

Research Article

Quaternionic Analysis of the Influences of Systematic Biases on Radar Data Processing

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For single radar tracking, Kalman filter (KF) can be used only when radar measurements have Gaussian white noises. However, when systematic biases exist, application of KF for single radar tracking is not trivial. Furthermore, in the existing radar network, the association (both measurement-to-measurement and measurement-to-track) problem and the systematic biases removal are mixed together, which further complicates radar network data fusion. The key to solving these problems is to find and apply a set of mathematical tools (quaternionic analysis) on the biased measurements to directly enable the application of KF. This paper provides a detailed discussion of quaternionic analysis that can be applied to solve this problem. Measurement frame (MF) is proposed where target states contain radar systematic biases. It is proved that for stationary and first kind of mobile radar (installed on gyro-stabilized platform) target kinematic model (TKM) in MF has the same form with that in East-North-Up (ENU) frame. And KF can be used directly for the biased measurements to obtain the biased target track in the sense of minimum mean square error (MMSE). However, for the second kind of mobile radars (rigidly connected and sways with platform), a first-order approximation yields large errors therefore they (or second kind of mobile radars) cannot be applied well when attitude biases are greater than 1° . This situation will be investigated and discussed further in future publications.

1. Introduction

Systematic errors are biases in measurement where *the mean of many separate measurements differs significantly from the actual value of the measured attribute*. All measurements are prone to systematic errors, often of several different types such as *imperfect calibration of measurement instruments* (zero error), *changes in the environment which interfere with*

the measurement process and sometimes *imperfect methods of observation* can be either constant error or percentage error [1]. For example, *distance measured by radar will be systematically overestimated if the slight slowing down of the waves in air is not accounted for*.

Systematic biases (SBs) may also be present as the result of an estimate based on a mathematical model or physical law [1]. In radar registration, biases typically include: (1) *offset*

biases (OBs) such as range, gain of the range, azimuth, and elevation biases, and (2) attitude biases (ABs) such as yaw, pitch, and roll biases which are generated from accumulated errors in Inertial Measurement Units (IMUs) of the Inertial Navigation System (INS). *All biases in measurements can cause significant deviation of the estimated target locations from the true locations.* When biased measurements of different radars are sent to fusion center to form Integrated Situation Awareness Picture (ISAP), besides inaccurate target location reports, ghost targets or miss targets always exist, which are harmful to ISAP. In order to overcome the adverse influences, many registration methods were proposed to run before fusion [2]-[5].

Usually, the registration methods used raw measurements to establish registration equation. However, these methods are unpractical because the measurement pairs of the same target cannot be found easily in the SB environment. Then, the methods using local tracks (LTs) obtained from Kalman filter (KF) are needed [2]-[5].

For KF, as shown in Figure 1(a), the measurement noises must be zero-mean Gaussian white noises; however, for the biased measurements, the “total” noises will be the sum of random noises (RNs) and corresponding biases, and their means will not be zeroes; hence, for biased measurements, KF cannot be applied directly. In order to use KF, RNs should be considered as the only outer noises, and the SBs are contained in target states, as shown in Figure 1(b), we call this frame as *measurement frame* (MF), where radar does not know that the biases exist. Then the key issue left for KF is how to describe target kinematic equation (TKE, specific mathematic expression for target movement) in MF mathematically [6]-[9]. As we all know, target kinematic models (TKMs, the type of the target movement in space) such as constant velocity (CV) / acceleration (CA) or coordinate turn (CT) etc., has different mathematical expressions in ENU (inertial) frame [6]-[9]. If true target coordinates (TTCs) in MF can be expressed by the coordinates in ENU frame and radar SBs, then, TKE in MF can be obtained by substituting these relational expressions to TKE in ENU [6]-[9].

Intuitively, just as in the previous discussion in [2]-[5], the relational expressions of TTCs between MF and ENU could be obtained by using trigonometric functions according to the geometry shown in Figure 2. In fact, this method does not work because concise TTCs in ENU cannot be written out in this way. However, quaternion can be used. All the rotations

from ENU frame to MF caused by angular biases can be expressed by corresponding unit quaternion. According to quaternion operation rules, the total unit quaternion expressing the effects of all the rotations can be written out easily as a rotation matrix (RM) using corresponding quaternion elements. As for the range biases, trigonometric functions can be used. The great advantage here using quaternion is that quaternion can express rotation of any direction, however, for trigonometric functions, the angle only defined referenced by three orthogonal axes as x , y , and z , respectively. Unfortunately, for this compound rotation (sequential rotates azimuth bias about z axis and elevation bias about the line perpendicular to visual axis), its resultant rotation axis can be in any direction [11]-[14], [17]-[45].

When TTCs relational expressions are obtained, all kinds of TKMs in MF can be written out by substituting these expressions to corresponding TKE in ENU frame. RM plays a major role to determine whether the TKM in FM is the same as that in ENU frame. The existence of nonlinear function in terms of RM causes the changes of TKM in MF. If the terms of RM has simpler form or the smaller magnitudes of nonlinear variables in RM terms, the smaller changes of TKM between MF and ENU. This applied for the stationary and the first kind mobile radar (detailed description see Section 2.1) situation well. This interesting result also manifests that, in MF, KF can be used directly and completely the same as in ENU frame as if no biases ever existed. Furthermore, it manifests that target states and SBs are dependent, they cannot be estimated separately by single radar itself.

However, unfortunately, for the second kind mobile radar, the first-order approximation of the terms of RM have large errors and complicated nonlinear functions which makes TKM in FM quite different and difficult with that in ENU in form. This situation will be investigated and discussed in future publications based on discussion similar to the additive dual quaternion error (ADQE) model and multiplicative dual quaternion error (MDQE) model proposed by Wu et al. [45] in 2006.

Since the stationary radar is a special case of the first kind mobile radar when attitude biases are zeros and the second kind mobile radar is a special case of the first kind when attitude angles are zeros¹, this relationship is applied to test the correctness of the derivations in the article.

Quaternion method demonstrates its superiority by successfully and widely-used for strapdown inertial

navigation system to estimate platform's attitudes via Euler Matrix Generation (EMG) Nagai et al. [22]. In this application, the outputs of IMU sensors are angle or angle rate, so, it is very convenient to use quaternion to express rotation [11]-[14], [17]-[45]. As a state variable, many estimation methods were developed and used for quaternion estimate such as quaternion estimation (QUEST) by Psiaki [20], [21] which is based on eigenvalue decomposition of the Solution of the generalized quadratic program. The QUEST algorithm could be used in context of the generalized eigenvalue problem in [8] which could be a future topic for a future publication. Shin [17] proposed a Quaternion-Based Unscented Kalman Filter for the Integration of GPS and MEMS INS in 2004. For proper modeling of the antenna locations relative to the center-of-mass, star camera-based attitude quaternions describing the orientation of the spacecraft body frame relative to the celestial reference frame have been provided by the TerraSAR-X/TanDEM-X flight dynamics system. Montenbruck et al. [40] in 2011 have employed star camera based attitude quaternions describing the orientation of the spacecraft body frame relative to the celestial reference frame via the TerraSAR-X/TanDEM-X flight dynamics system. Sooner or later quaternion method has found and will be employed in future geolocation applications Progrid [7]-[9] 2015, in robotics applications Bosse et al. [44], in accelerometer- and magnetometer-triads Elkaim [28]-[30] 2000-2007, magnetometer applications by Rogers et al. [39] in 2011 just to name a few. But even so, quaternion method is seldom used for radar data processing. In fact, according to the superiority of quaternion, it can be used well when radar has angle/attitude biases or for mobile radar (radar installs on mobile platform and the platform has attitude biases).

