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Research Article

VBOC1(α) ACF Pure Signal Optimization

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This journal paper examines common optimization algorithms, such as sum, product, and mean square sense applied to the optimization of the autocorrelation functions (ACF) of GPS L1C and M-code like signals such as the variable binary offset carrier (VBOC). Before the different versions are examined for efficiency, on GPS L1C and M-code like signals such as VBOC1 which vary with the parameter of signal design and optimization alpha, they are checked to make sure that their corresponding ACFs are valid via a set of conditions known as *continuity theorems*. Afterwards, by employing any of the optimization algorithms such as sum, product, and mean square sense on GPS L1C and M-code like signals such as VBOC1 we are able to produce ACFs that are one *hundred percent more efficient* than the corresponding ACFs of the GPS L1C and M-code signals.

Index Terms—Pulse generation, pulse amplitude modulation, pulse width modulation, multidimensional sequences, signal design, signal analysis, generalized functions, time-frequency analysis, minimization methods, optimization methods.

1 Introduction

The main objective of this paper is discuss $VBOC1(\alpha)$: pure signal optimization: (1) for the first time ever in a journal publication is the discussion of a complete set of continuity theorems; (2) for the first time ever in a journal publication is the discussion of a complete set of optimization theorems; as a function of the signal design and optimization parameter, α, and generalized parameter, p which denotes the order of the $VBOC(\alpha)$.

What are the ACF continuity theorems and why are they so important? Is just to be sure that the derivation of the ACF is well derived? If so, it is enough with the continuity theorems?; i.e. could a wrong ACF that fulfils with the continuity theorems exist? The ACF continuity theorems check the accuracy, integrity, and uniqueness of the closed form expression of the generalized ACF as a function of the signal design and optimization parameter, α, and generalized parameter, p which denotes the order of the $VBOC(\alpha)$; i.e., they are there to be sure that the wrong ACF who fulfils the contiguity theorems does not exist. They are very important because if a function fails the ACF continuity theorems it will most certainly fail the optimization theorems or the end user generalized objective functions (Progri et al. 2006) [10], (Michalson and Progri 2006), (Progri et al. 2009) [12], (Progri 2003) [13], and (Progri et al. 2007) [7].

The argument can be made the for $VBOC1(\alpha)$ it may appear as straight forward for the ACF to obey the ACF continuity theorems and thus produce good results from optimization theorems as a function of the signal design and optimization parameter, α, and generalized parameter, p which denotes the order of the $VBOC(\alpha)$.

However, such was not initially the case for the ACF of $VBOC2(\alpha, 1-\alpha)$ (Progri 2015b and c) [2], [3]. The process of coming up with a generalized expression of the ACF for $VBOC2(\alpha, 1-\alpha)$ that would obey the continuity theorems discussed here was very difficult and very frustrating. One can only imagine how difficult it might be to come up with a generalized expression of the ACF of the *k*th generation $VBOCk(\alpha_1, \alpha_2, \cdots, \alpha_k)$ that would obey the continuity theorems discussed here.

There are four *continuity theorems* discussed in this paper. The first continuity theorem ensures that $R_{VBOC1_{(m=np,n,\alpha)}}(\tau)$, *given by* (22) of (Progri 2015a) [1] is a continuous function for every values of τ, $p = \{1,2,3,4\}$, and continuous values of $0 \le \alpha \le 1$; i.e., this is the fundamental continuity theorem. The second *continuity theorem*; ensures that $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], is a continuous function for *every values of* τ, $p = \{1,2,3,4\}$, and only for discrete values of $\alpha = \{0,1\}$.

What are the optimization theorems and why are they so important? Optimization theorems ensure that we can find the optimum value of the signal design and optimization parameter, α, regardless of the generalized parameter, p, and optimization criteria. This journal paper provides an awesome opportunity to illustrate that even though four different optimization criteria have been applied to $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$; the optimum value of the signal design and optimization parameter, α, is $\alpha = 0.5$ regardless of the generalized parameter, p.

Finally, as a result of this work it is shown that $\alpha = 0.5$ offers 100% efficiency or for $\alpha = 0.5$ $VBOC1_{(n,n,\alpha)}$ is 100 % more efficient than either $BOC_{(m,n)}$ or $BPSK_0$ in terms of mean square sense in terms of out of phase ACF.

This paper is organized as follows. In Section 2 $VBOC1(\alpha)$ ACF pure signal optimization is discussed. Section 3 contains numerical results. Conclusion is provided in Section 4 and the paper is concluded with a list of references.

2 $VBOC1(\alpha)$ ACF Pure Signal Optimization

Detailed discussion on $VBOC1(\alpha)$ pure signal design is given in (Progri 2015a) [1]. $VBOC1(\alpha)$ ACF pure signal optimization includes: (1) $VBOC1(\alpha)$ generalized ACF continuity theorems and (2) generalized ACF optimization theorems.

2.1 Generalized ACF continuity theorems.

Theorem 1: *Prove that the generalized ACF* $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], obeys the *continuity theorem of the first kind;* i.e., *it is a continuous function for every values of* τ, $p = \{1,2,3,4\}$, *and continuous values of* $0 \le \alpha \le 1$.

Proof of theorem 1: The *proof of theorem* 1 is straightforward. In order to prove that *the* ACF, $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$, *given* by (22) of (Progri 2015a) [1], obeys the *continuity theorem of the first kind;* i.e., *it is a continuous function for every values of* τ, $p = \{1,2,3,4\}$, *and continuous values of* $0 \le \alpha \le 1$; it suffices to prove that $R_{VBOC1_{(pn=m,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], and $p = \{1,2,3,4\}$ obeys the *continuity theorem of the first kind;* i.e., *it is a continuous function for every values of* τ.

Because the ACF $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$ is an even function of τ it is sufficient to check the continuity of $R_{VBOC1_{(pn=m,n,\alpha)}}(\tau)$ for values of τ given by

$$\underbrace{\tau_{\mp i}(\alpha)}_{i=\{1,2,\cdots 3p\}} = \underbrace{\underbrace{\underbrace{\underbrace{\begin{bmatrix} \underbrace{\frac{T_c(1-\alpha)}{2p} \mp}_{\tau_{\mp 1}(\alpha)} \\ \underbrace{\frac{T_c(1+\alpha)}{2p} \mp}_{\tau_{\mp 2}(\alpha)} \\ \underbrace{\frac{T_c}{p} \mp}_{\tau_{\mp 3}(\alpha)} \end{bmatrix}}_{p=1} \begin{matrix} \underbrace{\frac{T_c(3-\alpha)}{2p} \mp}_{\tau_{\mp 4}(\alpha)} \\ \underbrace{\frac{T_c(3+\alpha)}{2p} \mp}_{\tau_{\mp 5}(\alpha)} \\ \underbrace{\frac{2T_c}{p} \mp}_{\tau_{\mp 6}(\alpha)} \end{matrix}}_{p=2} \begin{matrix} \underbrace{\frac{T_c(5-\alpha)}{2p} \mp}_{\tau_{\mp 7}(\alpha)} \\ \underbrace{\frac{T_c(5+\alpha)}{2p} \mp}_{\tau_{\mp 8}(\alpha)} \\ \underbrace{\frac{3T_c}{p} \mp}_{\tau_{\mp 9}(\alpha)} \end{matrix}}_{p=3} \begin{matrix} \underbrace{\frac{T_c(7-\alpha)}{2p} \mp}_{\tau_{\mp 10}(\alpha)} \\ \underbrace{\frac{T_c(7+\alpha)}{2p} \mp}_{\tau_{\mp 11}(\alpha)} \\ \underbrace{\frac{4T_c}{p} \mp}_{\tau_{\mp 12}(\alpha)} \end{matrix}}_{p=4}. \quad (1)$$

First, we substitute values of $\tau = \tau_{\mp 1}(\alpha)$ in $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], and we get

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau = \tau_{-1}(\alpha)] = \left| \begin{array}{l} 1 - \dfrac{(4p-1)(1-\alpha)}{2p} \\ \dfrac{1-2p+(4p-1)\alpha}{2p} \end{array} \right.$$

$$=\left\{\underbrace{\frac{-1+3\alpha}{2}}_{p=1},\underbrace{\frac{-3+7\alpha}{4}}_{p=2},\underbrace{\frac{-5+11\alpha}{6}}_{p=3},\underbrace{\frac{-7+15\alpha}{8}}_{p=4}\right\} \quad (2)$$

On the other hand

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{+1}(\alpha)]=\left|\begin{array}{l}\frac{-(p-1)+(2p-1)\alpha}{p}-\frac{1-\alpha}{2p}\\ =\underbrace{\frac{1-2p+(4p-1)\alpha}{2p}}_{=R_{VBOC1_{(pn,n,\alpha)}}[\tau_{-1}(\alpha)]}\end{array}\right.$$

$$=\left\{\underbrace{\frac{-1+3\alpha}{2}}_{p=1},\underbrace{\frac{-3+7\alpha}{4}}_{p=2},\underbrace{\frac{-5+11\alpha}{6}}_{p=3},\underbrace{\frac{-7+15\alpha}{8}}_{p=4}\right\} \quad (3)$$

