

Research Article

VBOC1(α) Generalized Multidimensional Geolocation Modulation Waveforms

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This paper presents the complete original definition of first generation Variable Binary Offset Carrier VBOC1(α) generalized multidimensional geolocation modulation waveforms, to improve the standardization of the United States DoD GPS, European Galileo, Russian GLONASS, Chinese Compass, Indian IRNSS in the L-band (1-2 GHz), and the United Nations International Telecommunications Union (ITU) GNSS or geolocation waveforms in the S-band (2-4 GHz) and C-band (4-8 GHz).

In the paper it is argued that the selection of BOC(1,1) on the GPS L1 civil data code and BOC(10,5) (or the military code or M-Code) on both GPS L1 and L2 frequencies is entirely arbitrary because BOC modulation is a special case of VBOC1(α) for $\alpha = 0$ or $\alpha = 1$; hence, all the current state-of-the-art GNSS waveforms exhibit sub-optimal signal design performance even at the end-user when generalized global objective functions are applied.

VBOC1(α) pure signal design or broad definition of generalized autocorrelation function (ACF) and power spectral density (PSD) offers a unique signal design methodology and provides the necessary framework for VBOC1(α) ACF pure signal optimization to fill in substantial signal design gaps; hence, improving the GNSS signal design and standardization.

Index Terms—Pulse generation, pulse amplitude modulation, pulse width modulation, multidimensional sequences, signal design, signal analysis, generalized functions, time-frequency analysis, minimization methods, optimization methods.

1 Introduction

The main objective of this paper is to introduce the first

generation VBOC1(α) generalized multidimensional geolocation modulation waveforms so as to fill in substantial signal design [1]-[37] methodology gaps created over the years

as a results of incomplete signal design methodologies.

In the past, signal design methodology was mainly motivated on performance metrics, such as sharper ACFs and user equipment performance measured by the signal ability to mitigate multipath, mitigate interference, jamming and product's ability to produce a working system [25]-[29] or based on a symmetric signal design waveform; i.e., symmetric non-return-to-zero (NRZ) [5] (or S-NRZ): the time (or duration) for which (+1) signal amplitude voltage occurs is equal to the time (or duration) for which (-1) signal amplitude voltage occurs.

The achieved level of success is based mostly on user segment performance metrics and very little on improvements from the signal design methodology. Hence, from the system, design, user equipment engineering point of view we have achieved substantial outstanding milestones; one would argue that from the rigorous signal design methodology (asymmetric NRZ or As-NRZ) point of view we have achieved reasonably good intermediate steps; hence, the main objective of this paper.

The main issue here is not why S-NRZ BOC (or BOC) is better than BPSK due to inclusion of a sub-carrier frequency of the BOC waveform [11]-[23]. The issue; however, is: Are As-NRZ BOC (or VBOC) waveforms better than the BOC waveform regardless of the sub-carrier frequency [1]-[3]? and if there are: (1) Which VBOC waveforms are there [1]-[3]? and (2) What are the criteria to determine which VBOC waveform is the best [1]-[3]?

For example $VBOC1(\alpha)$, which is a type of As-NRZ BOC for $0 \leq \alpha \leq 1$, generalizes the transition from the BPSK waveform $BPSK(t) = VBOC1_{(m,n,\alpha=1)}(t)$ to the current S-NRZ BOC modulation, used extensively in GPS L1 and L2 frequencies, because $BOC_{(m,n)}(t) = VBOC1_{(m,n,\alpha=0)}(t)$; however, in the current GNSS standard the choice $\alpha = 0$ in the context of $VBOC1_{(m,n,\alpha)}(t)$ is made entirely arbitrary [11]-[23]. Because in the interval from zero to one there are an infinite number of α s one cannot arbitrary select $\alpha = 0$ as the best waveform and make $BOC_{(m,n)}(t)$ the standard for all GPS III, IV and other GNSS users without a single explanation whatsoever regardless of integer values of m and n ; i.e., the sub-carrier frequency [11]-[23].

Initially, the signal design approach for pseudolite applications [7], [30], [31]-[37] was primarily driven by the mentality of achieving user performance metrics; however, it was not until very recently that various signal waveforms articulate new objectives of various signal design teams in the

21st century in *Indoor Geolocation Systems—Theory and Applications* [5], and *Geolocation of RF Signals—Principles and Simulations* [37].

Why is the above discussion so important? First, signal design and optimization parameter, α , exploits a particular asymmetry of the As-NRZ signal coding modulation in order to improve $VBOC1(\alpha)$ signal design and optimization. Second, the derivation of generalized ACFs and PSDs as a function of α and p provides for the first time the opportunity to understand the properties of all individual $VBOC1(\alpha)$ such as for all positive integer values of p without having to analyze all individual $VBOC1(\alpha)$ separately. Third, for the first time ever, optimization theorems show the consistency between the sum and mean-square criteria [1].

$VBOC1(\alpha)$ pure signal design or broad definition of generalized ACF and PSD offers a unique signal design methodology and provides the necessary framework for $VBOC1(\alpha)$ ACF pure signal optimization [1].

This paper is organized as follows: in Section II $VBOCk(\alpha_1, \alpha_2, \dots, \alpha_k)$ general discussion. $VBOC1(\alpha)$ pure signal design is discussed in Section III. Section IV contains numerical results; Conclusion is provided in Section V along with a list of references.

2 $VBOCk(\alpha_1, \alpha_2, \dots, \alpha_k)$ General Discussion

Generalized multidimensional geolocation modulation waveforms include: (a) first generation $VBOC1(\alpha)$, $VBOC2(\alpha, 1 - \alpha)$; (b) second generation $VBOC2(\alpha_1, \alpha_2)$; and (c) k th generation $VBOCk(\alpha_1, \alpha_2, \dots, \alpha_k)$.

First Generation $VBOC1(\alpha)$, $VBOC2(\alpha, 1 - \alpha)$: In an effort to overcome major signal design shortcomings of the current BOC modulation waveforms and offer an opportunity to our readers to understand signal design “secrets” I invented the variable binary offset carrier modulation also known as VBOC; found in Chap. 7 of [5].

Because VBOC modulation is a generalized waveform it includes BOC modulation [11]-[14] as a special case. Moreover, there are two types of VBOC modulations: (1) $VBOC1(\alpha)$ and (2) $VBOC2(\alpha, 1 - \alpha)$ [2] where α is known as the single dimensional signal design and optimization parameter [3]. We are briefly going to introduce these modulations here.

Second Generation $2(\alpha_1, \alpha_2)$: The second generation $VBOC2(\alpha_1, \alpha_2)$ is a generalized modulation of the first generation $VBOC1(\alpha)$ and $VBOC2(\alpha, 1 - \alpha)$ or the first

generation $VBOC1(\alpha)$ and $VBOC2(\alpha, 1 - \alpha)$ are special cases of the second generation $VBOC2(\alpha_1, \alpha_2)$ where α_1, α_2 are known as the two dimensional signal design and optimization parameters.

The k th Generation $(\alpha_1, \alpha_2, \dots, \alpha_k)$: The k th generation $VBOCk(\alpha_1, \alpha_2, \dots, \alpha_k)$ is a generalized modulation of the first and second generation $VBOC1(\alpha)$ and $VBOC2(\alpha_1, \alpha_2)$ or first and second generation $VBOC1(\alpha)$ and $VBOC2(\alpha_1, \alpha_2)$ are special cases of the second generation $VBOCk(\alpha_1, \alpha_2, \dots, \alpha_k)$ where $\alpha_1, \alpha_2, \dots, \alpha_k$ are known as the k th dimensional signal design and optimization parameters.

This paper discusses a subset of the first generations $VBOC1(\alpha)$ due to laborious investigation involved.

Although the derivations provided here are only for a subset of k th generation $VBOCk(\alpha_1, \alpha_2, \dots, \alpha_k)$; however, the methodology of $VBOC1(\alpha)$ pure signal design and optimization [1]-[3], followed here, may asymptotically lead towards discovering the whole truth of k th generation $VBOCk(\alpha_1, \alpha_2, \dots, \alpha_k)$ signal design and optimization methodology.

3 $VBOC1(\alpha)$ Pure Signal Design

Detailed discussion on $VBOC1(\alpha)$ pure signal design includes:

- (1) $VBOC1(\alpha)$ signal definition and discussion; and (2) $VBOC1(\alpha)$ generalized ACF definition and discussion; and (3) $VBOC1(\alpha)$ generalized PSD definition and discussion.

3.1 $VBOC1(\alpha)$ Signal Definition and Discussion

Definition 1: $VBOC1_{(m,n,\alpha)}(t)$ waveform is a generalized, periodic function of the $BOC_{(m,n)}(t)$ waveform with period $2T_s$, known as the subcarrier period, for all values of $t - \infty < t < \infty$.

$$VBOC1_{(m,n,\alpha)}(t) = VBOC1_{(m,n,\alpha)}(t \pm 2T_s), \quad (1)$$

and the relation of integers m, n is given by

$$2mT_s = nT_c, \quad (2)$$

where T_c is the defined as the chipping period.