This paper is organized as follows: in Sect. 2, the relational expressions of TTCs between ENU and MF are derived by using quaternionic analysis, which include 2-D radar, stationary and mobile 3-D radars. In Sect. 3, the biased TKE are derived and the state estimation performance by using KF directly for biased measurements is analyzed. Derivations and results of simulated tracks in two different scenarios are assessed in Sect. 4 which includes a maneuvering target and interacting multiple model (IMM). Finally, Sect. 5 summarizes the results of this study.

2. Quaternionic analysis of TTCs between MF and ENU frame

In order to obtain the TKE in MF, it is necessary to first derive the relational expressions of TTCs between MF and ENU. The quaternionic analysis at present is the most reasonable way to achieve this objective. In Sects. 2 and 3, "biased measurements" means measurements contaminated by SBs only and without considering RNs. Situations of 2-D, stationary and mobile 3-D radar expressions are discussed, respectively. For moving 3-D radar, two situations are analyzed according to different radar installation methods [2]. The first kind, radar is installed on the gyro-stabilized platform which can follow ENU [6], [7] frame; the second kind, radar is installed directly on the platform and has the same attitude angles with the platform.

For convenience of discussion, let r_t , θ_t , and ε_t represent the true target range, azimuth, and elevation, respectively; r_b , θ_b , and ε_b represent the corresponding biased measurements. SBs include the range bias, Δr , the gain of the range, k_r , azimuth bias, $\Delta\theta$, and elevation bias, $\Delta\varepsilon$, they can also be called as OBs. The subscript t and b denote the true and biased values, respectively.

2.1. Complex analysis for stationary 2-D radar

In 2-D plane, all the rotations are about the axis perpendicular to the plane. The quaternion in this situation contains only one angular variable. Since complex analysis are precursor to quaternionic analysis it is useful to illustrate quaternionic analysis via complex analysis in 2-D radar first to enable the reader to see the evolution of the quaternionic analysis.

For 2-D radar, 2-D plane rectangular coordinates are typically employed to describe the target coordinates. Let $\mathbf{x}_i = [x_i, y_i]^T$ $i = \{t, b\}$ represent true or biased coordinates, respectively; and the superscript $\dagger$ denotes the matrix transposition. Since the input variables for 2D radar are

$$\begin{cases} r_b = r_t + \Delta r + k_r r_t, \\ \theta_b = \theta_t + \Delta\theta \end{cases}, \mathbf{x}_i = r_i \boldsymbol{\Theta}_i, \text{ and } \boldsymbol{\Theta}_i = \begin{bmatrix} \sin\theta_i \\ \cos\theta_i \end{bmatrix}, \quad (1)$$

then the rectangular coordinates obtained from the measurements can be written as

$$\mathbf{x}_b = \boldsymbol{\Omega} \mathbf{x}_t + \Delta \mathbf{x}_b,$$

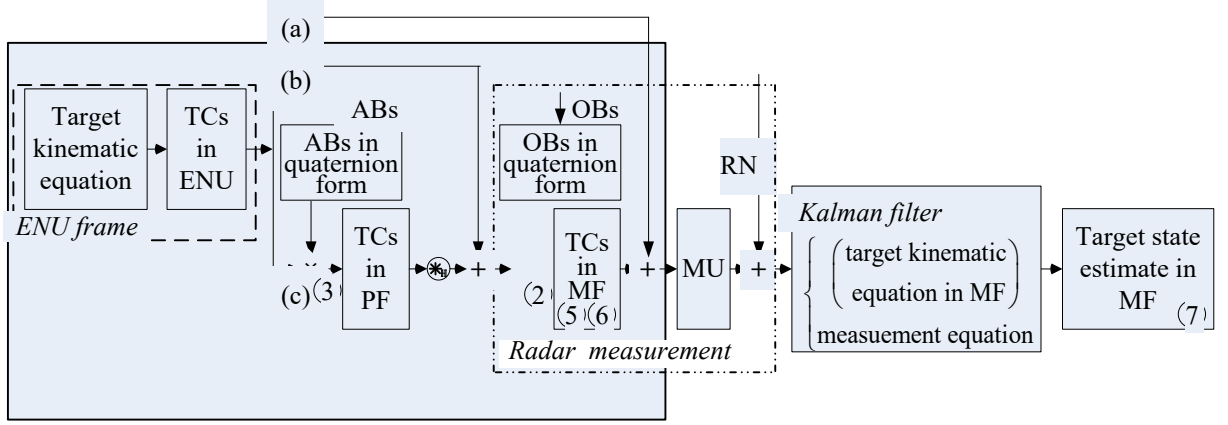


FIGURE 1: Generation of local tracks of 3-D radar via: channel (a) complex analysis for bias-free; channel (b) quaternionic analysis for biased measurements of stationary radar; channel (c) quaternionic analysis for biased measurements of mobile radar. (n in the circle, for $n = \{2, 3, \dots, 7\}$ corresponds to the figure number in the paper that contains detailed information on quaternionic analysis, decomposition, transformation, results etc.)

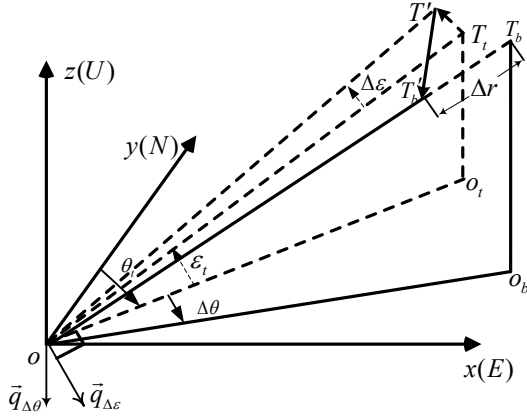


FIGURE 2: Quaternion components description: range, azimuth, and elevation biases on radar measurement.

$$\Omega = \begin{bmatrix} \cos\Delta\theta & \sin\Delta\theta \\ -\sin\Delta\theta & \cos\Delta\theta \end{bmatrix}, \Delta\mathbf{x}_b = (\Delta r + k_r r_t) \mathbf{\Theta}_b. \quad (2)$$

Equation (2)ⁱⁱ demonstrates that the azimuth bias rotate and all the range biases translate the target location, which can be seen in Figure 2 xoy horizontal plane.

2.2. Quaternionic analysis for stationary 3-D radar

For 3-D radar situation, as shown in Figure 1(b), the rotation caused by angle biases can be described by quaternionⁱⁱⁱ [11]-[41] which denotes the rotation along arbitrary axis in the space. And the translations caused by the range biases can be calculated independently from rotation conversions.

(a) Rotation conversion induced by the angle biases

Figure 2 illustrates the influences of the range, azimuth, and elevation biases on radar measurements in ENU frame.

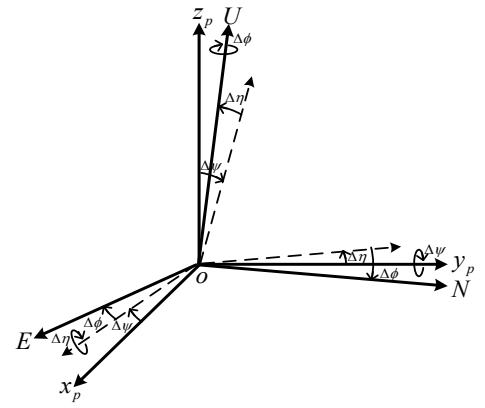


FIGURE 3: Conversion from the PF to ENU.

In Figure 2, radar locates at the origin o , the true target is denoted by T_t and T_b denotes its measurement. Their projections on the horizontal plane xoy are denoted as o_t and o_b , respectively. The true azimuth and azimuth biases are denoted as $\angle yoo_t$ and $\angle yoo_b$, respectively.

Typically, the true north corresponds to zero azimuth and the clockwise direction represents the increment of θ and $\Delta\theta$. The ghost location of the target under the influence of the elevation bias is denoted by T' which locates in the vertical plane $o_t o T_t$; and T'_b denotes the ghost location influenced by azimuth and elevation biases simultaneously. The true elevation and elevation bias are denoted as $\angle o_t o T_t$ and $\angle T_t o T'$, respectively, and the up direction represents their increments.