Second, we substitute values of $\tau=\tau_{\mp 2}(\alpha)$ in $R_{VBOC1_{(pn=n,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], and we get

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{-2}(\alpha)]=\left|\begin{array}{l}\frac{-(p-1)+(2p-1)\alpha}{p}-\frac{1+\alpha}{2p}\\ =\frac{1-2p+(4p-3)\alpha}{2p}\end{array}\right.$$

$$=\left\{\underbrace{\frac{-1+\alpha}{2}}_{p=1},\underbrace{\frac{-3+5\alpha}{4}}_{p=2},\underbrace{\frac{-5+9\alpha}{6}}_{p=3},\underbrace{\frac{-7+13\alpha}{8}}_{p=4}\right\} \quad (4)$$

On the other hand

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{+2}(\alpha)]=\left|\begin{array}{l}-\frac{3p-2}{p}+\frac{(4p-3)(1+\alpha)}{2p}\\ =\frac{1-2p+(4p-3)\alpha}{2p}\end{array}\right.$$

$$=\underbrace{\left\{\underbrace{\frac{-1+\alpha}{2}}_{p=1},\underbrace{\frac{-3+5\alpha}{4}}_{p=2},\underbrace{\frac{-5+9\alpha}{6}}_{p=3},\underbrace{\frac{-7+13\alpha}{8}}_{p=4}\right\}}_{=R_{VBOC1_{(pn,n,\alpha)}}[\tau_{-2}(\alpha)]}. \quad (5)$$

Third, we substitute values of $\tau=\tau_{\mp 3}$ in $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], and we get

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{-3}(\alpha)]=\frac{2-3p}{p}+\frac{4p-3}{p}$$

$$=\frac{p-1}{p}=\left\{\underbrace{0}_{p=1},\underbrace{\frac{1}{2}}_{p=2},\underbrace{\frac{2}{3}}_{p=3},\underbrace{\frac{3}{4}}_{p=4}\right\}. \quad (6)$$

On the other hand we have

$$\underbrace{R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{+3}(\alpha)]=\frac{5p-6-4p+5=p-1}{p}}_{=R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{-3}(\alpha)]}$$

$$=\left\{\underbrace{0}_{p=1},\underbrace{\frac{1}{2}}_{p=2},\underbrace{\frac{2}{3}}_{p=3},\underbrace{\frac{3}{4}}_{p=4}\right\}. \quad (7)$$

Fourth, we substitute values of $\tau=\tau_{\mp 4}(\alpha)$ in $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], and we get

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{-4}(\alpha)]=\left|\begin{array}{l}\frac{5p-6}{p}-\frac{(4p-5)(3-\alpha)}{2p}\\ =\frac{3-2p+(4p-5)\alpha}{2p}\end{array}\right.$$

$$=\left\{\underbrace{\frac{-1+3\alpha}{4}}_{p=2},\underbrace{\frac{-3+7\alpha}{6}}_{p=3},\underbrace{\frac{-5+11\alpha}{8}}_{p=4}\right\}. \quad (8)$$

On the other hand we have

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{+4}(\alpha)]=\left|\begin{array}{l}\frac{-(p-3)+(2p-3)\alpha}{p}-\frac{3-\alpha}{2p}\\ =\frac{3-2p+(4p-5)\alpha}{2p}\end{array}\right.$$

$$=\underbrace{\left\{\underbrace{\frac{-1+3\alpha}{4}}_{p=2},\underbrace{\frac{-3+7\alpha}{6}}_{p=3},\underbrace{\frac{-5+11\alpha}{8}}_{p=4}\right\}}_{=R_{VBOC1_{(pn,n,\alpha)}}[\tau_{-4}(\alpha)]}. \quad (9)$$

Fifth, we substitute values of $\tau=\tau_{\mp 5}(\alpha)$ in $R_{VBOC1_{(np=2n,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], and we get

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{-5}(\alpha)]=\left|\begin{array}{l}\frac{-(p-3)+(2p-3)\alpha}{p}-\frac{3+\alpha}{2p}\\ =\frac{3-2p+(4p-7)\alpha}{2p}\end{array}\right.$$

$$=\left\{\underbrace{\frac{-1+\alpha}{4}}_{p=2},\underbrace{\frac{-3+5\alpha}{6}}_{p=3},\underbrace{\frac{-5+9\alpha}{8}}_{p=4}\right\}. \quad (10)$$

On the other hand we have

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{+5}(\alpha)]=\left|\begin{array}{l}-\frac{7p-12}{p}+\frac{(4p-7)(3+\alpha)}{2p}\\ =\frac{-2p+3+(4p-7)\alpha}{2p}\end{array}\right.$$

$$=\underbrace{\left\{\underbrace{\frac{-1+\alpha}{4}}_{p=2},\underbrace{\frac{-3+5\alpha}{6}}_{p=3},\underbrace{\frac{-5+9\alpha}{8}}_{p=4}\right\}}_{=R_{VBOC1_{(pn,n,\alpha)}}[\tau_{-5}(\alpha)]}. \quad (11)$$

Sixth, we substitute values of $\tau=\tau_{\mp 6}$ in $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], and we get

$$R_{VBOC1_{(pn,n,\alpha)}}(\tau=\tau_{-6})=-\frac{7p-12}{p}+\frac{2(4p-7)}{p}=\frac{p-2}{p}$$

$$=\left\{\underbrace{0}_{p=2},\underbrace{\frac{1}{3}}_{p=3},\underbrace{\frac{1}{2}}_{p=4}\right\}. \quad (12)$$

On the other hand we have

$$R_{VBOC1_{(pn,n,\alpha)}}(\tau=\tau_{+6})=\frac{9p-20-2(4p-9)}{p}=\frac{p-2}{p}$$

$$=\left\{\underbrace{0}_{p=2},\underbrace{\tfrac{1}{3}}_{p=3},\underbrace{\tfrac{1}{2}}_{p=4}\right\}=R_{VBOC1_{(pn,n,\alpha)}}(\tau_{-6}). \quad (13)$$

Seventh, we substitute values of $\tau=\tau_{\mp 7}(\alpha)$ in $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], and we get

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{-7}(\alpha)]=\left|\begin{array}{l}\frac{9p-20}{p}-\frac{(4p-9)(5-\alpha)}{2p}\\ =\frac{5-2p+(4p-9)\alpha}{2p}\end{array}\right.$$

$$=\left\{\underbrace{\tfrac{-1+3\alpha}{6}}_{p=3},\underbrace{\tfrac{-3+7\alpha}{8}}_{p=4}\right\}. \quad (14)$$

On the other hand we have

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{+7}(\alpha)]=\left|\begin{array}{l}\frac{-(p-5)+(2p-5)\alpha}{p}-\frac{(5-\alpha)}{2p}\\ \underbrace{=\frac{5-2p+(4p-9)\alpha}{2p}}_{=R_{VBOC1_{(pn,n,\alpha)}}[\tau_{-7}(\alpha)]}\end{array}\right.$$

$$=\left\{\underbrace{\tfrac{-1+3\alpha}{6}}_{p=3},\underbrace{\tfrac{-3+7\alpha}{8}}_{p=4}\right\}. \quad (15)$$

Eight, we substitute values of $\tau=\tau_{\mp 8}(\alpha)$ in $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], and we get

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{-8}(\alpha)]=\left|\begin{array}{l}\frac{-(p-5)+(2p-5)\alpha}{p}-\frac{5+\alpha}{2p}\\ =\frac{5-2p+(4p-11)\alpha}{2p}\end{array}\right.$$

$$=\left\{\underbrace{\tfrac{-1+\alpha}{6}}_{p=3},\underbrace{\tfrac{-3+5\alpha}{8}}_{p=4}\right\}. \quad (16)$$

On the other hand we have

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{+8}(\alpha)]=\underbrace{-1+\frac{5+\alpha}{2p}=\frac{-2p+5+\alpha}{2p}}_{=R_{VBOC1_{(pn,n,\alpha)}}[\tau_{-8}(\alpha)]}$$

$$=\left\{\underbrace{\tfrac{-1+\alpha}{6}}_{p=3},\underbrace{\tfrac{-3+5\alpha}{8}}_{p=4}\right\}. \quad (17)$$

Ninth, we substitute values of $\tau=\tau_{\mp 9}$ in $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], we get

$$R_{VBOC1_{(pn,n,\alpha)}}(\tau=\tau_{-9})=\frac{13p-42}{p}-\frac{3(4p-13)}{p}$$

$$=\frac{p-3}{p}=\left\{\underbrace{0}_{p=3},\underbrace{\tfrac{1}{4}}_{p=4}\right\}. \quad (18)$$