Definition 2: One sub-carrier period of $VBOC1_{(m,n,\alpha)}(t)$ is the superposition of two pulses: a rising pulse and a falling pulse with amplitude/pulse widths, $+1/w_r$ and $-1/w_f$ respectively, or a falling pulse and a rising pulse amplitude/pulse widths $-1/w_f$ and $+1/w_r$ respectively that satisfy the following

$$w_r = T_s(1 \pm \alpha) = \frac{nT_c(1 \pm \alpha)}{2m} = \frac{T_c(1 \pm \alpha)}{2p}. \quad (3a)$$

$$w_f = T_s(1 \mp \alpha) = \frac{nT_c(1 \mp \alpha)}{2m} = \frac{T_c(1 \mp \alpha)}{2p}, \quad (3b)$$

$$VBOC1_{(m,n,\alpha)}(t) = \sum_{q=-\infty}^{\infty} [p_{w_r}(t_1, w_r) - p_{w_f}(t_2, w_f)] \quad (4)$$

where

$$t_1 = t - 0.5w_r - qT_s, \quad (5a)$$

$$t_2 = t - 0.5w_f - w_r - qT_s, \quad (5b)$$

and $p_w(t - t_0, w)$ is a unit rectangular pulse function with width w centered at t_0 with amplitude $+1$.

Although, definitions 1 and 2 completely define the waveform $VBOC1_{(m,n,\alpha)}(t)$; however, the acceptable range of values of α can be derived from the following theorem.

Theorem 1: Prove that α cannot be smaller than zero and greater than one; i.e., based on *Definitions 1* and *2*, the only acceptable range of α is given by

$$0 \leq \alpha \leq 1^1, \quad (6)$$

The *proof of theorem 1* is straightforward. Since, based on *definitions 1* and *2* the following holds

$$w_r + w_f = 2T_s = \frac{nT_c}{m} = \frac{T_c}{p}, \quad (7a)$$

$$w_f = T_s(1 + \alpha) = \frac{T_c(1 + \alpha)}{2p}. \quad (7b)$$

Hence, we can prove that α cannot be smaller than zero or greater than one in two different ways as follows

$$0 \leq w_r \leq T_s \text{ or } T_s \leq w_f \leq 2T_s \quad (8)$$

Either solution of (8) leads to the desired range of acceptable values of α (6).

Corollary 1: $BOC_{(m,n)}(t)$ is simply $VBOC1_{(m,n,\alpha)}(t)$ for $\alpha = 0$.

Corollary 2: $BPSK_0(t)$ (or $BOC_{(1,1/2)}(t)$: the basic or baseline waveform) is simply $VBOC1_{(m,n,\alpha)}(t)$ or for $\alpha = 1$.

Even though we have defined the $VBOC1_{(m,n,\alpha)}(t)$ signal waveform we want to understand the properties of the signal with itself for values of $0 \leq \tau \leq T$ with T being the observation interval. One means of achieving this by correlating the $VBOC1_{(m,n,\alpha)}(t)$ signal waveform with itself; hence, the definition of ACF, $R_{VBOC1_{(m,n,\alpha)}}(\tau)$.

This concludes $VBOC1(\alpha)$ signal definition and discussion; next we continue with $VBOC1(\alpha)$ generalized ACF definition and discussion.

¹ For ease of analysis we assume $+$ for w_r and $-$ for w_f .

3.2 $VBOC1(\alpha)$ Generalized ACF Definition and Discussion

Definition 3: The ACF of $VBOC1_{(m,n,\alpha)}(t)$ or $R_{VBOC1_{(m,n,\alpha)}}(\tau, T)$ with the variable coefficient $0 \leq \alpha \leq 1$ for all values of τ ; $t_0 - T \leq \tau \leq t_0 + T$, t_0 any arbitrary reference time, is given by

$$R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0) = \frac{v(t) * v(t)}{T}, \quad (9)$$

or in integral form

$$R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0) = \frac{\int_{t_0}^{t_0+T} v(t)v(t \pm \tau) dt}{T}, \quad (10)$$

$$v(t) = VBOC1_{(m,n,\alpha)}(t), \quad (11)$$

There are a couple of things that we should understand about the ACF, $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$, with respect to $\tau = 0$ and $\tau = \infty$. Hence, the following theorem holds.

Theorem 2: Show that $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$ has the maximum value for $(\tau = 0, T < \infty)$; and the minimum value for $(\tau = \infty, T < \infty)$ or $(\tau = \infty, T \rightarrow \infty)$; i.e.,

$$R_{VBOC1_{(m,n,\alpha)}}(\tau = 0, T < \infty, t_0 < \infty) = \max, \quad (12)$$

$$R_{VBOC1_{(m,n,\alpha)}}(\tau \rightarrow \infty, T < \infty, t_0 < \infty) = 0, \quad (13)$$

$$R_{VBOC1_{(m,n,\alpha)}}(\tau < \infty, T \rightarrow \infty, t_0 < \infty) = 0, \quad (14)$$

$$R_{VBOC1_{(m,n,\alpha)}}(\tau \rightarrow \infty, T \rightarrow \infty, t_0 < \infty) = 0, \quad (15)$$

The *proof of theorem 2* is straightforward. First, based on the following inequality

$$\int_{t_0}^{t_0+T} v(t)v(t \pm \tau) dt \leq \int_{t_0}^{t_0+T} |v(t)|^2 dt. \quad (16)$$

we obtain (12). The proof of (13), (14), and (15) are pretty straight forward also. When $T < \infty$ we have $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0) = 0$ for $|\tau| \geq T$ hence (13). Moreover, when $T \rightarrow \infty$ then (14) and (15) hold; i.e., $R_{VBOC1_{(m,n,\alpha)}}(\tau < \infty, T \rightarrow \infty, t_0) \rightarrow 0$ and $R_{VBOC1_{(m,n,\alpha)}}(\tau \rightarrow \infty, T \rightarrow \infty, t_0) \rightarrow 0$ just because $\frac{1}{T} \rightarrow 0$.

The reader can recognize the two most important values of the ACF: (a) max according to (12); (b) and zero according to (13), (14), and (15).

Theorem 3: Show that $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$ is in general not periodic function of t_0 for any values of (τ, T)

$$R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0) \neq R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0 \pm T), \quad (17)$$

$$T \neq nT_c = 2mT_s, m, n \text{ integers.} \quad (17a)$$

It is; however, periodic function of (τ, T) only when

$$R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0) = R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0 \pm T), \quad (18)$$

$$T = nT_c = 2mT_s, m, n \text{ integers.} \quad (18a)$$

moreover, show that when $T = nT_c = 2mT_s$ is only a special case of the definition of the ACF, $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$,

corresponding to the observation interval $t_0 \leq \tau \leq t_0 + nT_c$. The *proof of theorem 3* is straightforward. Based on *definitions 3* we can express T as follows

$$\left| \begin{matrix} T = 2mT_s + T_\varepsilon, \text{ and } 0 \leq T_\varepsilon < \\ m, n \text{ integers} \end{matrix} \right| \begin{matrix} nT_c & 2m \leq n \\ 2mT_s & 2m \geq n \end{matrix} \quad (18b)$$

Based on (18b) and (17), (10) can be written as

$$R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0) = \frac{\int_{t_0}^{t_0+2mT_s} v(t)v(t \pm \tau) dt + \int_{t_0+2mT_s}^{t_0+2mT_s+T_\varepsilon} v(t)v(t \pm \tau) dt}{T}. \quad (19)$$

Since, $v(t)$ is in general not periodic for values of $0 \leq t < 2mT_s$; hence, $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$ is in general not periodic if $T_\varepsilon > 0$; or $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$ is periodic only when $T = nT_c$.

Corollary 3: $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$ is periodic with period $nT_c = 2mT_s$ corresponding to the observation interval $t_0 \leq \tau \leq t_0 + T_c$ and in fact $R_{VBOC1_{(m,n,\alpha)}}(\tau)$ is only one period of ACF $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$ when

$$R_{VBOC1_{(m,n,\alpha)}}(\tau) = \frac{\int_0^{nT_c} v(t)v(t \pm \tau) dt}{nT_c}, \quad (20a)$$

which is the definition of the ACF whose shape we are most familiar with and it has the largest autocorrelation peak.ⁱⁱ

Corollary 4: The ACF of $VBOC1_{(m,n,\alpha)}(t)$ or $R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)$ is in general an even function with respect to τ with the variable coefficient $0 \leq \alpha \leq 1$ for all values of τ ; $-\infty < \tau < \infty$; i.e.,

$$R_{VBOC1_{(m,n,\alpha)}}(-\tau) = R_{VBOC1_{(m,n,\alpha)}}(\tau), \quad (20b)$$

The closed form expression of the generalized ACF, $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$, with respect to p , is subject to *theorem 4*.