As shown in Figure 2, when the range biases are not considered, the biased measurements can be obtained from two sequential rotations around radar from TTCs. The first is the rotation of the elevation bias $\Delta\epsilon$ about the line

perpendicular to oo_t in xoy plane as denoted by $\vec{q}_{\Delta\epsilon}$; the second is the rotation of the azimuth bias $\Delta\theta$ about z -axis as denoted by $\vec{q}_{\Delta\theta}$.

Since the first rotation is not about any axis of the ENU frame, the quaternion can be used to represent the rotation; typically, a quaternion or $\vec{q}$ is composed of rotation axis $\vec{l}$ and rotation angle a as [18]

$$\vec{q} = \cos(a/2) + \vec{l}\sin(a/2) = q_0 + \sum_{i=1}^3 q_i \vec{l}_i, \quad (3)$$

where $\vec{l}_1$, $\vec{l}_2$, and $\vec{l}_3$ represent x , y , and z coordinates, respectively, and q_i are the corresponding vector components of rotation axis $\vec{l}$

$$\vec{l} = \frac{\sum_{i=1}^3 q_i \vec{l}_i}{q_{sr}}, \quad q_{sr} = \sqrt{\sum_{i=1}^3 q_i^2}, \quad (4)$$

$$\cos(a/2) = \frac{q_0}{q_{sr0}}, \quad \sin(a/2) = \frac{q_{sr}}{q_{sr0}}, \quad q_{sr0} = \sqrt{\sum_{i=0}^3 q_i^2}. \quad (5)$$

According to the above analyses

$$\vec{q}_{\Delta\epsilon} = \cos(\Delta\epsilon/2) + (\cos\theta_t \vec{l}_1 - \sin\theta_t \vec{l}_2)\sin(\Delta\epsilon/2), \quad (6a)$$

$$\vec{q}_{\Delta\epsilon} = q_{\Delta\epsilon0} + (\vec{q}_{\Delta\epsilon123} = \vec{q}_{\Delta\epsilon12}), \quad (6b)$$

$$\vec{q}_{\Delta\theta} = \cos(\Delta\theta/2) - \vec{l}_3 \sin(\Delta\theta/2), \quad (7a)$$

$$\vec{q}_{\Delta\theta} = q_{\Delta\theta0} + (\vec{q}_{\Delta\theta123} = \vec{q}_{\Delta\theta3}). \quad (7b)$$

The total rotation vector can be obtained as follows

$$\vec{q} = [\vec{q}_{\Delta\theta} \equiv q_{\Delta\theta0} + \vec{q}_{\Delta\theta12}] \star_{\mathbb{H}} [\vec{q}_{\Delta\epsilon} \equiv q_{\Delta\epsilon0} + \vec{q}_{\Delta\epsilon3}] \quad (8a)$$

where $\star_{\mathbb{H}}$ represents Hamilton [11], [12] (or Grassman

$$\mathbf{M} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}, \quad (9b)$$

or in detailed components substituting (8a) into (9a) and (9b), (9a) can be written as

$$\mathbf{x}'_b = \mathbf{M} \equiv \begin{bmatrix} \underbrace{\sin(\tilde{\theta}_b \equiv \theta_b - \Delta\theta) \sin\theta_b [\cos\Delta\epsilon - 1] + \cos\Delta\theta}_{m_{11}} & \underbrace{\sin\theta_b \cos\Delta\epsilon \cos\tilde{\theta}_b - \cos\theta_b \sin\tilde{\theta}_b}_{m_{12}} & \underbrace{-\sin\theta_b \sin\Delta\epsilon}_{m_{13}} \\ \underbrace{\cos\Delta\epsilon \cos\theta_b \sin\tilde{\theta}_b - \sin\theta_b \cos\tilde{\theta}_b}_{m_{21}} & \underbrace{\cos\theta_b \cos\Delta\epsilon \cos\tilde{\theta}_b + \sin\theta_b \sin\tilde{\theta}_b}_{m_{22}} & \underbrace{-\cos\theta_b \sin\Delta\epsilon}_{m_{23}} \\ \underbrace{\sin\tilde{\theta}_b \sin\Delta\epsilon}_{m_{31}} & \underbrace{\cos\tilde{\theta}_b \sin\Delta\epsilon}_{m_{32}} & \underbrace{\cos\Delta\epsilon}_{m_{33}} \end{bmatrix} \mathbf{x}_t \quad (10)$$

All the elements of $\mathbf{M}$ in (10) are further employed in Appendix A. The correctness of the derivation of $\mathbf{M}$ can be easily testified by substitute polar coordinates of $\mathbf{x}'_b$ and $\mathbf{x}_t$ into (9a). Quaternions have had a revival since the late 20th century, primarily due to their utility in describing spatial rotations. The representations of rotations by quaternions are more compact and quicker to compute than the representations by matrices. Hence, the summary formula (10) is a lot easier to understand its derivations via quaternions than via conventional vector or matrix algebra [11].

Moreover, the inverse of $\mathbf{M}$ can be written as $\mathbf{M}^{-1}$, whose

[42]) product^v, given by [41], [42]

$$\vec{q} = \begin{bmatrix} q_{\Delta\epsilon0}q_{\Delta\theta0} - \vec{q}_{\Delta\epsilon12} \cdot \vec{q}_{\Delta\theta3} + q_{\Delta\epsilon0}\vec{q}_{\Delta\theta3} \\ q_{\Delta\theta0}\vec{q}_{\Delta\epsilon12} + \vec{q}_{\Delta\epsilon12} \times \vec{q}_{\Delta\theta3} \end{bmatrix}, \quad (8b)$$

or

$$\vec{q} = \vec{q}_{\Delta\theta} \star_{\mathbb{H}} \vec{q}_{\Delta\epsilon} (= q_0 + \sum_{i=1}^3 q_i \vec{l}_i), \quad (8c)$$

and

$$q_0 = \cos(\tilde{\Delta\theta} \equiv \frac{\Delta\theta}{2}) \cos(\tilde{\Delta\epsilon} \equiv \frac{\Delta\epsilon}{2}), \quad (8d)$$

$$q_{01} = \cos\tilde{\Delta\theta} \sin\tilde{\Delta\epsilon}, \quad (8e)$$

$$q_{02} = \sin\tilde{\Delta\theta} \sin\tilde{\Delta\epsilon}, \quad (8f)$$

$$q_1 = q_{01} \cos\theta_t - q_{02} \sin\theta_t, \quad (8g)$$

$$q_2 = -q_{01} \sin\theta_t - q_{02} \cos\theta_t, \quad (8h)$$

$$q_3 = -\sin\tilde{\Delta\theta} \cos\tilde{\Delta\epsilon}. \quad (8i)$$

According to the operational rules of quaternion [18], in the ENU frame, the relational expression between the true target location $\mathbf{x}_t = [x_t, y_t, z_t]^T$ and the location obtained from the biased measurement (without considering the range biases) $\mathbf{x}'_b = [x'_b, y'_b, z'_b]^T$ can be written as

$$\mathbf{x}'_b = \mathbf{M} \cdot \mathbf{x}_t \quad (9a)$$

where $\mathbf{M}$ is the *quaternion component rotation matrix from ENU to MF conversion* (Appendix A for its inverse)

and is given by Gebre-Egziabher et al. [28] in 2000, Farrell [25] in 2005 for example and others,

elements are derived in Appendix A.