On the other hand we have

$$R_{VBOC1_{(pn,n,\alpha)}}(\tau=\tau_{+9})=\underbrace{\frac{13p-42}{p}-\frac{3(4p-13)}{p}=\frac{p-3}{p}}_{p=4,=R_{VBOC1_{(pn,n,\alpha)}}(\tau_{-9})}$$

$$=\frac{p-3}{p}=\left\{\underbrace{0}_{p=3},\underbrace{\tfrac{1}{4}}_{p=4}\right\}. \quad (19)$$

Tenth, we substitute values of $\tau=\tau_{\mp 10}(\alpha)$ in $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], we get

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{-10}(\alpha)]=\left|\begin{array}{l}\frac{13p-42}{p}-\frac{(7-\alpha)(4p-13)}{2p}\\ =\frac{-2p+7+(4p-13)\alpha}{2p}\end{array}\right.$$

$$=\left\{\underbrace{\tfrac{-1+3\alpha}{8}}_{p=4}\right\}. \quad (20)$$

On the other hand we have

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{+10}(\alpha)]=\left|\begin{array}{l}\frac{-(p-7)+(2p-7)\alpha}{p}-\frac{7-\alpha}{2p}\\ \underbrace{=\frac{7-2p+(4p-13)\alpha}{2p}}_{=R_{VBOC1_{(pn,n,\alpha)}}[\tau_{-10}(\alpha)]}\end{array}\right.$$

$$=\left\{\underbrace{\tfrac{-1+3\alpha}{8}}_{p=4}\right\}. \quad (21)$$

Eleventh, we substitute values of $\tau=\tau_{\mp 11}(\alpha)$ in $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], we get

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{-11}(\alpha)]=\left|\begin{array}{l}\frac{-(p-7)+(2p-7)\alpha}{p}-\frac{7+\alpha}{2p}\\ =\frac{7-2p+(4p-15)\alpha}{2p}\end{array}\right.$$

$$=\left\{\underbrace{\tfrac{-1+\alpha}{8}}_{p=4}\right\}. \quad (22)$$

On the other hand we have

$$R_{VBOC1_{(pn,n,\alpha)}}[\tau=\tau_{+11}(\alpha)]=\left|\begin{array}{l}-\frac{(15p-56)}{p}+\frac{(4p-15)(7+\alpha)}{2p}\\ \underbrace{=\frac{7-2p+(4p-15)\alpha}{2p}}_{=R_{VBOC1_{(pn,n,\alpha)}}[\tau_{-11}(\alpha)]}\end{array}\right.$$

$$=\left\{\underbrace{\tfrac{-1+\alpha}{8}}_{p=4}\right\}. \quad (23)$$

Twelfth and finally, we substitute values of $\tau=\tau_{\mp 12}$ in $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], we get

$$R_{VBOC1_{(pn,n,\alpha)}}(\tau=\tau_{-12})=-\frac{(15p-56)}{p}+\frac{(4p-15)4}{p}=\frac{p-4}{p}$$

$$=0=\left\{\underbrace{0}_{p=4}\right\}. \quad (24)$$

On the other hand we have

$$R_{VBOC1_{(pn,n,\alpha)}}(\tau = \tau_{+12}) = 0 = R_{VBOC1_{(pn,n,\alpha)}}(\tau_{-12}). \quad (25)$$

Equations (1) through (25) complete the proof of *theorem 1*.

Theorem 2: *Prove that the ACF* $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], obeys the *continuity theorem of the second kind;* i.e., *it is a continuous function for every values of* τ *and* $p = \{1,2,3,4\}$ and $\alpha = \{0,1\}$.

Proof of theorem 2: The *proof of theorem* 2 is straightforward. In order to prove that *the* ACF, $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], obeys the *continuity theorem of the second kind;* i.e., *it is a continuous function for every values of* τ , $p = \{1,2,3,4\}$ and $\alpha = \{0,1\}$ it suffices to prove $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], obeys the *continuity theorem of the second kind;* i.e., *it is a continuous function for every values of* τ.

Since, we already proved in *theorem* 1; i.e., *continuity theorem of the first kind* that $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], for *or every values of* τ, $p = \{1,2,3,4\}$ and for $\alpha = [0,1]$ we can employ the results of *theorem 1 to prove theorem* 2.

First, we prove that $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], obeys the *continuity theorem of the second kind* for $\{\tau_1(\alpha), \tau_2(\alpha)\}$.

From (1) and (2), and $p = \{1,2,3,4\}$ we have

$$R_{VBOC1_{(pn,n,\alpha=0)}}[\tau_1(0)] = \frac{1-2p+(4p-1)\alpha}{2p} = -1 + \frac{1}{2p}$$

$$= \left\{\frac{-1}{2}, \frac{-3}{4}, \frac{-5}{6}, \frac{-7}{8}\right\}. \quad (26)$$

One the other hand, from (3) and (4) and $p = \{1,2,3,4\}$ we have

$$R_{VBOC1_{(pn,n,\alpha=0)}}[\tau_2(0)] = \frac{1-2p+(4p-3)\alpha}{2p} = -1 + \frac{1}{2p}$$

$$= \underbrace{\left\{\frac{-1}{2}, \frac{-3}{4}, \frac{-5}{6}, \frac{-7}{8}\right\}}_{=R_{VBOC1_{(pn,n,\alpha=0)}}[\tau_1(0)]}. \quad (27)$$

Again, from (1) and (2) and $p = \{1,2,3,4\}$ we have

$$R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_1(1)] = \frac{1-2p+4p-1}{2p} = 1 = R_{BPSK_0}[0], \quad (28)$$

Equation (28) results from *definition* 2 Appendix A, Chapter 7 of (Progri 2015) [1].

Finally from (3), (4), (6) and (7) for $p = \{1,2,3,4\}$ we have

$$R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_2(1)] = \frac{1-2p+4p-3}{2p}$$

$$= 1 - \frac{1}{p} = \left\{0, \frac{1}{2}, \frac{4}{6}, \frac{6}{8}\right\} = R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_3(1)], \quad (29)$$

Second, we prove that $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], obeys the *continuity theorem of the second kind* for $\{\tau_4(\alpha), \tau_5(\alpha)\}$.

From (8) and (9), for $p = \{2,3,4\}$, we have

$$R_{VBOC1_{(pn,n,\alpha=0)}}[\tau_4(0)] = \frac{3-2p+(4p-5)\alpha}{2p} = -1 + \frac{3}{2p}$$

$$= \left\{\frac{-1}{4}, \frac{-3}{6}, \frac{-5}{8}\right\}. \quad (30)$$

One the other hand, from (8), (9), (10), and (11), for $p = \{2,3,4\}$, we have

$$R_{VBOC1_{(pn,n,\alpha=0)}}[\tau_4(0)] = \frac{3-2p+(4p-5)\alpha}{2p} = -1 + \frac{3}{2p}$$

$$= \left\{\frac{-1}{4}, \frac{-3}{6}, \frac{-5}{8}\right\}. \quad (31)$$

Again from (6), (7), (8), and (9), for $p = \{2,3,4\}$, we have

$$R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_4(1)] = \frac{3-2p+4p-5}{2p} = 1 - \frac{1}{p} = \left\{\frac{1}{2}, \frac{4}{6}, \frac{6}{8}\right\}$$

$$= R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_3(1)], \quad (32)$$

One the other hand, from (10) and (11), for $p = \{2,3,4\}$, we have

$$R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_5(1)] = \frac{3-2p+4p-7}{2p} = 1 - \frac{2}{p} = \left\{0, \frac{2}{6}, \frac{4}{8}\right\}$$

$$= R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_6(1)], \quad (33)$$

Third, we prove that $R_{VBOC1_{(np,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], obeys the *continuity theorem of the second kind* for $\{\tau_7(\alpha), \tau_8(\alpha)\}$.

From (14) and (15), for $p = \{3,4\}$, we have

$$R_{VBOC1_{(pn,n,\alpha=0)}}[\tau_7(0)] = \frac{5-2p+(4p-9)\alpha}{2p} = -1 + \frac{5}{2p}$$

$$= \left\{\frac{-1}{6}, \frac{-3}{8}\right\}. \quad (34)$$

One the other hand, from (16), and (17), for $p = \{3,4\}$, we have

$$R_{VBOC1_{(pn,n,\alpha=0)}}[\tau_8(0)] = \frac{5-2p+(4p-11)\alpha}{2p} = \frac{5}{2p} - 1$$

$$= \left\{\frac{-1}{6}, \frac{-3}{8}\right\} = R_{VBOC1_{(pn,n,\alpha=0)}}[\tau_7(0)]. \quad (35)$$

Again from (12), (13), (14), and (15), for $p = \{3,4\}$ we have

$$R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_7(1)] = \frac{5-2p+(4p-9)\alpha}{2p} = 1 - \frac{2}{p}$$

$$= \left\{\frac{2}{6}, \frac{4}{8}\right\} = R_{VBOC1_{(pn,n,\alpha)}}[\tau_6(1)]. \quad (36)$$

One the other hand, from (16), (17), (18), and (19), for $p = \{3,4\}$, we have

$$R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_8(1)] = \frac{5-2p+(4p-11)\alpha}{2p} = 1 - \frac{3}{p}$$

$$= \left\{0, \frac{2}{8}\right\} = R_{VBOC1_{(pn,n,\alpha)}}[\tau_9(1)], \quad (37)$$

Fourth, we prove that $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], obeys the *continuity theorem of the second kind* for $\{\tau_{10}(\alpha), \tau_{11}(\alpha)\}$.