Theorem 4: Show that, for values of $p = \{1, 2, 3, 4\}$, $R_{VBOC1_{(m=pn,n,\alpha)}}(\tau)$ is the generalized ACF with respect to p ; i.e., (1) $R_{VBOC1_{(n,n,\alpha)}}(\tau) = R_{VBOC1_{(pn=n,n,\alpha)}}(\tau)$; $p = 1$; (2) $R_{VBOC1_{(2n,n,\alpha)}}(\tau) = R_{VBOC1_{(pn=2n,n,\alpha)}}(\tau)$; $p = 2$; (3) $R_{VBOC1_{(3n,n,\alpha)}}(\tau) = R_{VBOC1_{(pn=3n,n,\alpha)}}(\tau)$; $p = 3$; and (4) $R_{VBOC1_{(4n,n,\alpha)}}(\tau) = R_{VBOC1_{(pn=4n,n,\alpha)}}(\tau)$; $p = 4$; assuming that

$$\tau_{1(\alpha)} = \frac{T_c(1-\alpha)}{2p} = T_s(1-\alpha); \tau_{2(\alpha)} = \frac{T_c(1+\alpha)}{2p}; \tau_3 = \frac{T_c}{p}, \quad (21a)$$

$$\tau_{4(\alpha)} = \frac{T_c(3-\alpha)}{2p}; \tau_{5(\alpha)} = \frac{T_c(3+\alpha)}{2p}; \tau_6 = \frac{2T_c}{p}, \quad (21b)$$

$$\tau_{7(\alpha)} = \frac{T_c(5-\alpha)}{2p}; \tau_{8(\alpha)} = \frac{T_c(5+\alpha)}{2p}; \tau_9 = \frac{3T_c}{p}, \quad (21c)$$

$$\tau_{10(\alpha)} = \frac{T_c(7-\alpha)}{2p}; \tau_{11(\alpha)} = \frac{T_c(7+\alpha)}{2p}; \tau_{12} = \frac{4T_c}{p}, \quad (21d)$$

$R_1(\tau) = R_{VBOC1_{(m=np,n,\alpha)}}(\tau)$ is given by the expression below:

$$\begin{aligned}
R_1(\tau) = & \left[\begin{array}{l} \underbrace{\frac{1}{a_{1(p,n,\alpha)}} + \frac{-(4p-1)|\tau|}{b_{1(p,n,\alpha)} T_c}}_{r_{1(p,n,\alpha)}; p=\{1,2,3,4\}} |\tau|, |\tau| \leq \tau_{1(\alpha)} \\ \underbrace{\frac{-(p-1)+(2p-1)\alpha}{a_{2(p,n,\alpha)} p} + \frac{-1}{b_{2(p,n,\alpha)} T_c}}_{r_{2(p,n,\alpha)}; p=\{1,2,3,4\}} |\tau|, \tau_{1(\alpha)} \leq |\tau| \leq \tau_{2(\alpha)} \\ \underbrace{-\frac{3p-2}{a_{3(p,n,\alpha)} p} + \frac{4p-3}{b_{3(p,n,\alpha)} T_c}}_{r_{3(p,n,\alpha)}; p=\{1,2,3,4\}} |\tau|, \tau_{2(\alpha)} \leq |\tau| \leq \tau_3 \\ \underbrace{\frac{5p-6}{a_{4(p,n,\alpha)} p} + \frac{-(4p-5)|\tau|}{b_{4(p,n,\alpha)} T_c}}_{r_{4(p,n,\alpha)}; p=\{2,3,4\}} |\tau|, \tau_3 \leq |\tau| \leq \tau_{4(\alpha)} \\ \underbrace{\frac{-(p-3)+(2p-3)\alpha}{a_{5(p,n,\alpha)} p} + \frac{-1}{b_{5(p,n,\alpha)} T_c}}_{r_{5(p,n,\alpha)}; p=\{2,3,4\}} |\tau|, \tau_{4(\alpha)} \leq |\tau| \leq \tau_{5(\alpha)} \\ \underbrace{-\frac{7p-12}{a_{6(p,n,\alpha)} p} + \frac{(4p-7)|\tau|}{b_{6(p,n,\alpha)} T_c}}_{r_{6(p,n,\alpha)}; p=\{2,3,4\}} |\tau|, \tau_{5(\alpha)} \leq |\tau| \leq \tau_6 \\ \underbrace{\frac{9p-20}{a_{7(p,n,\alpha)} p} + \frac{-(4p-9)|\tau|}{b_{7(p,n,\alpha)} T_c}}_{r_{7(p,n,\alpha)}; p=\{3,4\}} |\tau|, \tau_6 \leq |\tau| \leq \tau_{7(\alpha)} \\ \underbrace{\frac{-(p-5)+(2p-5)\alpha}{a_{8(p,n,\alpha)} p} + \frac{-1}{b_{8(p,n,\alpha)} T_c}}_{r_{8(p,n,\alpha)}; p=\{3,4\}} |\tau|, \tau_{7(\alpha)} \leq |\tau| \leq \tau_{8(\alpha)} \\ \underbrace{-\frac{(11p-30)}{a_{9(p,n,\alpha)} p} + \frac{(4p-11)|\tau|}{b_{9(p,n,\alpha)} T_c}}_{r_{9(p,n,\alpha)}; p=\{3,4\}} |\tau|, \tau_{8(\alpha)} \leq |\tau| \leq \tau_9 \\ \underbrace{\frac{13p-42}{a_{10(p,n,\alpha)} p} + \frac{-(4p-13)|\tau|}{b_{10(p,n,\alpha)} T_c}}_{r_{10(p,n,\alpha)}; p=\{4\}} |\tau|, \tau_9 \leq |\tau| \leq \tau_{10(\alpha)} \\ \underbrace{\frac{-(p-7)+(2p-7)\alpha}{a_{11(p,n,\alpha)} p} + \frac{-1}{b_{11(p,n,\alpha)} T_c}}_{r_{11(p,n,\alpha)}; p=\{4\}} |\tau|, \tau_{10(\alpha)} \leq |\tau| \leq \tau_{11(\alpha)} \\ \underbrace{-\frac{(15p-56)}{a_{12(p,n,\alpha)} p} + \frac{(4p-15)|\tau|}{b_{12(p,n,\alpha)} T_c}}_{r_{12(p,n,\alpha)}} |\tau|, \tau_{11(\alpha)} \leq |\tau| \leq \tau_{12} \\ 0, |\tau| \geq T_c \end{array} \right], \quad (22)
\end{aligned}$$

$$T = nT_c = 2mT_s, \text{ and } m, n \text{ integers}, \quad (22a)$$

The *proof of theorem 4* is straightforward.

First, we recognize that substituting $p = 1$ in (22) we get

$$R_1(\tau) = \left[\begin{array}{l} 1 - \frac{(4p-1)|\tau|}{T_c}, |\tau| \leq \tau_{1(\alpha)} \\ \frac{-(p-1)+(2p-1)\alpha}{p} - \frac{|\tau|}{T_c}, \tau_{1(\alpha)} \leq |\tau| \leq \tau_{2(\alpha)} \\ -\frac{3p-2}{p} + \frac{(4p-3)|\tau|}{T_c}, \tau_{2(\alpha)} \leq |\tau| \leq \tau_3 \\ 0, |\tau| \geq T_c \end{array} \right], \quad (23)$$

Equation (23), for $p = 1$, is the same as in Chap. 7 [5] and (78) in [6].

Second, we recognize that substituting $p = 2$ in (22) we get

$$R_1(\tau) = \left[\begin{array}{l} 1 - \frac{(4p-1)|\tau|}{T_c}, |\tau| \leq \tau_{1(\alpha)} \\ \frac{-(p-1)+(2p-1)\alpha}{p} - \frac{|\tau|}{T_c}, \tau_{1(\alpha)} \leq |\tau| \leq \tau_{2(\alpha)} \\ -\frac{3p-2}{p} + \frac{(4p-3)|\tau|}{T_c}, \tau_{2(\alpha)} \leq |\tau| \leq \tau_3 \\ \frac{5p-6}{p} - \frac{(4p-5)|\tau|}{T_c}, \tau_3 \leq |\tau| \leq \tau_{4(\alpha)} \\ \frac{-(p-3)+(2p-3)\alpha}{p} - \frac{|\tau|}{T_c}, \tau_{4(\alpha)} \leq |\tau| \leq \tau_{5(\alpha)} \\ -\frac{7p-12}{p} + \frac{(4p-7)|\tau|}{T_c}, \tau_{5(\alpha)} \leq |\tau| \leq \tau_6 \\ 0, |\tau| \geq T_c \end{array} \right]. \quad (24)$$

Equation (24), for $p = 2$, is the same as in Chap. 7 [5] and (81) in [6].

Third, we recognize that that for up to values $p = 3$ in (22) we get

$$R_1(\tau) = \left[\begin{array}{l} 1 - \frac{(4p-1)|\tau|}{T_c}, |\tau| \leq \tau_{1(\alpha)} \\ \frac{-(p-1)+(2p-1)\alpha}{p} - \frac{|\tau|}{T_c}, \tau_{1(\alpha)} \leq |\tau| \leq \tau_{2(\alpha)} \\ -\frac{3p-2}{p} + \frac{(4p-3)|\tau|}{T_c}, \tau_{2(\alpha)} \leq |\tau| \leq \tau_3 \\ \frac{5p-6}{p} - \frac{(4p-5)|\tau|}{T_c}, \tau_3 \leq |\tau| \leq \tau_{4(\alpha)} \\ \frac{-(p-3)+(2p-3)\alpha}{p} - \frac{|\tau|}{T_c}, \tau_{4(\alpha)} \leq |\tau| \leq \tau_{5(\alpha)} \\ -\frac{7p-12}{p} + \frac{(4p-7)|\tau|}{T_c}, \tau_{5(\alpha)} \leq |\tau| \leq \tau_6 \\ \frac{9p-20}{p} - \frac{(4p-9)|\tau|}{T_c}, \tau_6 \leq |\tau| \leq \tau_{7(\alpha)} \\ \frac{-(p-5)+(2p-5)\alpha}{p} - \frac{|\tau|}{T_c}, \tau_{7(\alpha)} \leq |\tau| \leq \tau_{8(\alpha)} \\ -\frac{(11p-30)}{p} + \frac{(4p-11)|\tau|}{T_c}, \tau_{8(\alpha)} \leq |\tau| \leq \tau_9 \\ 0, |\tau| \geq T_c \end{array} \right]. \quad (25)$$

Equation (25), for $p = 3$, is the same as in Chap. 7 [5].

