indicated in *corollary* 1 as depicted in Figs. 42 and 90 of [6], also

$$G_{VBOC2_{(2,1,\alpha=1)}}(f)=\left|\begin{array}{c}\frac{4T_c}{\left(\frac{\omega T_c}{2}\right)^2}\sin^4\left(\frac{\omega T_c}{4}\right)=\\T_c\mathrm{sinc}^2(fT_c)\tan^2\left(\frac{\pi f T_c}{2}\right)=\\G_{BOC_{(1,1)}}(f)\end{array}\right. \tag{83}$$

as indicated in *corollary* 2 as depicted in Figs. 42, 90 of [6].

In Figure 1 the ACF of $VBOC2_{(2,1,\alpha=0.4)}(t)$ or $r_{VBOC2_{(2,1,\alpha=0.4)}}(\tau)$ is shown with solid green has lower secondary peaks than the ACF of $VBOC1_{(2,1,\alpha=0.5)}(t)$ or $r_{VBOC1_{(2,1,\alpha=0.5)}}(\tau)$ in dotted red; hence, $VBOC2_{(2,1,\alpha=0.4)}(t)$ should offer better interference protection than $VBOC1_{(2,1,\alpha=0.5)}(t)$ or the proposed replacement of the GPS III M-code on both L1 and L2 frequencies [1], [2], [4], [5]. In Figure 1 also the PSD of $VBOC2_{(2,1,\alpha=0.4)}(t)$ or $r_{VBOC2_{(2,1,\alpha=0.4)}}(\tau)$ is illustrated with solid green has lower secondary peaks than the ACF of $BOC_{(2,1)}(t)$ or $r_{BOC_{(2,1)}}(\tau)$ in dotted red; hence, $VBOC2_{(2,1,\alpha=0.4)}(t)$ should offer better interference protection than $BOC_{(2,1)}(t)$ or GPS M-code [3]-[5], [8]-[10].

Similarly, in Figure 2 the PSD of $VBOC2_{(2,1,\alpha=0.4)}(t)$ or $G_{VBOC2_{(2,1,\alpha=0.4)}}(f)$ is depicted with solid green is quasi-flatter and wider than the PSD of $VBOC1_{(2,1,\alpha=0.5)}(t)$ or $G_{VBOC1_{(2,1,\alpha=0.5)}}(f)$ in dotted red; hence, $VBOC2_{(2,1,\alpha=0.4)}(t)$ should offer better interference protection than $VBOC1_{(2,1,\alpha=0.5)}(t)$ or the proposed replacement of the GPS III M-code on both L1 and L2 frequencies [1], [2], [4], [5].

In Figure 2 also the PSD of $VBOC2_{(2,1,\alpha=0.5)}(t)$ or $G_{VBOC2_{(2,1,\alpha=0.4)}}(f)$ is displayed with solid green is quasi-flatter and wider than the PSD of $BOC_{(2,1)}(t)$ or $G_{BOC_{(2,1)}}(f)$ in dotted red; hence, $VBOC2_{(2,1,\alpha=0.4)}(t)$ should offer better interference protection than $BOC_{(2,1)}(t)$ or GPS M-code [3]-[5], [8]-[10].

Interference with the current GPS L1 BPSK is avoided by having PSD of $VBOC2_{(2,1,\alpha=0.4)}(t)$ orthogonal with the PSD of GPS L1 BPSK signal [5].

This completes the detailed discussion on $VBOC2(\alpha,1-\alpha)$ ACFs and PSDs which contains seven definitions, four theorems, and sixteen corollaries.

The $VBOC2(\alpha,1-\alpha)$ continuity theorems and optimization theorems are provided in [3].

$VBOC1(\alpha)$ [1], [2] and $VBOC2(\alpha,1-\alpha)$ [3] results are provided in the companion manuscript given here as a reference [4].

4 Conclusions

This paper is the first complete discussion on pure signal design for the first generation $VBOC2(\alpha,1-\alpha)$ generalized multidimensional geolocation modulation waveforms.

Contrast the results of this paper with previous signal design methodologies, this paper offers for the first time a complete pure signal design methodology subject to both signal design and optimization parameter α and generalized signal design and optimization parameter p.

Signal parameters α and p not only define the waveform $VBOC2(\alpha,1-\alpha)$ and generalized ACFs and PSDs but they also play a very important role in the optimization of $VBOC2(\alpha,1-\alpha)$ generalized ACFs and PSDs. The computational technique offers a unique and original description of the generalized ACFs and PSDs of $VBOC2(\alpha,1-\alpha)$ as functions of both α and p.

In the paper it is argued that the selection of $BOC(10,5)$ (or the military code or M-Code) on both GPS L1 and L2 frequencies is entirely arbitrary because BOC modulation is a special case of $VBOC2(\alpha,1-\alpha)$ for $\alpha=0$ or $\alpha=1$; hence, all the current state-of-the-art GNSS waveforms exhibit sub-optimal signal design performance even at the end-user when generalized global objective functions are applied.

The above is based on a discussion of $VBOC2(\alpha,1-\alpha)$ pure signal optimization in [1]: (1) the criteria for validating the closed form expression of the generalized ACF of $VBOC2(\alpha,1-\alpha)$ known as a set of *continuity theorems*; and (2) the criteria for selecting the optimum $0\le\alpha\le1$ based on a set of criteria known as *optimization theorems* regardless of generalized parameter p (or subcarrier frequency).

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