From (20) and (21), for $p = 4$, we have

$$R_{VBOC1_{(pn,n,\alpha=0)}}[\tau_{10}(0)] = \frac{7 - 2p + (4p - 13)\alpha}{2p}$$

$$= -1 + \tfrac{7}{2p} = \left\{\tfrac{-1}{8}\right\}. \quad (38)$$

One the other hand, from and (22) and (23), for $p = 4$, we have

$$R_{VBOC1_{(pn,n,\alpha=0)}}[\tau_{11}(0)] = \frac{7 - 2p + (4p - 15)\alpha}{2p}$$

$$= -1 + \tfrac{7}{2p} = \left\{\tfrac{-1}{8}\right\}. \quad (39)$$

Again from (18), (19), (20), and (21), for $p = 4$ we have

$$R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_{10}(1)] = \frac{7 - 2p + (4p - 13)\alpha}{2p}$$

$$= 1 - \tfrac{3}{p} = \left\{\tfrac{2}{8}\right\} = R_{VBOC1_{(pn,n,\alpha)}}[\tau_9(1)]. \quad (40)$$

One the other hand, from (22), (23), (24) and (25), for $p = 4$, we have

$$R_{VBOC1_{(pn,n,\alpha=1)}}[\tau_{11}(1)] = \frac{7 - 2p + (4p - 15)\alpha}{2p} = 1 - \frac{4}{p} = \{0\}$$

$$= R_{VBOC1_{(pn,n,\alpha)}}[\tau_{12}(1)]. \quad (41)$$

Equations (26) through (41) complete the proof of *theorem 2* or *continuity theorem of the second kind*.

Corollary 1: Based on *theorems* 2 in [1], 1 and 2 we conclude that $R_{VBOC1_{(pn,n,\alpha)}}[0] = 1 = \max$ regardless of p or α; i.e., $R_{VBOC1_{(pn,n,\alpha)}}[\tau_i(\alpha)] \leq 1$ regardless of p or α.

2.2 $VBOC1(\alpha)$ ACF Optimization Theorems

At this point we are certain that closed form expression of the ACF, $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, given by (22) of (Progri 2015a) [1], is the correct one and we are ready to find the optimum $0 \leq \alpha \leq 1$ that minimizes the out of phase ACF based on certain criteria known as the $VBOC1(\alpha)$ ACF Optimization Theorems.

Theorem 3*: Prove that the ACF* $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$*, given by* (22) of (Progri 2015a) [1], obeys the *optimization theorem of the first kind;* i.e., *find the values of* α that minimizes the out of phase ACF,

$$\min_{\alpha}\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^{2p}(\alpha)] \end{bmatrix} \to \min_{\alpha} \textstyle\sum_{i=1}^{2p} R_{VBOC1_{(pn,n,\alpha)}}[\tau^i(\alpha)], \quad (42)$$

$$\underbrace{\tau^i(\alpha)}_{i=\{1,2,\cdots 2p\}} = \underbrace{\underbrace{\underbrace{\underbrace{\begin{bmatrix} \underbrace{\frac{T_c(1-\alpha)}{2p}}_{\tau^1(\alpha)} \\ \underbrace{\frac{T_c(1+\alpha)}{2p}}_{\tau^2(\alpha)} \end{bmatrix}}_{p=1} \begin{matrix} \underbrace{\frac{T_c(3-\alpha)}{2p}}_{\tau^3(\alpha)} \\ \underbrace{\frac{T_c(3+\alpha)}{2p}}_{\tau^4(\alpha)} \end{matrix}}_{p=2} \begin{matrix} \underbrace{\frac{T_c(5-\alpha)}{2p}}_{\tau^5(\alpha)} \\ \underbrace{\frac{T_c(5+\alpha)}{2p}}_{\tau^6(\alpha)} \end{matrix}}_{p=3} \begin{matrix} \underbrace{\frac{T_c(7-\alpha)}{2p}}_{\tau^7(\alpha)} \\ \underbrace{\frac{T_c(7+\alpha)}{2p}}_{\tau^8(\alpha)} \end{matrix}}_{p=4}. \quad (43)$$

Proof of theorem 3: The *proof of theorem* 3 is also straightforward. Because the min-max of the ACF of $VBOC1_{(pn,n,\alpha)}(t)$, or $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, that depends α on for values of $p = \{1,2,3,4\}$ and $\underbrace{\tau^i(\alpha)}_{i=\{1,2,\cdots 2p\}}$ given by (42); hence, it suffices to show that show that (42) holds.

First, for $p = 1$ find the value of α that minimizes $\tau^1(\alpha)$ and $\tau^2(\alpha)$

$$\min_{\alpha}\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^2(\alpha)] \end{bmatrix} = \frac{\min_{\alpha}\begin{bmatrix} 1-2p-(1-4p)\alpha \\ 1-2p-(3-4p)\alpha \end{bmatrix}}{2p}. \quad (44)$$

Optimization theorem of the first kind says that proving (43) is equivalent with proving the following

$$\min_{\alpha} \textstyle\sum_{i=1}^{2} R_{VBOC1_{(pn,n,\alpha)}}[\tau^i(\alpha)] = \frac{1}{2}\min_{\alpha}\left[\frac{2-4p-(4-8p)\alpha}{2p}\right]$$

$$\to -1 + 2\alpha = 0; 0 \leq \alpha \leq 1 \quad (45)$$

Hence, we find minimum $\alpha = 0.5$. This completes the *proof of theorem* 3 and it is independent of p. Substituting, $\alpha = 0.5$ in (44) we obtain

$$\begin{vmatrix} R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 0.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 1.5}{2p}\right] \end{vmatrix} = \frac{\begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}}{2p} \quad (46)$$

Second, for $p = 2$ find the value of α that minimizes $\tau^1(\alpha)$, ..., $\tau^4(\alpha)$

$$\min_{\alpha}\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau_4(\alpha)] \end{bmatrix} = \frac{\min_{\alpha}\begin{bmatrix} 1-2p-(1-4p)\alpha \\ 1-2p-(3-4p)\alpha \\ 3-2p-(5-4p)\alpha \\ 3-2p-(7-4p)\alpha \end{bmatrix}}{2p} \quad (47)$$

Optimization theorem of the first kind says that proving (47) is equivalent with proving the following

$$\min_{\alpha} \textstyle\sum_{i=1}^{4} R_{VBOC1_{(pn,n,\alpha)}}[\tau^i(\alpha)] = \frac{1}{2}\min_{\alpha}\left[\frac{8-8p-(16-16p)\alpha}{2p}\right]$$

$$\to -1 + 2\alpha = 0, 0 \leq \alpha \leq 1 \quad (48)$$

Hence, we find minimum $\alpha = 0.5$. This completes the *proof of theorem 3* and it is independent of p. Substituting, $\alpha = 0.5$ in (47) we obtain

$$\underbrace{\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 0.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 1.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 2.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 3.5}{2p}\right] \end{bmatrix}}_{\min_{\alpha}} = \frac{\begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}}{2p}. \tag{49}$$

Third, for $p = 3$ find the value of α that minimizes $\tau^1(\alpha)$, ..., $\tau^6(\alpha)$

$$\min_{\alpha}\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^6(\alpha)] \end{bmatrix} = \frac{\min_{\alpha}\begin{bmatrix} 1-2p-(1-4p)\alpha \\ 1-2p-(3-4p)\alpha \\ 3-2p-(5-4p)\alpha \\ 3-2p-(7-4p)\alpha \\ 5-2p-(9-4p)\alpha \\ 5-2p-(11-4p)\alpha \end{bmatrix}}{2p} \tag{50}$$

Optimization theorem of the first kind says that proving (50) is equivalent with proving the following

$$\min_{\alpha} \sum_{i=1}^{6} R_{VBOC1_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right] = \frac{1}{2}\min_{\alpha}\left[\frac{18-12p-(36-24p)\alpha}{2p}\right]$$

$$\rightarrow -1 + 2\alpha = 0, 0 \le \alpha \le 1. \tag{51}$$

Hence, we find minimum $\alpha = 0.5$. This completes the *proof of theorem* 3 and it is independent of p. Substituting, $\alpha = 0.5$ in (50) we obtain

$$\min_{\alpha}\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 0.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 1.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 2.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 3.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 4.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 5.5}{2p}\right] \end{bmatrix} \rightarrow \frac{\begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}}{2p}. \tag{52}$$