Equation (22), for $p = 4$, is the same as in Chap. 7 [5]. This completed the *proof of theorem 4*.

Corollary 5: From *corollary 1* $VBOC1_{(m,n,\alpha=0)}(t) = BOC_{(m,n)}(t)$ and from *theorem 4*, the ACF $R_{VBOC1_{(m,n,\alpha=0)}}(\tau) = R_{BOC_{(m,n)}}(\tau)$ hence, prove that the generalized ACF with respect to p ; i.e., is: (1) $R_{BOC_{(n,n)}}(\tau) = R_{VBOC1_{(pn=n,n,\alpha=0)}}(\tau)$; $p = 1$; (2) $R_{BOC_{(2n,n)}}(\tau) = R_{VBOC1_{(pn=2n,n,\alpha=0)}}(\tau)$; $p = 2$; (3) $R_{BOC_{(3n,n)}}(\tau) = R_{VBOC1_{(pn=3n,n,\alpha=0)}}(\tau)$; $p = 3$; (4) $R_{BOC_{(4n,n)}}(\tau) = R_{VBOC1_{(pn=4n,n,\alpha=0)}}(\tau)$; $p = 4$.

$$R_{BOC(p,n)}(\tau) = \begin{cases} \underbrace{\frac{1}{p} + \frac{-(4p-1)}{T_c} |\tau|}_{a_{1(p,n)} \quad b_{1(p,n)}}, & |\tau| \leq \frac{T_c}{2p} = T_s \\ \underbrace{-\frac{3p-2}{p} + \frac{4p-3}{T_c} |\tau|}_{a_{3(p,n)} \quad b_{3(p,n)}}, & \frac{T_c}{2p} \leq |\tau| \leq \frac{T_c}{p} \\ \underbrace{\frac{5p-6}{p} + \frac{-(4p-5)}{T_c} |\tau|}_{a_{5(p,n)} \quad b_{5(p,n)}}, & \frac{T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ \underbrace{-\frac{7p-12}{p} + \frac{(4p-7)}{T_c} |\tau|}_{a_{7(p,n)} \quad b_{7(p,n)}}, & \frac{3T_c}{2p} \leq |\tau| \leq \frac{2T_c}{p} \\ \underbrace{-\frac{(11p-30)}{p} + \frac{(4p-11)}{T_c} |\tau|}_{a_{9(p,n)} \quad b_{9(p,n)}}, & \frac{5T_c}{2p} \leq |\tau| \leq \frac{3T_c}{p} \\ \underbrace{\frac{13p-42}{p} + \frac{-(4p-13)}{T_c} |\tau|}_{a_{10(p,n)} \quad b_{10(p,n)}}, & \frac{3T_c}{p} \leq |\tau| \leq \frac{7T_c}{2p} \\ \underbrace{-\frac{(15p-56)}{p} + \frac{(4p-15)}{T_c} |\tau|}_{a_{12(p,n)} \quad b_{12(p,n)}}, & \frac{7T_c}{2p} \leq |\tau| \leq \frac{4T_c}{p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (26)$$

$$T = nT_c = 2mT_s, \text{ and } m, n \text{ integers}, \quad (26a)$$

Proof of corollary 5: The *proof of corollary 5* is straightforward.

First, we recognize that substituting $p = 1$ in (26) we get

$$R_{BOC(p,n)}(\tau) = \begin{cases} 1 - \frac{(4p-1)|\tau|}{T_c}, & |\tau| \leq \frac{T_c}{2p} = T_s \\ -\frac{3p-2}{p} + \frac{(4p-3)|\tau|}{T_c}, & \frac{T_c}{2p} \leq |\tau| \leq \frac{T_c}{p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (27)$$

$$T = nT_c = 2mT_s, \text{ and } m = np, n \text{ integers}. \quad (27a)$$

Equation (27), for $p = 1$, is the same as in Chap. 7 [5] and (43) and (79) in [6].

Second, we recognize that substituting $p = 2$ in (26) we get

$$R_{BOC(p,n)}(\tau) = \begin{cases} 1 - \frac{(4p-1)|\tau|}{T_c}, & |\tau| \leq \frac{T_c}{2p} = T_s \\ -\frac{3p-2}{p} + \frac{(4p-3)|\tau|}{T_c}, & \frac{T_c}{2p} \leq |\tau| \leq \frac{T_c}{p} \\ \frac{5p-6}{p} - \frac{(4p-5)|\tau|}{T_c}, & \frac{T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ -\frac{7p-12}{p} + \frac{(4p-7)|\tau|}{T_c}, & \frac{3T_c}{2p} \leq |\tau| \leq \frac{2T_c}{p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (28)$$

$$T = nT_c = 2mT_s, \text{ and } m = np, n \text{ integers}. \quad (28a)$$

Equation (28), for $p = 2$, is the same as in Chap. 7 [5] and (82) in [6].

Third, we recognize that substituting $p = 3$ in (26) we get

$$R_{BOC(p,n)}(\tau) = \begin{cases} 1 - \frac{(4p-1)|\tau|}{T_c}, & |\tau| \leq \frac{T_c}{2p} = T_s \\ -\frac{3p-2}{p} + \frac{(4p-3)|\tau|}{T_c}, & \frac{T_c}{2p} \leq |\tau| \leq \frac{T_c}{p} \\ \frac{5p-6}{p} - \frac{(4p-5)|\tau|}{T_c}, & \frac{T_c}{p} \leq |\tau| \leq \frac{3T_c}{2p} \\ -\frac{7p-12}{p} + \frac{(4p-7)|\tau|}{T_c}, & \frac{3T_c}{2p} \leq |\tau| \leq \frac{2T_c}{p} \\ \frac{9p-20}{p} - \frac{(4p-9)|\tau|}{T_c}, & \frac{2T_c}{p} \leq |\tau| \leq \frac{5T_c}{2p} \\ -\frac{(11p-30)}{p} + \frac{(4p-11)|\tau|}{T_c}, & \frac{5T_c}{2p} \leq |\tau| \leq \frac{3T_c}{p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (29)$$

$$T = nT_c = 2mT_s, \text{ and } m = np, n \text{ integers}. \quad (29a)$$

Equation (25), for $p = 3$, is the same as in Chap. 7 [5].

Equation (26), for $p = 4$, is the same as in Chap. 7 [5]. This completed the *proof of corollary 5*.

Corollary 6: From *corollary 2* $VBOC1_{(m,n,\alpha=1)}(t) = BPSK_0(t)$ and from *theorem 4*, the ACF $R_{VBOC1_{(m,n,\alpha=1)}}(\tau) = R_{BPSK_0}(\tau)$ hence, prove that the generalized ACF with respect to p ; i.e., is

- (1) $R_{BPSK_0}(\tau) = R_{VBOC1_{(pn=n,n,\alpha=1)}}(\tau); p = 1;$
- (2) $R_{BPSK_0}(\tau) = R_{VBOC1_{(pn=2n,n,\alpha=1)}}(\tau); p = 2;$
- (3) $R_{BPSK_0}(\tau) = R_{VBOC1_{(pn=3n,n,\alpha=1)}}(\tau); p = 3;$
- (4) $R_{BPSK_0}(\tau) = R_{VBOC1_{(pn=4n,n,\alpha=1)}}(\tau); p = 4.$

$$R_{BPSK_0}(\tau) = \begin{cases} \underbrace{\frac{1}{p} + \frac{-1}{T_c} |\tau|}_{a_{2(p,n)} \quad b_{2(p,n)}}, & 0 \leq |\tau| \leq \frac{T_c}{p} \\ \underbrace{-\frac{3p-2}{p} + \frac{(4p-3)|\tau|}{T_c}}_{a_{5(p,n)} \quad b_{5(p,n)}}, & \frac{T_c}{p} \leq |\tau| \leq \frac{2T_c}{p} \\ \underbrace{\frac{5p-6}{p} - \frac{(4p-5)|\tau|}{T_c}}_{a_{8(p,n)} \quad b_{8(p,n)}}, & \frac{2T_c}{p} \leq |\tau| \leq \frac{3T_c}{p} \\ \underbrace{-\frac{7p-12}{p} + \frac{(4p-7)|\tau|}{T_c}}_{a_{11(p,n)} \quad b_{11(p,n)}}, & \frac{3T_c}{p} \leq |\tau| \leq \frac{4T_c}{p} \\ 0, & |\tau| \geq T_c \end{cases} \quad (30)$$

$$T = nT_c = 2mT_s, \text{ and } m = np, n \text{ integers}. \quad (30a)$$

The *proof of corollary 6* is straightforward so we leave that as an exercise to the reader.

This concludes $VBOC1(\alpha)$ generalized PSD definition and discussion; next, we continue with $VBOC1(\alpha)$ generalized PSD definition and discussion.