(b) *Translation induced by the range biases*

When all OBs are considered, the biased target rectangular coordinates $\mathbf{X}_b = [x_b, y_b, z_b]^T$ can be obtained by using biased measurements as (see Figure 2)

$$\begin{cases} x_b = r_b \sin\theta_b \cos\epsilon_b \\ y_b = r_b \cos\theta_b \cos\epsilon_b \\ z_b = r_b \sin\epsilon_b \end{cases} \quad \begin{cases} r_b = r_t + \Delta r + k_r r_t \\ \theta_b = \theta_t + \Delta\theta \\ \epsilon_b = \epsilon_t + \Delta\epsilon \end{cases} \quad (11)$$

Substituting (9a) and (10) into (11), we have

$$\mathbf{x}_b = (\mathbf{x}'_b \equiv \mathbf{M} \mathbf{x}_t) + \Delta \mathbf{x}, \quad (12a)$$

where

$$\Delta \mathbf{x} = [\Delta r + k_r r_t] \begin{bmatrix} \sin \theta_b \cos \varepsilon_b \\ \cos \theta_b \cos \varepsilon_b \\ \sin \varepsilon_b \end{bmatrix}. \quad (12b)$$

Equation (12) is analytical which manifests that all the angle biases make TTCs rotate and the range biases make TTCs translate.

2.3. Quaternionic analysis for mobile 3-D radar

As discussed in [4], two different situations ((a) Radar installed on the gyro-stabilized platform, and (b) Radar installed on the platform directly) are discussed respectively.

(a) Radar installed on the gyro-stabilized platform [4]

In this situation, as shown in Figure 1(c), 3-D radar is installed on the gyro-stabilized platform, which can steadily follow ENU frame through IMU and the servo-system of the platform. However, a set of Euler angles exist between the axis of the gyro-stabilized platform frame (PF) and the ENU frame which are usually called as attitude biases (ABs) and they include the yaw bias $\Delta\phi$, pitch bias $\Delta\eta$, and roll bias $\Delta\psi$.

As shown in Figure 3, the PF has the same origin with ENU, the axes of PF can be denoted as x_p , y_p , and z_p , respectively. The transformation of the target coordinates from PF to ENU is accomplished by first rotating about the y_p -axis by the roll angle $\Delta\psi$, then rotating about the intermediate x_p -axis by the pitch angle $\Delta\eta$, and rotating about the final z_p -axis by the yaw angle $\Delta\phi$. The axes drawn in dashed lines are the intermediate axes. Customarily, the polarity definitions of $\Delta\phi$ and $\Delta\psi$ abide by the left-hand rule, and $\Delta\eta$ abides by the right-hand rule.

Similarly to (6a) and (7a), the quaternion representation for attitude biases can be written as:

$$\vec{q}_{\Delta\psi} = \cos(\Delta\psi/2) - \vec{i}_2 \sin(\Delta\psi/2), \quad (13a)$$

$$\vec{q}_{\Delta\eta} = \cos(\Delta\eta/2) + \vec{i}_1 \sin(\Delta\eta/2), \quad (13b)$$

$$\vec{q}_{\Delta\phi} = \cos(\Delta\phi/2) - \vec{i}_3 \sin(\Delta\phi/2), \quad (13c)$$

The resultant rotation quaternion from ENU to MF can be written as:

$$\vec{q}_{ENU2MF} = \vec{q} *_{\mathbb{H}} \vec{q}_{\Delta} \quad (14)$$

$$\vec{q}_{\Delta} = \vec{q}_{(-\Delta\psi)} *_{\mathbb{H}} \vec{q}_{(-\Delta\eta)} *_{\mathbb{H}} \vec{q}_{(-\Delta\phi)} \quad (14a)$$

Substitute (6a), (7a), and (13a)~(13c) into (14a), similar to (9b), the first-order approximation of the first kind mobile radar quaternion component RM from ENU to MF conversion $\mathbf{M}_{q1}$ can be written as:

$$\mathbf{M}_{q1} \approx \begin{bmatrix} 1 & \Delta\theta - \Delta\phi & \Delta\psi - \sin\theta_t \Delta\varepsilon \\ \Delta\phi - \Delta\theta & 1 & \Delta\eta - \cos\theta_t \Delta\varepsilon \\ \sin\theta_t \Delta\varepsilon - \Delta\psi & \cos\theta_t \Delta\varepsilon - \Delta\eta & 1 \end{bmatrix} \quad (15)$$

Analogous to (12a), the relational expression in this situation can be written as:

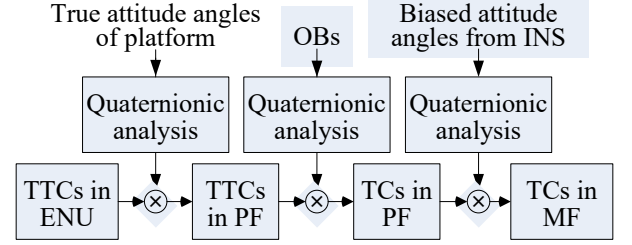


FIGURE 4 Flow chart of TTCs from ENU to MF of the second kind of mobile radar.

$$\mathbf{x}_b = \mathbf{M}_{q1} \mathbf{x}_t + \Delta \mathbf{x}_{q1}, \quad (16a)$$

where

$$\Delta \mathbf{x}_{q1} = [\Delta r + k_r r_t] \begin{bmatrix} \sin \theta_{p_b} \cos \varepsilon_{p_b} \\ \cos \theta_{p_b} \cos \varepsilon_{p_b} \\ \sin \varepsilon_{p_b} \end{bmatrix}, \quad (16b)$$

the subscript “ p_b ” represents biased measurement from platform frame.

In reference [2], the equivalent offset bias expressions caused by platform attitude biases were derived, these equivalent offset biases can be merged into radar offset biases, then the resultant offset biases are substituted into stationary radar relational expressions obtained by quaternion analysis above, and this is the alternative method for obtaining relational expressions for the first mobile radars.

For mobile radar, the ABs can be converted to radar measurement errors such as the range error Δr_c , azimuth error $\Delta\theta_c$, and elevation error $\Delta\varepsilon_c$, respectively, with the expressions of [2]

$$\Delta r_c = 0 + o(\Delta\phi, \Delta\eta, \Delta\psi), \quad (17)$$

$$\Delta\theta_c = -\Delta\phi + \frac{y_t z_t \Delta\psi - x_t z_t \Delta\eta}{x_t^2 + y_t^2} + o(\Delta\phi, \Delta\eta, \Delta\psi), \quad (18)$$

$$\Delta\varepsilon_c = \frac{-x_t \Delta\psi - y_t \Delta\eta}{\sqrt{x_t^2 + y_t^2}} + o(\Delta\phi, \Delta\eta, \Delta\psi), \quad (19)$$

where $\mathbf{x}_t = [x_t, y_t, z_t]^T$ denotes TTCs in the PF where no OBs are considered.

Substituting (17)-(19) into (12), the relational expression of the target location for the first kind of mobile radar can be written as

$$\mathbf{x}_b = \mathbf{M}_1 \mathbf{x}_t + \Delta \mathbf{x}_1, \quad (20a)$$

$$\Delta \mathbf{x}_1 = \Delta r_{rt} \begin{bmatrix} \sin \theta'_t \cos \varepsilon'_t \\ \cos \theta'_t \cos \varepsilon'_t \\ \sin \varepsilon'_t \end{bmatrix}, \quad (20b)$$

$$\begin{cases} \Delta r_{rt} = \Delta r + k_r r_t \\ \theta'_t = \theta_b = \theta_t + \Delta \theta', \\ \varepsilon'_t = \varepsilon_b = \varepsilon_t + \Delta \varepsilon' \end{cases} \quad (20c)$$

where $\mathbf{M}_1 = \mathbf{M}_{q1}$ which represents the quaternion components RM. $\mathbf{M}_1$ can also be obtained by substituting $\Delta \theta$ and $\Delta \varepsilon$ in $\mathbf{M}$ of (10) with $\Delta \theta'$ and $\Delta \varepsilon'$, respectively. And

$$\Delta \theta' = \Delta \theta - \Delta \phi + \frac{y_t z_t \Delta \psi - x_t z_t \Delta \eta}{x_t^2 + y_t^2} + o(\Delta \phi, \Delta \eta, \Delta \psi), \quad (21)$$

$$\Delta \varepsilon' = \Delta \varepsilon + \frac{-x_t \Delta \psi - y_t \Delta \eta}{\sqrt{x_t^2 + y_t^2}} + o(\Delta \phi, \Delta \eta, \Delta \psi). \quad (22)$$

(b) Radar installed on the platform directly

For this kind of mobile radars, they are installed directly on the platform and sway with the platform simultaneously with the same attitude angles. Radar measurement, in this situation, should be rectified by the biased attitude angles obtained from INS. The flow char of TTCs from ENU to MF is given in Figure 4. The conversion from PF to ENU is the same with the first kind of mobile radar as shown in Figure 3, the only difference is to substitute ABs in Figure 3 with true attitude angles.