Fourth, for $p = 4$ find the value of α that minimizes $\tau^1(\alpha)$, ..., $\tau^8(\alpha)$

$$\min_{\alpha}\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^8(\alpha)] \end{bmatrix} = \frac{\min_{\alpha}\begin{bmatrix} 1-2p-(1-4p)\alpha \\ 1-2p-(3-4p)\alpha \\ 3-2p-(5-4p)\alpha \\ 3-2p-(7-4p)\alpha \\ 5-2p-(9-4p)\alpha \\ 5-2p-(11-4p)\alpha \\ 7-2p-(13-4p)\alpha \\ 7-2p-(15-4p)\alpha \end{bmatrix}}{2p}. \tag{53}$$

Optimization theorem of the first kind says that proving (53) is equivalent with proving the following

$$\min_{\alpha} \sum_{i=1}^{8} R_{VBOC1_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right] = \frac{1}{2}\min_{\alpha}\left[\frac{32-16p-(64-32p)\alpha}{2p}\right]$$

$$= -1 + 2\alpha = 0; 0 \le \alpha \le 1. \tag{54}$$

Hence, we find minimum $\alpha = 0.5$. This completes the *proof of theorem* 3 and it is independent of p. Substituting, $\alpha = 0.5$ in (53) we obtain

$$\min_{\alpha}\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 0.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 1.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 2.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 3.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 4.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 5.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 6.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 7.5}{2p}\right] \end{bmatrix} = \frac{\begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}}{2p}. \tag{55}$$

Hence, the *optimization theorem of the first kind* proves that $\alpha = 0.5$ minimizes the out-of-phase autocorrelation peaks of the ACF of $VBOC1_{(pn,n,\alpha)}(t)$, or $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, for all values of p.

Corollary 2: It follows immediately from (55) that

$$\lim_{p\to\infty}\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 0.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 1.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 2.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 3.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 4.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 5.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 6.5}{2p}\right] \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\frac{T_c 7.5}{2p}\right] \end{bmatrix} = \lim_{p\to\infty}\frac{\begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}}{2p} = 0. \tag{56}$$

which is another reason why $\alpha = 0.5$ is the optimum delay for the out-of-phase autocorrelation peaks of the ACF of $VBOC1_{(pn,n,\alpha)}(t)$, or $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, for all values of p.

Theorem 4: *Prove that the ACF* $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$*, given by* (22) of (Progri 2015a) [1], obeys the *optimization theorem of the second kind;* i.e.*, find the values of* α that minimizes the out of phase ACF,

$$\min_{\alpha}\begin{vmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^{2p}(\alpha)] \end{vmatrix} \rightarrow$$

$$\min_{\alpha}\begin{vmatrix} \sum_{i=1}^{2} R_{VBOC1_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right] \\ \vdots \\ \sum_{i=2p-1}^{2p} R_{VBOC1_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right] \end{vmatrix} \tag{57}$$

$\tau^i(\alpha)$ for $i = \{1,2,\cdots 2p\}$ given by (43) in *theorem 3* and $p = \{1,2,3,4\}$.

Proof of theorem 4: The *proof of theorem 4* is also straightforward and it results in exactly the same value of $\alpha = 0.5$ as from *theorem* 3. Because the proof is so obvious we leave it as an exercise to the reader.

Hence, the optimization theorems of the first and second kind are both the necessary and sufficient conditions of optimization that involves the sum operator of $\underbrace{\tau^i(\alpha)}_{i=\{1,2,\cdots 2p\}}$ given by (43) because of the symmetry of the ACF.

Theorem 5: *Prove that the ACF* $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$, *given by* (22) of (Progri 2015a) [1], obeys the *optimization theorem of the third kind;* i.e., *find the values of* α that minimizes the out-of-phase ACF

$$\min_{\alpha}\begin{vmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^{2p}(\alpha)] \end{vmatrix} \rightarrow \prod_{i=1}^{2p} R_{VBOC1_{(pn,n,\alpha)}}[\tau_i(\alpha)] = 0. \quad (58)$$

$\tau^i(\alpha)$ for $i = \{1,2,\cdots 2p\}$ given by (44) in *theorem* 3 and $p = \{1,2,3,4\}$.

Proof of theorem 5: The *proof of theorem* 5 is also straightforward. Because the min-max of the ACF of $VBOC1_{(pn,n,\alpha)}(t)$, or $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$, that depends α on for values of $p = \{1,2,3,4\}$ and $\underbrace{\tau^i(\alpha)}_{i=\{1,2,\cdots 2p\}}$ given by (44); hence, it suffices to show that (58) holds.

First, for $p = 1$ find the value of α that minimizes $\tau^1(\alpha)$ and $\tau^2(\alpha)$

$$\min_{\alpha}\begin{vmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^2(\alpha)] \end{vmatrix} = \frac{\min_{\alpha}\begin{bmatrix} 1-2p-(1-4p)\alpha \\ 1-2p-(3-4p)\alpha \end{bmatrix}}{2p}. \quad (59)$$

Optimization theorem of the third kind says that proving (59) is equivalent with proving the following

$$\min_{\alpha}\left\{R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \cdot R_{VBOC1_{(pn,n,\alpha)}}[\tau^2(\alpha)]\right\}$$

$$\rightarrow \left[\frac{1-2p-(1-4p)\alpha}{2p}\right]\left[\frac{1-2p-(3-4p)\alpha}{2p}\right] = 0. \quad (60)$$

or

$$\begin{vmatrix} 1-2p-(1-4p)\alpha = 0 \\ 1-2p-(3-4p)\alpha = 0 \end{vmatrix} \rightarrow \begin{vmatrix} \alpha = \left.\frac{1-2p}{1-4p}\right|_{p=1} = \frac{1}{3} \\ \alpha = \left.\frac{1-2p}{3-4p}\right|_{p=1} = 1 \end{vmatrix}. \quad (61)$$

There are three observations from (61). The first is that α depends on p. And the second is that minimum $\alpha = \{1/3\,,1\}$ for $p = 1$. Substituting, minimum $\alpha = \{1/3\,,1\}$ in (59) we obtain

$$\min_{\alpha}\begin{vmatrix} R_{VBOC1_{(pn,n,\alpha=\frac{1}{3})}}\left[\frac{T_c}{3p}\right] \\ R_{VBOC1_{(pn,n,\alpha=1)}}\left[\frac{T_c}{p}\right] \end{vmatrix} = \begin{bmatrix} \frac{1-p}{3p} \\ \frac{p-1}{p} \end{bmatrix}_{p=1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (62)$$

or

$$\min_{\alpha}\left\{R_{VBOC1_{(pn,n,\alpha)}}[\tau_1(\alpha)] \cdot R_{VBOC1_{(pn,n,\alpha)}}[\tau_2(\alpha)]\right\} = 0 \times 0 = 0. \quad (63)$$

The third observation is that minimum value of (63) according to optimization theorem of third kind is 0 because we set it to be equal to 0.

Second, it is also straight forward to show that for $p = 2$

$$\alpha = \left.\frac{1-2p}{1-4p}\right|_{p=2} = \frac{3}{7};\ \alpha = \left.\frac{1-2p}{3-4p}\right|_{p=2} = \frac{3}{5};$$

$$\alpha = \left.\frac{3-2p}{5-4p}\right|_{p=2} = \frac{1}{3};\ \alpha = \left.\frac{3-2p}{7-4p}\right|_{p=2} = 1. \quad (64)$$

There are three observations from (64). The first is that α depends on p. And the second is that minimum $\alpha = \{3/7\,,3/5\,,1/3\,,1\}$ for $p = 2$. Substituting, minimum $\alpha = \{3/7\,,3/5\,,1/3\,,1\}$ in (58) we obtain

$$\min_{\alpha}\begin{vmatrix} R_{VBOC1_{(pn,n,\alpha=\frac{3}{7})}}\left[\frac{2T_c}{7p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{3}{5})}}\left[\frac{4T_c}{5p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{1}{3})}}\left[\frac{4T_c}{6p}\right] \\ R_{VBOC1_{(pn,n,\alpha=1)}}\left[\frac{2T_c}{p}\right] \end{vmatrix} = \begin{bmatrix} \frac{2-p}{7p} \\ \frac{p-2}{5p} \\ \frac{2-p}{3p} \\ \frac{p-2}{p} \end{bmatrix}_{p=2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (65)$$

The third observation is that minimum value of (65) according to optimization theorem of third kind is 0 because we set it to be equal to 0.