3.3 $VBOC1(\alpha)$ Generalized PSD Definition and Discussion

Definition 4: The PSD of $VBOC1_{(m,n,\alpha)}(t)$ or $G_{VBOC1_{(m,n,\alpha)}}(f)$ with the variable coefficient $0 \leq \alpha \leq 1$ for all values of f ; $-\infty < f < \infty$ is given by

$$G_{VBOC1_{(m,n,\alpha)}}(f, T) = \mathcal{F} \left\{ \underbrace{R_{VBOC1_{(m,n,\alpha)}}(\tau, T, t_0)}_{\frac{V(f)V(f)}{T} = \frac{V^2(f)}{T}} \right\}, \quad (31)$$

where $\mathcal{F}\{v(t)\}$ denotes the Fourier Transform (FT) of $v(t)$; i.e.,

$$V(f) = \mathcal{F}\{VBOC1_{(m,n,\alpha)}(t)\} = \mathcal{F}\{v(t)\}. \quad (32)$$

Now that we have defined the PSD of $VBOC1_{(m,n,\alpha)}(t)$ we formulate *theorem 5* in the same manner as *theorem 2*.

Theorem 5: Show that $G_{VBOC1_{(m,n,\alpha)}}(f, T)$ has the maximum value for $(f = \pm f_0 < \infty, T < \infty)$; and the minimum value for $(\tau = \infty, T < \infty)$ or $(\tau = \infty, T \rightarrow \infty)$; i.e.,

$$G_{VBOC1_{(m,n,\alpha)}}(f = \pm f_0 < \infty, T < \infty, f_0 < \infty) = \max, \quad (33)$$

$$G_{VBOC1_{(m,n,\alpha)}}(f \rightarrow \infty, T < \infty, f_0 < \infty) = 0, \quad (34)$$

$$G_{VBOC1_{(m,n,\alpha)}}(f < \infty, T \rightarrow \infty, f_0 < \infty) = 0, \quad (35)$$

$$G_{VBOC1_{(m,n,\alpha)}}(f \rightarrow \infty, T \rightarrow \infty, f_0 < \infty) = 0. \quad (36)$$

The *proof of theorem 5* is straightforward and is very similar to the *proof of theorem 2* so we leave it as an exercise to the reader.

Corollary 7: The PSD of $VBOC1_{(m,n,\alpha)}(t)$; i.e., $G_{VBOC1_{(m,n,\alpha)}}(f, T)$ is in general NOT periodic simply because $V(f)$ is in general not periodic even if $v(t)$ is periodic.

Corollary 8: The PSD of $VBOC1_{(m,n,\alpha)}(t)$; i.e., $G_{VBOC1_{(m,n,\alpha)}}(f, T)$ is always an even function of $-\infty < f < \infty$; i.e., from (34) the following holds

$$G_{VBOC1_{(m,n,\alpha)}}(-f, T) = G_{VBOC1_{(m,n,\alpha)}}(f, T). \quad (37)$$

Theorem 6: Show that, for values of $p = \{1, 2, 3, 4\}$, $G_{VBOC1_{(m=np,n,\alpha)}}(f)$ is the generalized ACF with respect to p ; i.e., (1) $G_{VBOC1_{(n,n,\alpha)}}(f) = G_{VBOC1_{(pn=n,n,\alpha)}}(f)$; $p = 1$; (2) $G_{VBOC1_{(2n,n,\alpha)}}(f) = G_{VBOC1_{(pn=2n,n,\alpha)}}(f)$; $p = 2$; (3) $G_{VBOC1_{(3n,n,\alpha)}}(f) = G_{VBOC1_{(pn=3n,n,\alpha)}}(f)$; $p = 3$; (4) $G_{VBOC1_{(4n,n,\alpha)}}(f) = G_{VBOC1_{(pn=4n,n,\alpha)}}(f)$; $p = 4$.

Find the generalized PSD of $G_{VBOC1_{(m=np,n,\alpha)}}(f)$ given by

$$G_{VBOC1_{(pn,n,\alpha)}}(f) = \left| \mathcal{F} \left\{ R_{VBOC1_{(pn,n,\alpha)}}(\tau) \right\} \right| = \frac{V(f)V(f)}{T} = \frac{V^2(f)}{T}, \quad (38)$$

or

$$G_{VBOC1_{(pn,n,\alpha)}}(\omega) = 2 \int_0^\infty R_{VBOC1_{(pn,n,\alpha)}}(\tau) \cos(\omega\tau) d\tau, \quad (39)$$

where $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$ is given by (22), $T = nT_c = 2mT_s$, m, n integers and $\omega = 2\pi f$.

Proof of theorem 6: The *proof of theorem 6* is straightforward and is very similar to the *proof of theorem 4*.

Hence, we compute the generalized PSD of $VBOC1_{(pn,n,\alpha)}$, or $G_{VBOC1_{(pn,n,\alpha)}}(\omega)$, as a FT of the ACF such as

$$G_{VBOC1_{(pn,n,\alpha)}}(\omega) = 2 \int_0^\infty \underbrace{R_{VBOC1_{(pn,n,\alpha)}}(\tau) \cos(\omega\tau) d\tau}_{\sum_{i=1}^{12} \int_{\tau_{i-1}}^{\tau_i} \underbrace{r_{i(pn,n,\alpha)}(\tau) \cos(\omega\tau) d\tau}_{g_{i(pn,n,\alpha)}(\omega)}}, \quad (40)$$

where t_i and $r_{i(pn,n,\alpha)}(\tau)$ are given below in the recall of $R_{VBOC1_{(pn,n,\alpha)}}(\tau)$ given by (22).

As we can see from (40) and (22) that the computation of $G_{VBOC1_{(pn,n,\alpha)}}(\omega)$ is in general a laborious process because it involves the computation of twelve integrals to obtain the generalized expression of $G_{VBOC1_{(pn,n,\alpha)}}(\omega)$.

Let us provide the details for obtaining the twelve integrals that lead to the generalized expression of $G_{VBOC1_{(pn,n,\alpha)}}(\omega)$ given by (40) by considering the following integral:

$$g_{i(pn,n,\alpha)}(\omega) = \int_{\tau_{i-1}}^{\tau_i} r_{i(pn,n,\alpha)}(\tau) \cos(\omega\tau) d\tau, \quad (41)$$

where $r_{i(pn,n,\alpha)}(\tau)$ is given by

$$r_{i(pn,n,\alpha)}(\tau) = a_{i(p,n,\alpha)} + \tau b_{i(p,n,\alpha)}, \quad (42)$$

where $a_{i(p,n,\alpha)}$ and $b_{i(p,n,\alpha)}$ are independent of τ given in (22).

Substituting (42) into (41) the following is obtained:

$$g_{i(pn,n,\alpha)}(\omega) = \left| \begin{array}{l} a_{i(p,n,\alpha)} \underbrace{\int_{\tau_{i-1}}^{\tau_i} \cos(\omega\tau) d\tau}_{q_a(\tau_i, \tau_{i-1}, \omega)} + \\ b_{i(p,n,\alpha)} \underbrace{\int_{\tau_{i-1}}^{\tau_i} \tau \cos(\omega\tau) d\tau}_{q_b(\tau_i, \tau_{i-1}, \omega)} \end{array} \right|, \quad (43)$$

In order to compute (43) we compute two integrals $q_a(\tau_i, \tau_{i-1}, \omega)$ as a direct integral as follows:

$$q_a(\tau_i, \tau_{i-1}, \omega) = \int_{\tau_{i-1}}^{\tau_i} \cos(\omega\tau) d\tau = \frac{\sin(\omega\tau_i) - \sin(\omega\tau_{i-1})}{\omega}, \quad (44)$$

and $q_b(\tau_i, \tau_{i-1}, \omega)$ can be computed using integration by parts

$$q_b(\tau_i, \tau_{i-1}, \omega) = \left| \begin{array}{l} \int_{\tau_{i-1}}^{\tau_i} \tau \cos(\omega\tau) d\tau = \\ \frac{\tau_i \sin(\omega\tau_i) - \tau_{i-1} \sin(\omega\tau_{i-1})}{\omega} + \\ \frac{\cos(\omega\tau_i) - \cos(\omega\tau_{i-1})}{\omega^2} \end{array} \right|, \quad (45)$$

Substituting (45) and (44) into (43) we obtain the detailed expression of $g_{i(pn,n,\alpha)}(\omega)$ as follows:

$$g_{i(pn,n,\alpha)}(\omega) = \left| \begin{array}{l} a_{i(p,n,\alpha)} \frac{\sin(\omega\tau_i) - \sin(\omega\tau_{i-1})}{\omega} + \\ b_{i(p,n,\alpha)} \left[\frac{\tau_i \sin(\omega\tau_i) - \tau_{i-1} \sin(\omega\tau_{i-1})}{\omega} + \right. \\ \left. \frac{\cos(\omega\tau_i) - \cos(\omega\tau_{i-1})}{\omega^2} \right] \end{array} \right|, \quad (46)$$

or

$$g_{i(p,n,\alpha)}(\omega) = \left| \begin{array}{l} \frac{[a_{i(p,n,\alpha)} + b_{i(p,n,\alpha)}\tau_i]\sin(\omega\tau_i)}{\omega} - \\ \frac{[a_{i(p,n,\alpha)} + b_{i(p,n,\alpha)}\tau_{i-1}]\sin(\omega\tau_{i-1})}{\omega} +, \end{array} \right. \quad (47)$$

$$b_{i(p,n,\alpha)} \frac{\cos(\omega\tau_i) - \cos(\omega\tau_{i-1})}{\omega^2}$$