Quaternion representation of rotation angles in Figure 4 can be written as:

$$\vec{q}_\psi = \cos(\psi/2) - \vec{i}_2 \sin(\psi/2), \quad (23a)$$

$$\vec{q}_\eta = \cos(\eta/2) + \vec{i}_1 \sin(\eta/2), \quad (23b)$$

$$\vec{q}_\phi = \cos(\phi/2) - \vec{i}_3 \sin(\phi/2), \quad (23c)$$

where $i = i_t$ or i_b ($i = \psi, \eta, \phi$), and i_b represents biased angles given by INS, that is $i_b = i_t + \Delta i$.

According to Figure 4, the resultant rotation quaternion can be written as:

$$\vec{q}_{ENU2MF} = \vec{q}_b * \vec{q} * \vec{q}_t \quad (24)$$

$$\vec{q}_b = \vec{q}_{(-\phi_b)} * \vec{q}_{(-\eta_b)} * \vec{q}_{(-\psi_b)} \quad (24a)$$

$$\vec{q}_t = \vec{q}_{\psi_t} * \vec{q}_{\eta_t} * \vec{q}_{\phi_t} \quad (24b)$$

Using (9b), the first-order approximation of the RM of quaternion $\vec{q}_{ENU2MF}$ can be derived as:

$$\tilde{\mathbf{M}}_2 \approx \begin{bmatrix} \tilde{m}_{11} & \tilde{m}_{12} & \tilde{m}_{13} \\ -\tilde{m}_{12} & \tilde{m}_{22} & \tilde{m}_{23} \\ -\tilde{m}_{13} & -\tilde{m}_{23} & \tilde{m}_{33} \end{bmatrix}, \quad (25a)$$

where

$$\tilde{m}_{11} = \tilde{m}_{22} = \tilde{m}_{33} = 1, \quad (25b)$$

$$\tilde{m}_{12} = \begin{bmatrix} -\Delta \phi + \Delta \psi \sin \eta_t + \Delta \theta \cos \eta_t \cos \psi_t \\ -\Delta \varepsilon \sin \eta_t \sin \theta_t + \Delta \varepsilon \sin \psi_t \cos \eta_t \cos \theta_t \end{bmatrix}, \quad (25c)$$

$$\tilde{m}_{13} = \begin{bmatrix} -\Delta \eta \sin \phi_t + \Delta \psi \cos \eta_t \cos \phi_t - \Delta \theta \sin \psi_t \sin \phi_t \\ -\Delta \varepsilon \cos \eta_t \cos \phi_t \sin \theta_t - \Delta \theta \sin \eta_t \cos \psi_t \cos \phi_t \\ + \Delta \varepsilon \sin \phi_t \cos \psi_t \cos \theta_t \\ -\Delta \varepsilon \sin \eta_t \sin \psi_t \cos \theta_t \cos \phi_t \end{bmatrix}, \quad (25d)$$

$$\tilde{m}_{23} = \begin{bmatrix} \Delta \eta \cos \phi_t + \Delta \psi \cos \eta_t \sin \phi_t + \Delta \theta \cos \phi_t \sin \psi_t \\ -\Delta \varepsilon \cos \psi_t \cos \theta_t \cos \phi_t - \Delta \theta \cos \psi_t \sin \eta_t \sin \phi_t \\ -\Delta \varepsilon \sin \phi_t \cos \eta_t \sin \theta_t \\ -\Delta \varepsilon \sin \eta_t \sin \psi_t \sin \phi_t \cos \theta_t \end{bmatrix}. \quad (25e)$$

The approximation of $\tilde{\mathbf{M}}_2$ can be proved by special cases as shown in Appendix D.

The relational expression in this situation can be written as:

$$\mathbf{x}_b = \tilde{\mathbf{M}}_2 \mathbf{x}_t + \Delta \tilde{\mathbf{x}}_2, \quad (26a)$$

where

$$\Delta \tilde{\mathbf{x}}_2 = \mathbf{T}_{p2ENU_b} [\Delta r + k_r r_t] \begin{bmatrix} \sin \theta_{p_b} \cos \varepsilon_{p_b} \\ \cos \theta_{p_b} \cos \varepsilon_{p_b} \\ \sin \varepsilon_{p_b} \end{bmatrix}. \quad (26b)$$

$\mathbf{T}_{p2ENU_b}$ can be calculated from quaternion $\vec{q}_b$ of (24a) or from equation (35).

Similar to the indirect method for the first kind of mobile radar, the second method using first-order approximate quaternion calculations step by step will be given as follows:

Assuming that the true yaw ϕ_t , pitch η_t , and roll ψ_t of the platform are known, the conversion from ENU frame to PF can be described by three sequential rotation transformations. The roll/pitch/yaw angle rotation matrices representing each transformation can be written respectively as [4]

$$\mathbf{T}_{\psi_t} = \begin{bmatrix} \cos \psi_t & 0 & \sin \psi_t \\ 0 & 1 & 0 \\ -\sin \psi_t & 0 & \cos \psi_t \end{bmatrix}, \quad (27a)$$

$$\mathbf{T}_{\eta_t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \eta_t & \sin \eta_t \\ 0 & -\sin \eta_t & \cos \eta_t \end{bmatrix}, \quad (27b)$$

$$\mathbf{T}_{\phi_t} = \begin{bmatrix} \cos \phi_t & -\sin \phi_t & 0 \\ \sin \phi_t & \cos \phi_t & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (27c)$$

TTCs in the PF can be denoted as

$$\mathbf{x}_{p_t} = [x_{p_t}, y_{p_t}, z_{p_t}]^\top = \underbrace{\mathbf{T}_{\psi_t} \mathbf{T}_{\eta_t} \mathbf{T}_{\phi_t}}_{=\mathbf{T}_{ENU2p_t}} \mathbf{x}_t, \quad (28)$$

where $\mathbf{T}_{ENU2p_t}$ denotes the quaternion components rotation matrix^{vii} from the ENU frame to the PF, it can also be obtained from $\vec{q}_t$ of (24b), and

$$\mathbf{T}_{ENU2p_t} = \begin{bmatrix} \underbrace{\cos\psi_t \cos\phi_t - \sin\psi_t \sin\eta_t \sin\phi_t}_{t_{11}} & \underbrace{-\cos\psi_t \sin\phi_t - \cos\phi_t \sin\eta_t \sin\psi_t}_{t_{12}} & \underbrace{\cos\eta_t \sin\psi_t}_{t_{13}} \\ \underbrace{\cos\eta_t \sin\phi_t}_{t_{21}} & \underbrace{\cos\eta_t \cos\phi_t}_{t_{22}} & \underbrace{\sin\eta_t}_{t_{23}} \\ \underbrace{-\sin\psi_t \cos\phi_t - \cos\psi_t \sin\eta_t \sin\phi_t}_{t_{31}} & \underbrace{\sin\psi_t \sin\phi_t - \cos\psi_t \sin\eta_t \cos\phi_t}_{t_{32}} & \underbrace{\cos\psi_t \cos\eta_t}_{t_{33}} \end{bmatrix}. \quad (29)$$

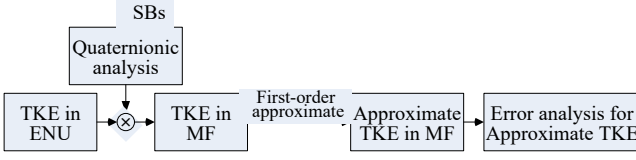


FIGURE 5: Flow chat for quaternionic analysis of the TKM in MF.