Third, it is also straight forward to show that for $p = 3$

$$\begin{vmatrix} \alpha = \left.\frac{1-2p}{1-4p}\right|_{p=3} = \frac{5}{11};\ \alpha = \left.\frac{1-2p}{3-4p}\right|_{p=3} = \frac{5}{9};\ \alpha = \left.\frac{3-2p}{5-4p}\right|_{p=3} = \frac{3}{7}; \\ \alpha = \left.\frac{3-2p}{7-4p}\right|_{p=3} = \frac{3}{5};\alpha = \left.\frac{5-2p}{9-4p}\right|_{p=3} = \frac{1}{3};\alpha = \left.\frac{5-2p}{11-4p}\right|_{p=3} = 1 \end{vmatrix} \quad (66)$$

There are three observations from (66). The first is that α depends on p. And the second is that minimum $\alpha = \{5/11\,,5/9\,,3/7\,,3/5\,,1/3\,,1\}$ for $p = 3$. Substituting, minimum $\alpha = \{5/11\,,5/9\,,3/7\,,3/5\,,1/3\,,1\}$ in (58) we obtain

$$\min_{\alpha}\begin{vmatrix} R_{VBOC1_{(pn,n,\alpha=\frac{5}{11})}}\left[\frac{3T_c}{11p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{5}{9})}}\left[\frac{7T_c}{9p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{3}{7})}}\left[\frac{9T_c}{7p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{3}{5})}}\left[\frac{9T_c}{5p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{1}{3})}}\left[\frac{7T_c}{3p}\right] \\ R_{VBOC1_{(pn,n,\alpha=1)}}\left[\frac{3T_c}{p}\right] \end{vmatrix} = \begin{bmatrix} \frac{3-p}{11p} \\ \frac{p-3}{9p} \\ \frac{3-p}{7p} \\ \frac{p-3}{5p} \\ \frac{3-p}{3p} \\ \frac{p-3}{p} \end{bmatrix}_{p=3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (67)$$

The third observation is that minimum value of (67) according to optimization theorem of third kind is 0 because we set it to be equal to 0.

Fourth, it is also straight forward to show that for $p = 4$

$$\min_{\alpha}\begin{vmatrix} R_{VBOC1_{(pn,n,\alpha=\frac{5}{11})}}\left[\frac{3T_c}{11p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{5}{9})}}\left[\frac{7T_c}{9p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{3}{7})}}\left[\frac{9T_c}{7p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{3}{5})}}\left[\frac{9T_c}{5p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{1}{3})}}\left[\frac{7T_c}{3p}\right] \\ R_{VBOC1_{(pn,n,\alpha=1)}}\left[\frac{3T_c}{p}\right] \end{vmatrix} = \begin{bmatrix} \frac{3-p}{11p} \\ \frac{p-3}{9p} \\ \frac{3-p}{7p} \\ \frac{p-3}{5p} \\ \frac{3-p}{3p} \\ \frac{p-3}{p} \end{bmatrix}_{p=3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (68)$$

There are three observations from (68). The first is that α depends on p. And the second is that minimum $\alpha = \{7/15, 7/13, 5/11, 5/9, 3/7, 3/5, 1/3, 1\}$ for $p = 4$.

For $\alpha = \{7/15, 7/13, 5/11, 5/9, 3/7, 3/5, 1/3, 1\}$ in (58) yields

$$\min_{\alpha}\left|\begin{matrix} R_{VBOC1_{(pn,n,\alpha=\frac{7}{15})}}\left[\frac{4T_c}{15p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{7}{13})}}\left[\frac{10T_c}{13p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{5}{11})}}\left[\frac{14T_c}{11p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{5}{9})}}\left[\frac{16T_c}{9p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{3}{7})}}\left[\frac{16T_c}{7p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{3}{5})}}\left[\frac{14T_c}{5p}\right] \\ R_{VBOC1_{(pn,n,\alpha=\frac{1}{3})}}\left[\frac{10T_c}{3p}\right] \\ R_{VBOC1_{(pn,n,\alpha=1)}}\left[\frac{4T_c}{p}\right] \end{matrix}\right. = \left.\begin{bmatrix} \frac{4-p}{15p} \\ \frac{p-4}{13p} \\ \frac{4-p}{11p} \\ \frac{p-4}{9p} \\ \frac{4-p}{7p} \\ \frac{p-4}{5p} \\ \frac{4-p}{3p} \\ \frac{p-4}{p} \end{bmatrix}\right|_{p=4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (69)$$

The third observation is that minimum value of (69) according to optimization theorem of third kind is 0 because we set it to be equal to 0.

Corollary 3: $\forall p$ half of the αs (or odd ones) are smaller than 0.5 and the remaining half (or even ones) are greater than 0.5. Indeed

(1) for $p = 1$, $\alpha = \{1/3\} < 0.5$ and $\alpha = \{1\} > 0.5$;
(2) for $p = 2$, $\alpha = \{3/7, 1/3\} < 0.5$ and $\alpha = \{3/5, 1\} > 0.5$;
(3) for $p = 3$, $\alpha = \{5/11, 3/7, 1/3\} < 0.5$ and $\alpha = \{5/9, 3/5, 1\} > 0.5$;
(4) for $p = 3$, $\alpha = \{7/15, 5/11, 3/7, 1/3\} < 0.5$ and $\alpha = \{7/13, 5/9, 3/5, 1\} > 0.5$.

Corollary 4: $\alpha = 0.5$ is the asymptotic value for which the product optimum αs from the *optimization theorem of the third kind* approaches the sum optimum α from the *optimization theorem of the first and second kinds*.

Indeed we can see that from the *optimization theorem of the third kind* $\alpha = \left.\frac{1-2p}{1-4p}\right|_{p\to\infty} = 0.5$ which is the asymptotic value for which the product optimum αs from the *optimization theorem of the third kind* approaches the sum optimum α from the *optimization theorem of the first and second kinds*.

Theorem 6: *Prove that the ACF* $R_{VBOC1_{(m=np,n,\alpha)}}(\tau)$, *given by* (22) of (Progri 2015a) [1], obeys the *optimization theorem of the fourth kind (or in the mean square sense);* i.e., *find the values of* α that minimizes the out-of-phase ACF,

$$\min_{\alpha}\left|\begin{matrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^{2p}(\alpha)] \end{matrix}\right| \to 0 \quad (70)$$

which is equivalent with

$$\alpha_{\min} = \frac{\sum_{i=1}^{2p} a_i(p)b_i(p)}{\sum_{i=1}^{2p}[a_i(p)]^2} = \frac{\mathbf{a}(p)^{\mathrm{T}}\mathbf{b}(p)}{\mathbf{a}(p)^{\mathrm{T}}\mathbf{a}(p)} = \frac{\mathbf{a}(p)^{\mathrm{T}}\mathbf{b}(p)}{\|\mathbf{a}(p)\|^2}. \quad (71)$$

$\tau^i(\alpha)$ for $i = \{1,2,\cdots 2p\}$ given by (43) in *theorem 3* and $p = \{1,2,3,4\}$ and $\mathbf{a}(p)$ and $\mathbf{b}(p)$ are given by

$$\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^8(\alpha)] \end{bmatrix} = \underbrace{\frac{\begin{bmatrix} \underbrace{\begin{matrix}1-2p \\ 1-2p\end{matrix}}_{p=\{1,2,3,4\}} \\ \underbrace{\begin{matrix}3-2p \\ 3-2p\end{matrix}}_{p=\{2,3,4\}} \\ \underbrace{\begin{matrix}5-2p \\ 5-2p\end{matrix}}_{p=\{3,4\}} \\ \underbrace{\begin{matrix}7-2p \\ 7-2p\end{matrix}}_{p=\{4\}} \end{bmatrix}}{2p}}_{\mathbf{b}(p)} - \underbrace{\frac{\begin{bmatrix} \underbrace{\begin{matrix}1-4p \\ 3-4p\end{matrix}}_{p=\{1,2,3,4\}} \\ \underbrace{\begin{matrix}5-4p \\ 7-4p\end{matrix}}_{p=\{2,3,4\}} \\ \underbrace{\begin{matrix}9-4p \\ 11-4p\end{matrix}}_{p=\{3,4\}} \\ \underbrace{\begin{matrix}13-4p \\ 15-4p\end{matrix}}_{p=\{4\}} \end{bmatrix}}{2p}}_{\mathbf{a}(p)}\alpha. \quad (72)$$

Proof of theorem 6: The *proof of theorem* 6 is also straight forward.

First for $p = 1$ from the *optimization theorem of the fourth kind (or in the mean square sense);* i.e., *find the values of* α that minimizes the out-of-phase ACF,

$$\min_{\alpha}\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^2(\alpha)] \end{bmatrix} = 0, \quad (73)$$

which is equivalent with

$$\alpha_{\min} = \frac{\sum_{i=1}^{2} a_i(p)b_i(p)}{\sum_{i=1}^{2}[a_i(p)]^2} = \frac{\mathbf{a}(p)^{\mathrm{T}}\mathbf{b}(p)}{\mathbf{a}(p)^{\mathrm{T}}\mathbf{a}(p)} = \frac{\mathbf{a}(p)^{\mathrm{T}}\mathbf{b}(p)}{\|\mathbf{a}(p)\|^2}, \quad (74)$$

where

$$\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^2(\alpha)] \end{bmatrix} = \underbrace{\frac{\begin{bmatrix}1-2p \\ 1-2p\end{bmatrix}}{2p}}_{\mathbf{b}[\boldsymbol{\tau}(\alpha)]} - \underbrace{\frac{\begin{bmatrix}1-4p \\ 3-4p\end{bmatrix}}{2p}}_{\mathbf{a}[\boldsymbol{\tau}(\alpha)]}\alpha. \quad (75)$$