Finally, substituting (47) into (40) we obtain the final expression for generalized PSD of $VBOC1_{(pn,n,\alpha)}$ or $G_{VBOC1_{(pn,n,\alpha)}}(\omega)$ as follows

$$G_{VBOC1_{(pn,n,\alpha)}}(\omega) = 2 \left[\begin{array}{l} \underbrace{\sum_{i=1}^3 g_{i(p,n,\alpha)}(\omega)}_{G_1; p=\{1,2,3,4\}} + \\ \underbrace{\sum_{i=4}^6 g_{i(p,n,\alpha)}(\omega)}_{G_2; p=\{2,3,4\}} + \\ \underbrace{\sum_{i=7}^9 g_{i(p,n,\alpha)}(\omega)}_{G_3; p=\{3,4\}} + \\ \underbrace{\sum_{i=10}^{12} g_{i(p,n,\alpha)}(\omega)}_{G_4; p=\{4\}} \end{array} \right], \quad (48)$$

The final expression of the generalized PSD of $G_{VBOC1_{(m=pn,n,\alpha)}}(f)$ as

$$G_{VBOC1_{(pn,n,\alpha)}}(\omega) = 2(G_S + G_C), \quad (49)$$

where

$$G_S = \sum_{i=1}^4 G_{is} \text{ and } G_C = \sum_{i=1}^4 G_{ic}, \quad (50)$$

or

$$G_S = \left[\begin{array}{l} \frac{(p-1)\sin(\frac{\omega T_c}{p})}{p} - \frac{(p-1)\sin(\frac{\omega T_c}{p})}{p} + \frac{(p-2)\sin(\frac{\omega 2T_c}{p})}{p} \\ \underbrace{p=\{1,2,3,4\}} \\ \frac{(p-2)\sin(\frac{\omega 2T_c}{p})}{p} + \frac{(p-3)\sin(\frac{\omega 3T_c}{p})}{p} \\ \underbrace{p=\{2,3,4\}} \\ \frac{(p-3)\sin(\frac{\omega 3T_c}{p})}{p} + \frac{(p-4)\sin(\frac{\omega 4T_c}{p})}{p} \\ \underbrace{p=\{3,4\}} \\ \frac{(p-4)\sin(\frac{\omega 4T_c}{p})}{p} \\ \underbrace{p=\{4\}} \end{array} \right] \omega = 0, \quad (51)$$

and

$$G_C = \left[\begin{array}{l} \frac{(2-4p)\left\{\cos\left[\frac{\omega T_c(1-\alpha)}{2p}\right] + \cos\left[\frac{\omega T_c(1+\alpha)}{2p}\right]\right\}}{p} + (4p-3)\cos\left(\frac{\omega T_c}{p}\right) + 4p-1 \\ \underbrace{p=\{1,2,3,4\}} \\ \frac{(6-4p)\left\{\cos\left[\frac{\omega T_c(3-\alpha)}{2p}\right] + \cos\left[\frac{\omega T_c(3+\alpha)}{2p}\right]\right\}}{p} + (4p-7)\cos\left(\frac{\omega 2T_c}{p}\right) + (4p-5)\cos\left(\frac{\omega T_c}{p}\right) \\ \underbrace{p=\{2,3,4\}} \\ \frac{(-4p+10)\left\{\cos\left[\frac{\omega T_c(5-\alpha)}{2p}\right] + \cos\left[\frac{\omega T_c(5+\alpha)}{2p}\right]\right\}}{p} + (4p-11)\cos\left(\frac{\omega 3T_c}{p}\right) + (4p-9)\cos\left(\frac{\omega 2T_c}{p}\right) \\ \underbrace{p=\{3,4\}} \\ \frac{(14-4p)\left\{\cos\left[\frac{\omega T_c(7-\alpha)}{2p}\right] + \cos\left[\frac{\omega T_c(7+\alpha)}{2p}\right]\right\}}{p} + (4p-15)\cos\left(\frac{\omega 4T_c}{p}\right) + (4p-13)\cos\left(\frac{\omega 3T_c}{p}\right) \\ \underbrace{p=\{4\}} \end{array} \right] \omega^2 T_c, \quad (52)$$

Since, $G_S = 0$ for any values of p ; hence, (49) becomes

$$G_{VBOC1_{(pn,n,\alpha)}}(\omega) = 2G_C, \quad (53)$$

Now, let us compute the individual expressions of $G_{VBOC1_{(pn,n,\alpha)}}(\omega)$ for values of $p = \{1,2,3,4\}$ based on (52)

and (53).

First, for $p = \{1\}$ from (52) and (53) we obtain

$$G_{VBOC1_{(n,n,\alpha)}}(\omega) = \frac{2 \left[-2 \left\{ \cos\left[\frac{\omega T_c(1-\alpha)}{2}\right] + \cos\left[\frac{\omega T_c(1+\alpha)}{2}\right] \right\} + \cos\left(\frac{\omega T_c}{1}\right) + 3 \right]}{\omega^2 T_c}. \quad (54)$$

Equation (54) is identical in Appendix A of Chap. 7 of [5].

Second, for $p = \{2\}$ from (52) and (53) we obtain

$$G_{VBOC1_{(2n,n,\alpha)}}(\omega) = \frac{2 \left[-6 \left\{ \cos\left[\frac{\omega T_c(1-\alpha)}{4}\right] + \cos\left[\frac{\omega T_c(1+\alpha)}{4}\right] \right\} + 8 \cos\left(\frac{\omega T_c}{2}\right) + 7 \right.}{\omega^2 T_c} \left. - 2 \left\{ \cos\left[\frac{\omega T_c(3-\alpha)}{4}\right] + \cos\left[\frac{\omega T_c(3+\alpha)}{4}\right] \right\} + \cos(\omega T_c) \right] \omega^2 T_c}. \quad (55)$$

Equation (55) is identical in Appendix A of Chap. 7 of [5].

Third, for $p = \{3\}$ from (52) and (53) we obtain

$$G_{VBOC1_{(3n,n,\alpha)}}(\omega) = \frac{2 \left[10 \left\{ \cos\left[\frac{\omega T_c(1-\alpha)}{6}\right] + \cos\left[\frac{\omega T_c(1+\alpha)}{6}\right] \right\} + 16 \cos\left(\frac{\omega T_c}{3}\right) + 11 + 8 \cos\left(\frac{\omega 2T_c}{3}\right) \right.}{\omega^2 T_c} \left. - 6 \left\{ \cos\left[\frac{\omega T_c(3-\alpha)}{6}\right] + \cos\left[\frac{\omega T_c(3+\alpha)}{6}\right] \right\} - 2 \left\{ \cos\left[\frac{\omega T_c(5-\alpha)}{6}\right] + \cos\left[\frac{\omega T_c(5+\alpha)}{6}\right] \right\} + \cos(\omega T_c) \right] \omega^2 T_c}. \quad (56)$$

Equation (56) is identical in Appendix A of Chap. 7 of [5].

Fourth and finally, for $p = \{4\}$ from (52) and (53) we obtain

$$G_{VBOC1_{(4n,n,\alpha)}}(\omega) = \frac{2 \left[14 \left\{ \cos\left[\frac{\omega T_c(1-\alpha)}{8}\right] + \cos\left[\frac{\omega T_c(1+\alpha)}{8}\right] \right\} + 24 \cos\left(\frac{\omega T_c}{4}\right) + 15 + 16 \cos\left(\frac{\omega 2T_c}{4}\right) \right.}{\omega^2 T_c} \left. - 10 \left\{ \cos\left[\frac{\omega T_c(3-\alpha)}{8}\right] + \cos\left[\frac{\omega T_c(3+\alpha)}{8}\right] \right\} - 6 \left\{ \cos\left[\frac{\omega T_c(5-\alpha)}{8}\right] + \cos\left[\frac{\omega T_c(5+\alpha)}{8}\right] \right\} + 8 \cos\left(\frac{\omega 3T_c}{4}\right) + \cos(\omega T_c) \right.}{\omega^2 T_c} \left. - 2 \left\{ \cos\left[\frac{\omega T_c(7-\alpha)}{8}\right] + \cos\left[\frac{\omega T_c(7+\alpha)}{8}\right] \right\} \right] \omega^2 T_c}. \quad (57)$$

Equation (57) is identical in Appendix A of Chap. 7 of [5].

Equations (40) through (57) complete the proof of *theorem 6*.

Definition 4, theorems 5 and 6 and corollaries 7 and 8 are the most important *definition, theorem and corollaries* of $VBOC1(\alpha)$ PSDs in general. This concludes $VBOC1(\alpha)$ generalized PSD definition and discussion; next, we continue with $VBOC1(\alpha)$ numerical results or examples.

4 Numerical, Theoretical Results

We discuss two examples: $VBOC1_{(1,1,\alpha)}(t)$ and $VBOC1_{(2,1,\alpha)}(t)$ because $VBOC1_{(1,1,\alpha)}(t)$ is the generalized $BOC_{(1,1)}(t) = VBOC1_{(1,1,\alpha=0)}(t)$ on GPS L1C data signal [5]-[16] and $VBOC1_{(2,1,\alpha)}(t)$ is the generalized $BOC_{(2,1)}(t) = BOC_{(10,5)}(t) = VBOC1_{(2,1,\alpha=0)}(t)$ or the GPS military M-code [5]-[16] on both GPS L1 and L2 frequencies perhaps the two most important waveforms in the GNSS community at the

present time.