According to (28), the true target azimuth in PF can be obtained as

$$\theta_{p_t} = \text{atan}\left(\frac{x_{p_t}}{y_{p_t}}\right) = \text{atan}\left[\frac{x_t t_{11} + y_t t_{12} + z_t t_{13}}{x_t t_{21} + y_t t_{22} + z_t t_{23}}\right]. \quad (30)$$

The true target elevation in PF can be obtained as

$$\varepsilon_{p_t} = \text{asin}\left(\frac{z_{p_t}}{r_{p_t}}\right), \quad (31a)$$

where

$$r_{p_t} = \sqrt{x_{p_t}^2 + y_{p_t}^2 + z_{p_t}^2} = \sqrt{x_t^2 + y_t^2 + z_t^2}, \quad (31b)$$

$$z_{p_t} = x_t t_{31} + y_t t_{32} + z_t t_{33}. \quad (31c)$$

On the analogy of (12a) which is derived for the stationary radar, the relational expression of the target coordinates for the second kind of mobile radar can be written as

$$\mathbf{x}_{p_b} = \begin{bmatrix} x_{p_b} \\ y_{p_b} \\ z_{p_b} \end{bmatrix} = \mathbf{M}_2 \mathbf{x}_{p_t} + \Delta \mathbf{x}_2 \quad (32a)$$

$$\Delta = \begin{bmatrix} 0 & \sin\psi_b \Delta\eta + \cos\eta_b \cos\psi_b \Delta\phi & -\sin\eta_b \Delta\phi - \Delta\psi \\ -\sin\psi_b \Delta\eta - \cos\eta_b \cos\psi_b \Delta\phi & 0 & -\cos\psi_b \Delta\eta + \cos\eta_b \sin\psi_b \Delta\phi \\ \sin\eta_b \Delta\phi + \Delta\psi & \cos\psi_b \Delta\eta - \cos\eta_b \sin\psi_b \Delta\phi & 0 \end{bmatrix}. \quad (36)$$

For detailed derivations of the *quaternion component rotation matrix from PF to ENU* (35) see Appendix B^{viii}.

Substituting (33), (35), and (36) into (34), the relational expression between TTCs and the biased coordinates in the ENU frame can be obtained as

$$\mathbf{x}_b = \mathbf{T}_{ENU2p_t}^{-1} \tilde{\Delta} \mathbf{M}_2 \mathbf{T}_{ENU2p_t} \mathbf{x}_t + \mathbf{T}_{ENU2p_t}^{-1} \tilde{\Delta} \Delta \mathbf{x}_2. \quad (37)$$

According to (35), we can obtain that

$$\mathbf{T}_{ENU2p_t} = \tilde{\Delta} \mathbf{T}_{ENU2p_t}^{-1}. \quad (38)$$

Comparing (37) with (26a), it is obvious that the two first-order approximate matrices:

$$\tilde{\mathbf{M}}_2 = \mathbf{T}_{p2ENU_b} \mathbf{M}_2 \tilde{\Delta} \mathbf{T}_{ENU2p_t}^{-1}. \quad (39)$$

Substituting (38) and (39) into (37), another form of (37) can be written as

$$\Delta \mathbf{x}_2 = \Delta r_{rt} \begin{bmatrix} \sin\theta'_{p_t} \cos\varepsilon'_{p_t} \\ \cos\theta'_{p_t} \cos\varepsilon'_{p_t} \\ \sin\varepsilon'_{p_t} \end{bmatrix}, \quad \begin{cases} \Delta r_{rt} = \Delta r + k_r r_t \\ \theta'_{p_t} = \theta_{p_t} + \Delta\theta \\ \varepsilon'_{p_t} = \varepsilon_{p_t} + \Delta\varepsilon \end{cases} \quad (32b)$$

where $\mathbf{M}_2$ denotes the RM which has the same form as the *quaternion components rotation matrix* $\mathbf{M}$ in (10). The only difference between two matrices is that θ_{p_t} is substituted for θ_b in $\mathbf{M}_2$. The difference between $\mathbf{x}_{p_b}$ and $\mathbf{x}_{p_t}$ is that the former contains radar OBs and the latter has no OBs.

Substituting (28) into (32a), (32a) can be written further as

$$\mathbf{x}_{p_b} = (\mathbf{M}_2 \mathbf{T}_{ENU2p_t} \approx \tilde{\mathbf{M}}_2) \mathbf{x}_t + \Delta \mathbf{x}_2. \quad (33)$$

The measurements here are usually rectified by the attitude angles provided by INS. These attitude angles are biased which include yaw angle ϕ_b , pitch angle η_b , and roll angle ψ_b , RNs in these attitude measurements are not considered for simplicity. The rectified target coordinates in ENU frame can be written as

$$\mathbf{x}_b = [x_b, y_b, z_b]^T = \mathbf{T}_{p2ENU_b} \mathbf{x}_{p_b}, \quad (34)$$

where

$$\mathbf{T}_{p2ENU_b} \left| \begin{aligned} &= \mathbf{T}_{ENU2p_b}^\dagger = \mathbf{T}_{\phi_b}^\dagger \mathbf{T}_{\eta_b}^\dagger \mathbf{T}_{\psi_b}^\dagger \\ &\approx (\mathbf{T}_{\phi_t}^\dagger \mathbf{T}_{\eta_t}^\dagger \mathbf{T}_{\psi_t}^\dagger = \mathbf{T}_{ENU2p_t}^{-1}) (\tilde{\Delta} \equiv \mathbf{I} + \Delta) \end{aligned} \right. \quad (35)$$

which denotes the *quaternion component rotation matrix from PF to ENU* frame by using biased navigation information, and Δ is given by

$$\mathbf{x}_b = \tilde{\mathbf{M}}_2 \mathbf{x}_t + \mathbf{T}_{p2ENU_b} \Delta \mathbf{x}_2. \quad (40)$$

Equation (40) can be tested by special cases as shown in Appendix D. Usually, TKE in ENU frame can be written easily, then (37) or (39) is substituted to it, and the analytical expression of TKE in MF can be obtained.

3. Quaternionic analysis of the TKM in MF

TKE is the mathematic expression for target spatial displacement. Usually in radar tracking field, TKE belongs to Newton's laws of motion which can be described as a polynomial with time as variable. TKEs as CV model ($n = 1$), CA model ($n = 2$) can be expressed by using summation

notation as: $\sum_{i=0}^n a_i t^i$. For two polynomials which have the same terms and term numbers, even the coefficients are different (because of different initial values), we call them belong to the same TKM.^{ix}

Proposition. If the distance of biased measurements of the same target at any two successive observation instances change little compared with the range, then the TKM in MF is the same as that in ENU frame.

Proof: For the sake of simplicity of analysis, a stationary 3-D radar and a target with CV model are considered in the paper. The flow chat for quaternionic analysis is shown in Figure 5. Assuming that TTCs in ENU frame (inertial coordinates system) at the observation instant k can be denoted by the vector $\mathbf{x}_t(k) = [x_t(k), y_t(k), z_t(k)]^\top$, then, the TKE can be written as

$$\mathbf{x}_t(k+1) = \mathbf{x}_t(k) + T\mathbf{v}_t(k) + \frac{T^2}{2}\mathbf{a}_t(k), \quad (41)$$

where $\mathbf{v}_t(k) = [v_{x_t}(k), v_{y_t}(k), v_{z_t}(k)]^\top$ denotes the true target velocity in ENU frame of the x , y , and z coordinates, respectively.