Hence,

$$\alpha_{\min} = \frac{\mathbf{a}(p)^{\mathrm{T}}\mathbf{b}(p)}{\|\mathbf{a}(p)\|^2} = \frac{4(2p-1)^2}{(4p-1)^2+(4p-3)^2} = \frac{4}{3^2+1^2} = 0.4. \quad (76)$$

Second, for $p = 2$ from the *optimization theorem of the fourth kind (or in the minimum mean square error (MMSE) sense (Anon. 2015));* i.e., *find the values of* α that minimizes the out-of-phase ACF,

$$\min_{\alpha}\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^4(\alpha)] \end{bmatrix} = 0, \quad (77)$$

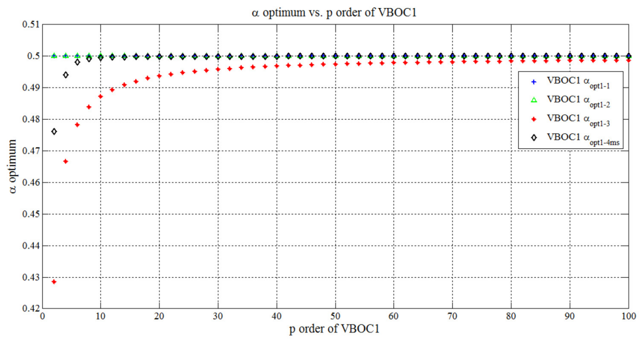
is equivalent with

$$\alpha_{\min} = \frac{\sum_{i=1}^{4} a_i(p)b_i(p)}{\sum_{i=1}^{2}[a_i(p)]^2} = \frac{\mathbf{a}(p)^{\mathrm{T}}\mathbf{b}(p)}{\mathbf{a}(p)^{\mathrm{T}}\mathbf{a}(p)} = \frac{\mathbf{a}(p)^{\mathrm{T}}\mathbf{b}(p)}{\|\mathbf{a}(p)\|^2}, \quad (78)$$

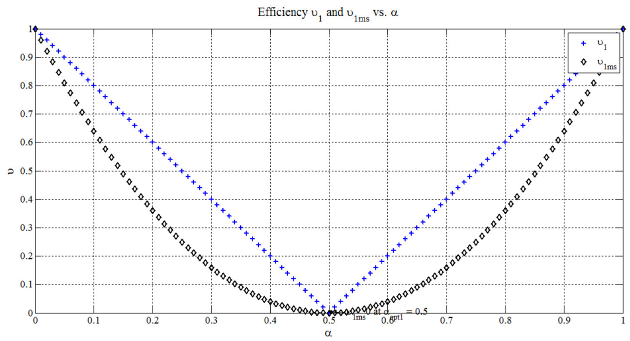
where

$$\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}[\tau^1(\alpha)] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}[\tau^4(\alpha)] \end{bmatrix} = \underbrace{\frac{\begin{bmatrix}1-2p \\ 1-2p \\ 3-2p \\ 3-2p\end{bmatrix}}{2p}}_{\mathbf{b}(p)} - \underbrace{\frac{\begin{bmatrix}1-4p \\ 3-4p \\ 5-4p \\ 7-4p\end{bmatrix}}{2p}}_{\mathbf{a}(p)}\alpha. \quad (79)$$

Hence,



(a) α optimum vs. order p of $VBOC1_{(pn,n,\alpha)}(t)$



(b) Efficiency $\Upsilon_{1(\alpha)}$ (+) and $\Upsilon_{1MS(\alpha)}$ (◊) vs α

FIGURE 1: Illustration of ACF optimization theorems.

$$\alpha_{\min}=\frac{\mathbf{a}(p)^{\mathbf{T}}\mathbf{b}(p)}{\|\mathbf{a}(p)\|^2}=\frac{4\left[(2p-1)^2+(2p-3)^2\right]}{(4p-1)^2+(4p-3)^2+(4p-5)^2+(4p-7)^2}, \quad (80)$$

or

$$\alpha_{\min}=\frac{\mathbf{a}(p)^{\mathbf{T}}\mathbf{b}(p)}{\|\mathbf{a}(p)\|^2}=\frac{4\left[3^2+1^2\right]}{7^2+5^2+3^2+1}=\frac{4\times 10}{84}=0.476. \quad (81)$$

Third, for $p=3$ from the *optimization theorem of the fourth kind (or in the mean square sense);* i.e., *find the values of* α that minimizes the out-of-phase ACF,

$$\min_{\alpha}\begin{vmatrix} R_{VBOC1_{(pn,n,\alpha)}}\left[\tau^1(\alpha)\right] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\tau^6(\alpha)\right]\end{vmatrix}=0, \quad (82)$$

which is equivalent with

$$\alpha_{\min}=\frac{\sum_{i=1}^{6}a_i(p)b_i(p)}{\sum_{i=1}^{2}[a_i(p)]^2}=\frac{\mathbf{a}(p)^{\mathbf{T}}\mathbf{b}(p)}{\mathbf{a}(p)^{\mathbf{T}}\mathbf{a}(p)}=\frac{\mathbf{a}(p)^{\mathbf{T}}\mathbf{b}(p)}{\|\mathbf{a}(p)\|^2}, \quad (83)$$

where

$$\begin{bmatrix} R_{VBOC1_{(pn,n,\alpha)}}\left[\tau^1(\alpha)\right] \\ \vdots \\ R_{VBOC1_{(pn,n,\alpha)}}\left[\tau^6(\alpha)\right]\end{bmatrix}=\underbrace{\frac{\begin{bmatrix}1-2p\\1-2p\\3-2p\\3-2p\\5-2p\\5-2p\end{bmatrix}}{2p}}_{\mathbf{b}(p)}-\underbrace{\frac{\begin{bmatrix}1-4p\\3-4p\\5-4p\\7-4p\\9-4p\\11-4p\end{bmatrix}}{2p}}_{\mathbf{a}(p)}\alpha. \quad (84)$$

Hence,

$$\alpha_{\min}=\frac{\mathbf{a}(p)^{\mathbf{T}}\mathbf{b}(p)}{\|\mathbf{a}(p)\|^2}=\frac{4\left[(2p-1)^2+(2p-3)^2+(2p-5)^2\right]}{\begin{bmatrix}(4p-1)^2+(4p-3)^2+(4p-5)^2+(4p-7)^2\\+(4p-9)^2+(4p-11)^2\end{bmatrix}}, \quad (85)$$

or

$$\alpha_{\min}=\frac{\mathbf{a}(p)^{\mathbf{T}}\mathbf{b}(p)}{\|\mathbf{a}(p)\|^2}=\frac{4\left[5^2+3^2+1^2\right]}{11^2+9^2+7^2+5^2+3^2+1}=\frac{4\times 35}{286}\cong 0.49. \quad (86)$$

Fourth, for $p=4$ from the *optimization theorem of the fourth kind (or in the mean square sense);* i.e., *find the values of* α that minimizes the out-of-phase ACF

$$\alpha_{\min}=\frac{\mathbf{a}(p)^{\mathbf{T}}\mathbf{b}(p)}{\|\mathbf{a}(p)\|^2}=\frac{4\left[(2p-1)^2+(2p-3)^2+(2p-5)^2+(2p-7)^2\right]}{\begin{bmatrix}(4p-1)^2+(4p-3)^2+(4p-5)^2+(4p-7)^2+\\(4p-9)^2+(4p-11)^2+(4p-13)^2+(4p-15)^2\end{bmatrix}}$$

$$=\frac{4\times 84}{680}\cong 0.494. \quad (87)$$

Equations (73) through (87) complete the *proof of theorem* 6.

Corollary 5: $\alpha=0.5$ is the asymptotic value for which the mean square optimum αs from the *optimization theorem of the fourth kind* approach the optimum α.

Indeed we can see that from the *optimization theorem of the fourth kind* $\alpha=\left.\frac{4\cdot p\cdot 4p^2}{2\cdot p\cdot 4^2p^2}\right|_{p\to\infty}=0.5$ which is the asymptotic value for which the mean square optimum αs from the *optimization theorem of the fourth kind* approach the optimum α.

Corollary 6: or *performance enhancement or efficiency* both in terms of optimization theorems of the *first kind* and *fourth kinds*.

Performance enhancement or efficiency based on the *optimization theorem of the first kind* or $\Upsilon_{1(p,\alpha)}$ is defined as absolute value of the ratio or the sum of all out-of-phase autocorrelation peaks that depend on α by their number as follows:

$$\Upsilon_{1(p,\alpha)}=\frac{\left|\sum_{i=1}^{2p}R_{VBOC1_{(pn,n,\alpha)}}\left[\tau^i(\alpha)\right]\right|}{2p}=\left|\frac{\frac{|32-16p||1-2\alpha|}{16p}}{\left|1-\frac{2}{p}\right||2\alpha-1|}\right|. \quad (88)$$

As we take the limit as $p\to\infty$ we find efficiency only as function of α as follows:

$$\Upsilon_{1(\alpha)}=\lim_{p\to\infty}\Upsilon_{1(p,\alpha)}=|2\alpha-1|. \quad (89)$$

or $|2\alpha-1|=0$ or $\alpha=0.5$ gives efficiency equal to zero or infinite improvements of the *performance enhancement*.