4.1 $VBOC1(\alpha)$ Examples

Corollary 9: From *definition 1*, $VBOC1_{(1,1,\alpha)}(t)$ is simply

$$VBOC1_{(m=1,n=1,\alpha)}(t) = VBOC1_{(m=1,n=1,\alpha)}(t \pm 2T_s), \quad (58)$$

and the relation between T_s and T_c is given by

$$2T_s = T_c, \quad (59)$$

hence,

$$VBOC1_{(m=1,n=1,\alpha)}(t) = VBOC1_{(m=1,n=1,\alpha)}(t \pm T_c), \quad (60)$$

Corollary 10: From *definition 2*, $VBOC1_{(1,1,\alpha)}(t)$ is simply

$$w_r = \frac{T_c(1-\alpha)}{2}, \quad w_f = \frac{T_c(1+\alpha)}{2}, \quad (61)$$

$$0 \leq w_r \leq \frac{T_c}{2} \text{ and } \frac{T_c}{2} \leq w_f \leq T_c; \quad 0 \leq \alpha \leq 1, \quad (62)$$

$$VBOC1_{(1,1,\alpha)}(t) = \sum_{p=-\infty}^{\infty} [p_{w_r}(t_1, w_r) + p_{w_f}(t_2, w_f)], \quad (63)$$

where

$$t_1 = t - 0.5w_r - 2pT_s = t - 0.5w_r - pT_c, \quad (64)$$

$$t_2 = t - 0.5w_f - w_r - 2pT_s = t - 0.5w_f - w_r - pT_c, \quad (65)$$

If we consider only one chipping period of $VBOC1_{(1,1,\alpha)}(t)$;

i.e., for $p = 0$ we get

$$VBOC1_{(1,1,\alpha)}(t) = p_{w_r}(t_1, w_r) + p_{w_f}(t_2, w_f), \quad (66)$$

$$t_1 = t - 0.5w_r - 2T_s = t - 0.5w_r - T_c, \quad (67)$$

$$t_2 = t - 0.5w_f - w_r - 2T_s = t - 0.5w_f - w_r - T_c, \quad (68)$$

Based on *definition 3* we find the ACF for $VBOC1_{(1,1,\alpha)}(t)$,

or $R_{VBOC1_{(1,1,\alpha)}}(\tau)$ given by

$$R_{VBOC1_{(1,1,\alpha)}}(\tau) = \begin{cases} 1 - \frac{3|\tau|}{T_c}, & |\tau| \leq \frac{T_c(1-\alpha)}{2} \\ \alpha - \frac{|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{2} \leq |\tau| \leq \frac{T_c(1+\alpha)}{2} \\ -1 + \frac{|\tau|}{T_c}, & \frac{T_c(1+\alpha)}{2} \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c \end{cases}, \quad (69)$$

Because $R_{VBOC1_{(1,1,\alpha)}}(\tau)$ is a special case of $R_{VBOC1_{(m,n,\alpha)}}(\tau)$ it is easy to see that it satisfied all conditions of *theorems 5 and 6*.

Corollary 11: Prove that the ACF of $VBOC1_{(1,1,\alpha)}(t)$ given by (69) will result in the $BOC_{(1,1)}(t)$ for $\alpha = 0$ as indicated in *definition 1* and $BOC_{(1/2,1)}(t)$ for $\alpha = 1$ as indicated in *definition 2*.

Proof of corollary 11: The *proof of corollary 11* is straightforward. First, we substitute values of $\alpha = 0$ in (69) and we get $R_{VBOC1_{(n,n,\alpha=0)}}(\tau) = R_{BOC_{(n,n)}}(\tau)$ as indicated in *corollary 1* (*definition 1* of $VBOC1_{(1,1,\alpha)}(t)$ in Appendix A of Chap. 7 of [5]) as shown in Fig.1 and Figs. 41 and 89 in [6].

$$R_{VBOC1_{(n,n,\alpha=0)}}(\tau) = \begin{cases} 1 - \frac{3|\tau|}{T_c}, & |\tau| \leq \frac{T_c}{2} \\ -1 + \frac{|\tau|}{T_c}, & \frac{T_c}{2} \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c \end{cases} = R_{BOC_{(n,n)}}(\tau) \quad (70)$$

Second, we substitute values of $\alpha = 1$ in (69) and we get $R_{VBOC1_{(n,n,\alpha=1)}}(\tau) = R_{BOC_{(1/2n,n)}}(\tau) = R_{BPSK}(\tau)$ as indicated in *corollary 2* (*definition 2* of $VBOC1_{(1,1,\alpha)}(t)$ in Appendix A of Chap. 7 of [5]) as shown in Figs. 5, 23, and 29 in [6], as follows

$$R_{VBOC1_{(n,n,\alpha=1)}}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T_c}, & 0 \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c \end{cases} = R_{BPSK_0}(\tau) \quad (71)$$

Equations (70) and (71) complete the *proof of corollary 11*.

The ACF for $VBOC1_{(2,1,\alpha)}(t)$ is defined as $R_{VBOC1_{(2,1,\alpha)}}(\tau)$ given by

$$R_{VBOC1_{(2,1,\alpha)}}(\tau) = \begin{cases} 1 - \frac{7|\tau|}{T_c}, & |\tau| \leq \frac{T_c(1-\alpha)}{4} \\ \frac{-1+3\alpha}{2} - \frac{|\tau|}{T_c}, & \frac{T_c(1-\alpha)}{4} \leq |\tau| \leq \frac{T_c(1+\alpha)}{4} \\ -2 + \frac{5|\tau|}{T_c}, & \frac{T_c(1+\alpha)}{4} \leq |\tau| \leq \frac{T_c}{2} \\ 2 - \frac{3|\tau|}{T_c}, & \frac{T_c}{2} \leq |\tau| \leq \frac{T_c(3-\alpha)}{4} \\ \frac{1+\alpha}{2} - \frac{|\tau|}{T_c}, & \frac{T_c(3-\alpha)}{4} \leq |\tau| \leq \frac{T_c(3+\alpha)}{4} \\ -1 + \frac{|\tau|}{T_c}, & \frac{T_c(3+\alpha)}{4} \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c = 4T_s \end{cases} \quad (72)$$

Corollary 12: Prove that the ACF of $VBOC1_{(2,1,\alpha)}(t)$ given by (72) will result in the $BOC_{(2,1)}(t)$ for $\alpha = 0$ as indicated in *corollary 1* (*definition 1* of $VBOC1_{(1,1,\alpha)}(t)$ in Appendix A of Chap. 7 of [5]) and $R_{BOC_{(1/2n,n)}}(\tau) = R_{BPSK}(\tau)$ for $\alpha = 1$ as indicated in *definition 2*.

Proof of corollary 12: The *proof of corollary 12* is straightforward. First, we substitute values of $\alpha = 0$ in (72) and we get $R_{VBOC1_{(2n,n,\alpha=0)}}(\tau) = R_{BOC_{(2n,n)}}(\tau)$ as shown in Fig.1 and indicated in *corollary 1* (*definition 1* of $VBOC1_{(2,1,\alpha)}(t)$ in Appendix A of Chap. 7 of [5]) as follows,

$$R_{VBOC1_{(2n,n,\alpha=0)}}(\tau) = \begin{cases} 1 - \frac{7|\tau|}{T_c}, & |\tau| \leq \frac{T_c}{4} = T_s \\ -2 + \frac{5|\tau|}{T_c}, & \frac{T_c}{4} \leq |\tau| \leq \frac{T_c}{2} \\ 2 - \frac{3|\tau|}{T_c}, & \frac{T_c}{2} \leq |\tau| \leq \frac{3T_c}{4} \\ -1 + \frac{|\tau|}{T_c}, & \frac{3T_c}{4} \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c \end{cases} = R_{BOC_{(2n,n)}}(\tau) \quad (73)$$

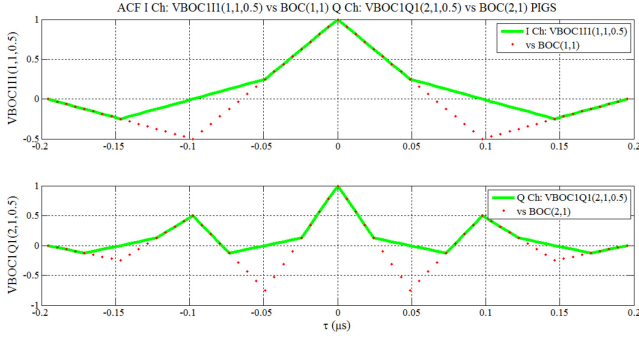


FIGURE 1: ACF of $VBOC1_{(n,n,0.5)}(t)$ vs $BOC_{(n,n)}(t)$ on I channel and $VBOC1_{(2n,n,0.5)}(t)$ vs $BOC_{(2n,n)}(t)$ on Q channel.