$\mathbf{a}_t(k) = [a_{x_t}(k), a_{y_t}(k), a_{z_t}(k)]^\top$ denotes the corresponding acceleration vector. Each acceleration is assumed to be zero-mean Gaussian white noise. T denotes the scanning period of radar.

According to (12a), the target coordinates under the influences of SBs can be written as

$$\mathbf{x}_b(k+1) = \mathbf{M}(k+1)\mathbf{x}_t(k+1) + \Delta\mathbf{x}(k+1), \quad (42a)$$

$$\mathbf{x}_t(k) = \mathbf{M}^{-1}(k)[\mathbf{x}_b(k) - \Delta\mathbf{x}(k)], \quad (42b)$$

where if we assume that the vector $\mathbf{m}(k)$ is given by

$$\mathbf{m}(k) = \begin{bmatrix} \sin\theta_b(k)\cos\varepsilon_b(k) \\ \cos\theta_b(k)\cos\varepsilon_b(k) \\ \sin\varepsilon_b(k) \end{bmatrix} \quad (43a)$$

then $\Delta\mathbf{x}(k)$ can be further written as

$$\Delta\mathbf{x}(k) = [k_r r_t(k) + \Delta r]\mathbf{m}(k) \quad (43b)$$

Substituting (41) into (42a), we have

$$\begin{aligned} \mathbf{x}_b(k+1) &= \mathbf{M}(k+1)\mathbf{M}^{-1}(k)\mathbf{x}_b(k) \\ &+ T\mathbf{M}(k+1)\mathbf{v}_t(k) + 0.5T^2\mathbf{M}(k+1)\mathbf{a}_t(k) \\ &+ \Delta\mathbf{x}(k+1) - \mathbf{M}(k+1)\mathbf{M}^{-1}(k)\Delta\mathbf{x}(k). \end{aligned} \quad (44)$$

Omitting the higher-order terms and minor error terms caused by SBs, (44) can be expanded and approximated as:

$$\mathbf{x}_b(k+1) \approx \mathbf{x}_b(k) + T\mathbf{v}_b(k) + \frac{T^2}{2}\mathbf{a}_b(k), \quad (45a)$$

where $\mathbf{v}, \mathbf{a}_b(k) = [v, a_{x_b}(k); v, a_{y_b}(k); v, a_{z_b}(k)]^\top$

$$\mathbf{v}, \mathbf{a}_b(k) = \mathbf{M}(k+1)\mathbf{v}, \mathbf{a}_t(k). \quad (45b)$$

The detailed derivations and analyses of (45a) can be seen in Appendix C. Though, in (45a), $\mathbf{v}_b(k)$ varies with $\theta_b(k+1)$, Equation (C15) and (C16) in Appendix C proves that $\mathbf{v}_b(k)$ approximate to be constant. And (45a) shows that the approximate TKM in MF is still a constant velocity model, which is the same with that in ENU frame. Since the proof of the *Proposition* is based on the target states of two successive observation instances, and has no special request for the TKM between the observation instances, this result can be extended to other kinematic forms such as CA or CT models etc.

It should be noted that (45a) is an equation, and the prerequisite for the approximation is that the target displacement between two successive observation instances is much smaller (an order of magnitude or more) than the target range. Fortunately, this prerequisite is usually satisfied.

According to the analyses above, the key factor determining the type of TKM in MF is RM. The existence of nonlinear function in terms of RM causes the changes of TKM in MF. If the terms of RM are all constants, it is obvious that TKM in MF is completely the same with that in ENU. In another word, with simpler form or the smaller magnitudes of nonlinear variables in RM terms, the smaller changes of TKM between MF and ENU. Comparing (12), (16), and (25), The *Proposition* can be applied well for stationary and the first kind of mobile radar, as for the second kind of mobile radar, only when the variables in (25) are small enough or change slowly, the *Proposition* can be applied.

Inference: KF can be used directly for the biased measurements just like the measurements only contain RNs. The estimate results of both situations are minimum mean square error (MMSE) estimations, however, the state estimation results of the biased measurements contain the influences of SBs.

In fact, the *inference* can be inferred from *proposition* directly without any proof. The following is also an exemplification of CV model.

When we analyze the estimation performances using KF, the biased measurements in this section mean that the measurements contain both SBs and random measurement noises. Taking CV model for example, the state vector in MF can be selected as $\mathbf{u}_b(k)$ and is given by

$$\mathbf{u}_b(k) = \begin{bmatrix} \mathbf{u}_{xb}(k) \\ \mathbf{u}_{yb}(k) \\ \mathbf{u}_{zb}(k) \end{bmatrix}, \quad \mathbf{u}_{ib}(k) = \begin{bmatrix} i_b(k) \\ v_{i_b}(k) \end{bmatrix}, \quad i \equiv \{x, y, z\} \quad (46)$$

According to the *proposition*, and omitting the minor terms, the state equation in MF can be obtained by rearranging (45a) as

$$\mathbf{u}_b(k+1) = \mathbf{F}\mathbf{u}_b(k) + \mathbf{G}\mathbf{a}_b(k), \quad (47a)$$

where

$$\mathbf{F} = \text{diag}([\mathbf{F}_1, \mathbf{F}_1, \mathbf{F}_1]), \quad \mathbf{G} = \text{diag}([\mathbf{G}_1, \mathbf{G}_1, \mathbf{G}_1]), \quad (47b)$$

$$\mathbf{F}_1 = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad \mathbf{G}_1 = \begin{bmatrix} 0.5T^2 \\ T \end{bmatrix}, \quad (47c)$$

where $\mathbf{B} = \text{diag}(\mathbf{A})$ denotes the elements of $\mathbf{A}$ are on the main diagonal of square matrix $\mathbf{B}$.

Since the “biased” target states include SBs, the measurement equations in MF can be written as

$$r(k) = \sqrt{x_b^2(k) + y_b^2(k) + z_b^2(k)} + \delta_r(k), \quad (48a)$$

$$\theta(k) = \text{atan}[x_b(k)/y_b(k)] + \delta_\theta(k), \quad (48b)$$

$$\varepsilon(k) = \text{atan}[z_b(k)/\sqrt{x_b^2(k) + y_b^2(k)}] + \delta_\varepsilon(k). \quad (48c)$$

Equations (47a) and (48a) ~ (48c) constitute the dynamic equations for the “biased” system, and they have the same form as bias free system. So, the state estimation method and correlation method for the “biased” single sensor system share the same theory and performances with no biases system.

When the “biased” state estimations are obtained, it is quite natural to compute TTCs and SBs by using (12a), however, it is obvious that the solutions are not unique. Combined with *proposition*, this result manifests that for single radar, SBs are dependent with the target states, they cannot be estimated separately, and only the “biased” state estimations can be obtained. In view of this, the successful registration methods usually use two radars’ simultaneous measurements of the same target to establish the registration equations where TTCs contained in both radars’ measurements can be eliminated and only SBs can be remained, so, the equations are observable.

4. Experimental results

In this section, three experiments will be presented to testify the *proposition* and the *inference*, respectively. For the first experiment, the *proposition* is exemplified by a typical scenario where we assume that a stationary 3-D radar and a mobile platform locates at the same place, the origin; a target is moving with constant speed and has no process noises (zero

accelerations).

The initial state of the target is [90 km, -80 m/s, 90 km, -30 m/s, 5 km, 1 m/s]. In the state vector, the variables denote x -coordinate, x -velocity, y -coordinate, y -velocity, z -coordinate, and z -velocity, respectively. Radar has no measurement noises and its OBs are $\Delta r = 300$ m, $k_r = 0.01$, and $\Delta\theta = \Delta\varepsilon = 2^\circ$, respectively. Figure 6 gives the errors between the approximate target kinematic model (45a) and the exact model (44) in measurement frame.