Performance enhancement or efficiency based on the *optimization theorem of the fourth kind* is defined as the ratio of the sum of all out-of-phase autocorrelation peaks squared by their number and denoted by $\Upsilon_{1MS(p,\alpha)}$ as follows:

$$\Upsilon_{1MS(p,\alpha)}=\frac{\sum_{i=1}^{2p}\left(R_{VBOC1_{(pn,n,\alpha)}}[\tau_i(\alpha)]\right)^2}{8}=\frac{[\mathbf{b}(p)-\mathbf{a}(p)\alpha]^T[\mathbf{b}(p)-\mathbf{a}(p)\alpha]}{8}, \quad (90)$$

or

$$\Upsilon_{1MS(p,\alpha)}=\frac{\mathbf{b}(p)^T\mathbf{b}(p)-[\mathbf{b}(p)^T\mathbf{a}(p)+\mathbf{a}(p)^T\mathbf{b}(p)]\alpha+\mathbf{a}(p)^T\mathbf{a}(p)\alpha^2}{8}, \quad (91)$$

or

$$\Upsilon_{1\mathrm{MS}(p,\alpha)}=\frac{\begin{bmatrix}(1-2p)^2-2(1-2p)(1-4p)\alpha+(1-4p)^2\alpha^2\\+(1-2p)^2-2(1-2p)(3-4p)\alpha+(3-4p)^2\alpha^2\\+(3-2p)^2-2(3-2p)(5-4p)\alpha+(5-4p)^2\alpha^2\\+(3-2p)^2-2(3-2p)(7-4p)\alpha+(7-4p)^2\alpha^2\\+(5-2p)^2-2(5-2p)(9-4p)\alpha+(9-4p)^2\alpha^2\\+(5-2p)^2-2(5-2p)(11-4p)\alpha+(11-4p)^2\alpha^2\\+(7-2p)^2-2(7-2p)(13-4p)\alpha+(13-4p)^2\alpha^2\\+(7-2p)^2-2(7-2p)(15-4p)\alpha+(15-4p)^2\alpha^2\end{bmatrix}}{8\times 4p^2}$$

$$=\frac{\begin{bmatrix}2(1-2p)^2-8(1-2p)^2\alpha\\+[(1-4p)^2+(3-4p)^2]\alpha^2\\+2(3-2p)^2-8(3-2p)^2\alpha\\+[(5-4p)^2+(7-4p)^2]\alpha^2\\+2(5-2p)^2-8(5-2p)^2\alpha\\+[(9-4p)^2+(11-4p)^2]\alpha^2\\+2(7-2p)^2-8\alpha\\+[(13-4p)^2+(15-4p)^2]\alpha^2\end{bmatrix}}{8\times 4p^2}. \quad (92)$$

Or

$$\Upsilon_{1\mathrm{MS}(p,\alpha)}=\frac{2\sum_{\substack{i=1\\i:odd}}^{7}(i-2p)^2-8\sum_{\substack{i=1\\i:odd}}^{7}(i-2p)^2\alpha+\sum_{\substack{i=1\\i:odd}}^{15}(i-4p)^2\alpha^2}{32p^2}, \quad (93)$$

or

$$\Upsilon_{1\mathrm{MS}(p,\alpha)}=\frac{\begin{bmatrix}2(84-64p+16p^2)-8(84-64p+16p^2)\alpha\\+8(85-64p+16p^2)\alpha^2\end{bmatrix}}{32p^2}$$

$$\cong\frac{2(84-64p+16p^2)(1-4\alpha+4\alpha^2)}{32p^2}. \quad (94)$$

As we take the limit as $p\to\infty$ we find efficiency only as function of α as follows:

$$\Upsilon_{1\mathrm{MS}(\alpha)}=\lim_{p\to\infty}\Upsilon_{1\mathrm{MS}(p,\alpha)}=1-4\alpha+4\alpha^2. \quad (95)$$

For $\alpha=\{0,1\}$ we have $\Upsilon_{1\mathrm{MS}(\alpha=\{0,1\})}=1$ as opposed to for $\alpha=0.5$ we have $\Upsilon_{1\mathrm{MS}(\alpha=0.5)}=0$; hence, $\alpha=0.5$ offers 100% efficiency or for $\alpha=0.5$ $VBOC1_{(pn,n,\alpha)}$ is 100% more efficient than either $BOC_{(pn,n)}$ or $BPSK_0$ in terms of mean square sense in terms of out-of-phase ACF.

3 Numerical, Theoretical Results

Very accurate simulation and interference results on $VBOC1(\alpha)$ and $VBOC2(\alpha,1-\alpha)$ ACFs and PSDs are given in Chap. 7 of (Progri 2015) [5] and (Progri 2012) [6].

However, detailed discussion on optimum α; i.e., the simulation results of the optimization theorems 3 through 6; corollaries 5 and 6; simulation results with optimum $\alpha=0.5$ $VBOC1_{(n,n,\alpha)}$ and $VBOC1_{(2n,n,\alpha)}$ are unique and original to this journal paper that the reader cannot find in (Progri 2015) [5] and (Progri 2012) [6].

Figure 1(a) presents the α optimum vs. order p of $VBOC1_{(pn,n,\alpha)}(t)$. There are four curves shown in Figure 1: (1)-(2) first through third correspond to $VBOC1_{(pn,n,\alpha)}(t)$ optimization theorems 3 and 4; (3) correspond to $VBOC1_{(pn,n,\alpha)}(t)$ optimization theorem 5; and (4) correspond to $VBOC1_{(pn,n,\alpha)}(t)$ optimization theorem 6 or in the mean square sense. The first curve which corresponds to $VBOC1_{(pn,n,\alpha)}(t)$ shows that as p changes from two to one hundred α_{opt} is either equal to or converges to 0.5 regardless of generalized optimization parameter p (or of sub-carrier frequency) and regardless of the optimization theorem(s).

The argument can be made that the choice we make for α will equally affect the standardization of the waveforms regardless of sub-carrier frequency; i.e., even though the choices on $BOC_{(1,1)}(t)=VBOC1_{(1,1,\alpha=0)}(t)$ on GPS L1 data signal and $BOC_{(2,1)}(t)=BOC_{(10,5)}(t)=VBOC1_{(2,1,\alpha=0)}(t)$ or the GPS military M-code (Progri 2015) [5], (Progri 2012) [6], (Progri et. al. 2007) [7], (Betz 2001-2002) [8], and (Sousa and Nunes, 2013) [9] on both GPS L1 and L2 frequencies were arbitrary it only happened by accident that they have the same efficiency. Next, we consider how efficient are these choices?

Figure 1(b) depicts the efficiency $\Upsilon_{1(\alpha)}$ and $\Upsilon_{1\mathrm{MS}(\alpha)}$ corresponding to $VBOC1_{(pn,n,\alpha)}(t)$ respectively vs. signal design and optimization parameter α. As shown in Figure 2 efficiency for either $\Upsilon_{1(\alpha)}$ or $\Upsilon_{1\mathrm{MS}(\alpha)}$ at $\alpha=0.5$ equals to $\Upsilon_{1(\alpha=0.5)}=\Upsilon_{1\mathrm{MS}(\alpha=0.5)}=0$ in corollary 6 of (Progri 2015b) [2].

The argument can be made that the choices on $BOC_{(1,1)}(t)=VBOC1_{(1,1,\alpha=0)}(t)$ on GPS L1 data signal and $BOC_{(2,1)}(t)=BOC_{(10,5)}(t)=VBOC1_{(2,1,\alpha=0)}(t)$ or the GPS military M-code (Progri 2015) [5], (Progri 2012) [6], (Progri et. al. 2007) [7], (Betz 2001-2002) [8], and (Sousa and Nunes, 2013) [9] on both GPS L1 and L2 frequencies are arbitrary and 100% inefficient in terms of out of phase ACF; i.e., BOC modulation is just as inefficient as the BPSK modulation in terms of out of phase ACF.

4 Conclusions

This journal paper examined common optimization algorithms, such as sum, product, and MMSE sense applied to the optimization of the ACF of GPS L1C and M-code like signals such as VBOC. Before the different versions were examined for efficiency, on GPS L1C and M-code like signals such as VBOC1 which vary with the parameter of signal design and optimization alfa, they were checked to make sure that their corresponding ACFs are valid via a set of conditions known as *continuity theorems*. Afterwards, by employing any of the optimization algorithms such as sum, product, and mean square sense on GPS L1C and M-code like signals such as VBOC1 we were able to produce ACFs that are one *hundred percent more efficient* than the corresponding ACFs of the GPS L1C and M-code signals.

Future work: Is it possible that from optimization of the system (transmitter, channel, and receiver) one can obtain a different optimum α than $\alpha = 0.5$ as a result of the optimization of the received out-of-phase ACF? The results of this laborious investigation will be presented in the future.

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