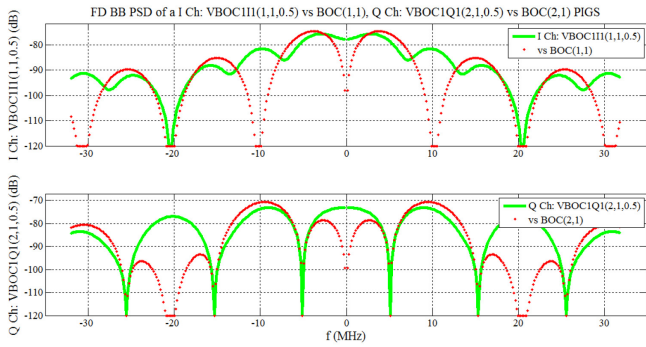


FIGURE 2: PSD of $VBOC1_{(n,n,0.5)}(t)$ vs $BOC_{(n,n)}(t)$ on I channel and $VBOC1_{(2n,n,0.5)}(t)$ vs $BOC_{(2n,n)}(t)$ on Q channel.

Second, we substitute values of $\alpha = 1$ in (72) and we get $R_{VBOC1_{(2n,n,\alpha=1)}}(\tau) = R_{BOC_{(1/2n,n)}}(\tau) = R_{BPSK}(\tau)$ as indicated in *corollary 2* (*definition 2* of $VBOC1_{(2,1,\alpha)}(t)$ in Appendix A of Chap. 7 of [5]) as follows,

$$R_{VBOC1_{(n,n,\alpha=1)}}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T_c}, & 0 \leq |\tau| \leq T_c \\ 0, & |\tau| \geq T_c \end{cases} = R_{BPSK}(\tau) \quad (74)$$

Equations (73) and (74) complete the *proof of corollary 12*.

Corollary 13: From *theorem 6* (54) the PSD of $VBOC1_{(1,1,\alpha)}(t)$ or $G_{VBOC1_{(1,1,\alpha)}}(\omega)$ is given by

$$G_{VBOC1_{(1,1,\alpha)}}(\omega) = \frac{6 - 4\cos\left[\frac{\omega T_c(1-\alpha)}{2}\right] - 4\cos\left[\frac{\omega T_c(1+\alpha)}{2}\right] + 2\cos(\omega T_c)}{\omega^2 T_c} \quad (75)$$

Proof of corollary 13: The *proof of corollary 13* is straightforward. After some simplifications of (75) and $\alpha = 0$ if we substitute $\omega = 2\pi f$ it can be shown that

$$G_{VBOC1_{(1,1,\alpha=0)}}(f) = \left| T_c \text{sinc}^2(f T_c) \tan^2\left(\frac{\pi f T_c}{2}\right) \right| = G_{BOC_{(1,1)}}(f) \quad (76)$$

as indicated in (*corollary 1*) as depicted in Fig. 2 and Figs. 42

and 90 of [6], also after some simplifications of (75) and $\alpha = 1$ we obtain

$$G_{VBOC1_{(1,1,\alpha=1)}}(f) = T_c \text{sinc}^2(f T_c) = G_{BPSK_0}(f) \quad (77)$$

as indicated in (*corollary 2*) as depicted in Figs. 6, 24, and 30 of [6].

From *theorem 6* (55) the PSD of $VBOC1_{(2,1,\alpha)}(t)$ or $G_{VBOC1_{(2,1,\alpha)}}(\omega)$ is given by

$$G_{VBOC1_{(2,1,\alpha)}}(\omega) = \frac{14 - 8\cos\left(\frac{\alpha\omega T_c}{4}\right)3\cos\left(\frac{\omega T_c}{4}\right) + 2\cos(\omega T_c) + 8\cos\left(\frac{\alpha\omega T_c}{4}\right)\cos\left(\frac{\omega 3T_c}{4}\right) + 16\cos\left(\frac{\omega T_c}{2}\right)}{\omega^2 T_c} \quad (78)$$

After some simplifications of (78) and $\alpha = 0$ it can be shown that

$$G_{VBOC1_{(2,1,\alpha=0)}}(\omega) = \frac{\frac{4}{T_c} \tan^2\left(\frac{\omega T_c}{8}\right) \sin^2\left(\frac{\omega T_c}{2}\right)}{\frac{1}{T_c} \tan^2\left(\frac{\omega T_c}{2}\right) \sin^2(2\omega T_c)} \quad (79)$$

as indicated in (*corollary 1*) as depicted in Fig. 2 and Figs. 42 and 90 of [6], also after some simplifications of (79) and $\alpha = 1$ we obtain

$$G_{VBOC1_{(2,1,\alpha=1)}}(f) = T_c \text{sinc}^2(f T_c) = G_{BPSK_0}(f) \quad (80)$$

as indicated in (*corollary 2*) and if we substitute $\omega = 2\pi f$ as depicted in Figs. 6, 24, and 30 of [6].

Equations (76) through (80) complete the proof of *corollary 13*.

In Fig.1 the ACF of $VBOC1_{(1,1,\alpha=0.5)}(t)$ or $r_{VBOC1_{(1,1,\alpha=0.5)}}(\tau)$ is shown with solid green has lower peaks than the ACF of $BOC_{(1,1)}(t)$ or $r_{BOC_{(1,1)}}(\tau)$ in dotted red; hence, $VBOC1_{(1,1,\alpha=0.5)}(t)$ should offer better interference protection than $BOC_{(1,1)}(t)$ or the current GPS III L1 data-code. In Fig.1 also the ACF of $VBOC1_{(2,1,\alpha=0.5)}(t)$ or $r_{VBOC1_{(2,1,\alpha=0.5)}}(\tau)$ is illustrated with solid green has lower peaks than the ACF of $BOC_{(2,1)}(t)$ or $r_{BOC_{(2,1)}}(\tau)$ in dotted red; hence, $VBOC1_{(2,1,\alpha=0.5)}(t)$ should offer better interference protection than $BOC_{(2,1)}(t)$ or GPS M-code.

Similarly, in Fig. 2 the PSD of $VBOC1_{(1,1,\alpha=0.5)}(t)$ or $G_{VBOC1_{(1,1,\alpha=0.5)}}(f)$ is depicted with solid green is quasi-flatter and wider than the PSD of $BOC_{(1,1)}(t)$ or $G_{BOC_{(1,1)}}(f)$ in dotted red; hence, $VBOC1_{(1,1,\alpha=0.5)}(t)$ should offer better interference protection than $BOC_{(1,1)}(t)$ or the current GPS III L1 data-code. In Fig. 2 also the PSD of $VBOC1_{(2,1,\alpha=0.5)}(t)$ or $G_{VBOC1_{(2,1,\alpha=0.5)}}(f)$ is displayed with solid green is quasi-flatter and wider than the PSD of $BOC_{(2,1)}(t)$ or $G_{BOC_{(2,1)}}(f)$ in dotted red; hence, $VBOC1_{(2,1,\alpha=0.5)}(t)$ should offer better interference protection than $BOC_{(2,1)}(t)$ or GPS M-code.

Interference with the current GPS L1 BPSK is avoided by having PSD of $VBOC1_{(\{1,2\},1,\alpha=0.5)}(t)$ orthogonal with the

PSD of GPS L1 BPSK signal [5].

This completes the detailed discussion on $VBOC1(\alpha)$ generalized ACFs and PSDs which contains four definitions, six theorems, and thirteen corollaries.

5 Conclusions

This paper is the first complete discussion on pure signal design for the first generation $VBOC1(\alpha)$ generalized multidimensional geolocation modulation waveforms.

Contrast the results of this paper with previous signal design methodologies, this paper offers for the first time a complete pure signal design methodology subject to both signal design and optimization parameter α and generalized signal design and optimization parameter p .

Signal parameters α and p not only define the waveform $VBOC1(\alpha)$ and generalized ACFs and PSDs but they also play a very important role in the optimization of $VBOC1(\alpha)$ generalized ACFs and PSDs. The computational technique offers a unique and original description of the generalized ACFs and PSDs of $VBOC1(\alpha)$ as functions of both α and p .

In the paper it is argued that the selection of BOC(1,1) on the GPS L1 civil data code and BOC(10,5) (or the military code or M-Code) on both GPS L1 and L2 frequencies is entirely arbitrary because BOC modulation is a special case of $VBOC1(\alpha)$ for $\alpha = 0$ or $\alpha = 1$; hence, all the current state-of-the-art GNSS waveforms exhibit sub-optimal signal design performance even at the end-user when generalized global objective functions are applied.

The above is based on a discussion of $VBOC1(\alpha)$ pure signal optimization in [1]: (1) the criteria for validating the closed form expression of the generalized ACF of $VBOC1(\alpha)$ known as a set of *continuity theorems*; and (2) the criteria for selecting the optimum $0 \leq \alpha \leq 1$ based on a set of criteria known as *optimization theorems* regardless of generalized parameter p (or subcarrier frequency).

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ⁱ The definition given here avoids that two possible signals are defined : one where the $+1/w_r$ part is longer than the $-1/w_f$ and one where the $-1/w_f$ part is longer than the positive $+1/w_r$; hence, avoiding allowing the parameter α to take on values in the range $-1 \leq \alpha \leq 1$; hence, reducing the complexity of continuity and optimization theorems by a factor of two.

ⁱⁱ In general, there are an infinite number of shapes of ACFs because T can be any arbitrary number; however, only one of them has the largest ACF peak the one for which $T = nT_c$.