In the second experiment, the *inference* is tested in a simulated scenario where a common track is generated for two stationary 3-D radars (named A and B, respectively) which located at the same place, the origin. Both radars have the same performances, the standard deviations of radars are $\sigma_{ri} = 50$ m and $\sigma_{\theta i} = \sigma_{\varepsilon i} = 0.5^\circ$ ($i \in \{A, B\}$), respectively.

For each observation instance, both radars have the same measurement noises, the only difference is that radar B has the same OBs with those in experiment 1, and radar A has no biases at all. The target moves in the horizontal plane with three different models as CV (model 1), left/right CT (model 2/3). The initial state of the target is [70 km, -100 m/s, 50 km, 100m/s, 5 km, 0 m/s]. The standard deviations of the target are set to $\sigma_x = \sigma_y = 1$ m/s², $\sigma_z = 0.1$ m/s², respectively. Both radars are synchronous and the sampling interval is $T = 1$ s. Two hundred scans of the target are simulated and the number of Monte-Carlo runs is one hundred. For each run, the target moves to left CT from approximately 51 to 70s, and to the right CT from approximately 131 to 150s, and CV in the remaining time periods (see Figure 7(a)). The angular velocity of CT is $3^\circ/\text{s}$. The common initialization method and Interactive Multiple Model (IMM)^x are used for both radars. The model transition probability of IMM [43] is set to

$$\mathbf{P}_{ij} = \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.1 & 0.8 & 0.1 \\ 0.05 & 0.15 & 0.8 \end{bmatrix}.$$

The initial probability of the model is set to $\mathbf{u} = [0.3, 0.3, 0.4]$. The simulation results of experiment 2 are given in Figure 7. In order to testify the correctness of (25a), the results of a special cases as $\phi = \eta = \psi = \Delta\phi = \Delta\eta = \Delta\psi = 0^\circ$ for the second kind of mobile radar are given simultaneously for experiment 1 and 2 respectively.

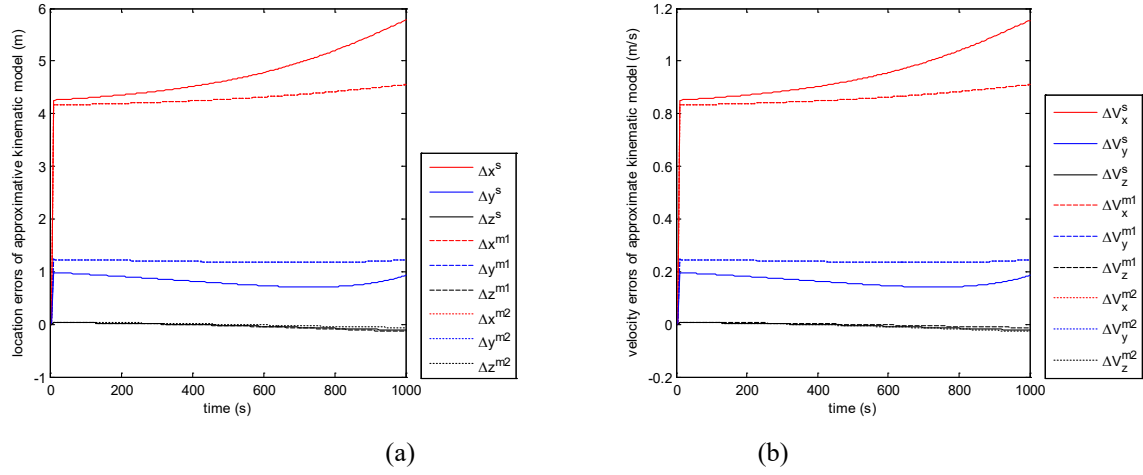


FIGURE 6: Errors of approximate kinematic model in measurement frame for stationary 3-D radar. (a) location errors; (b) velocity errors.

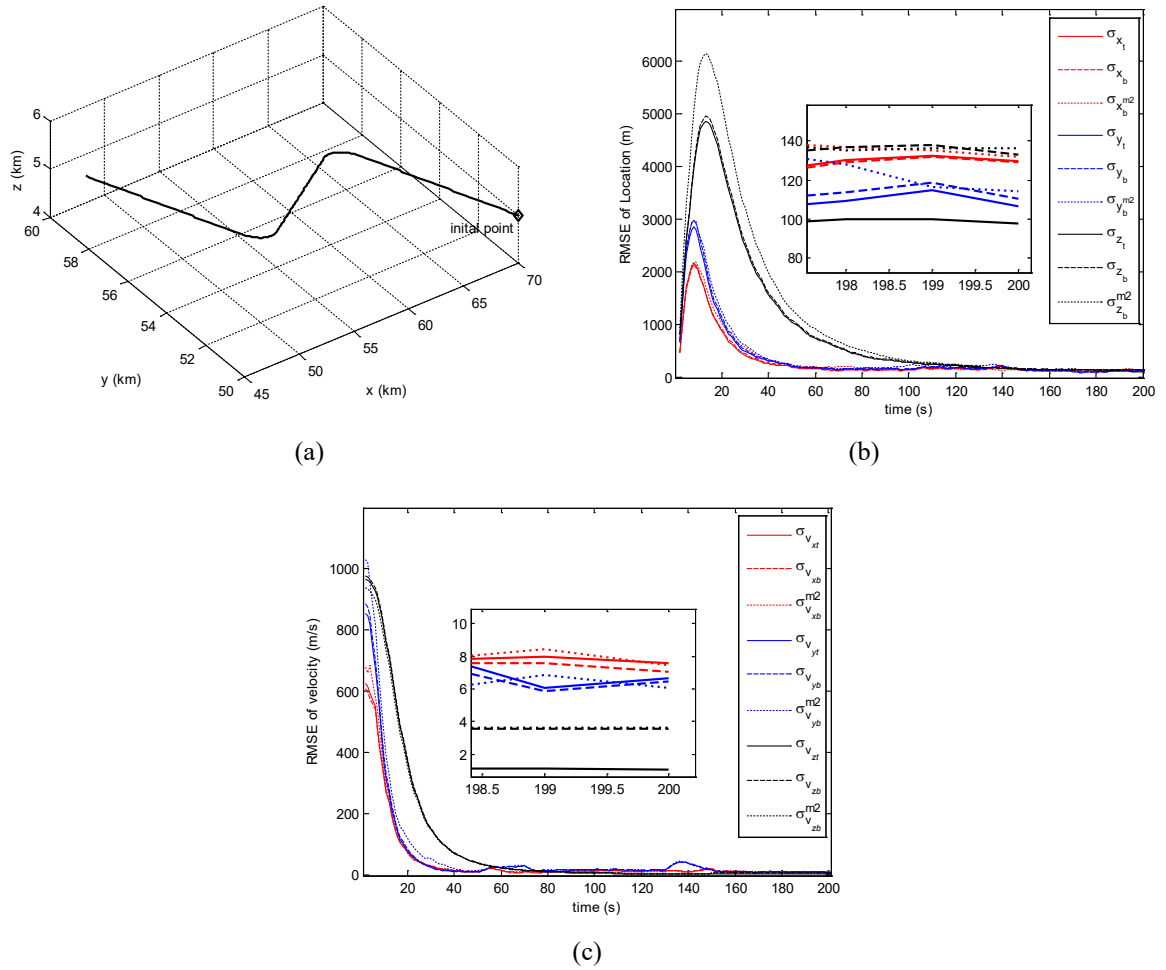


FIGURE 7: Performance comparison of the same filter for biased measurement (quaternionic analysis for stationary 3-D radar) and no bias measurement (complex analysis). (a) target track; (b) RMSE of location estimation; (c) RMSE of velocity estimation.

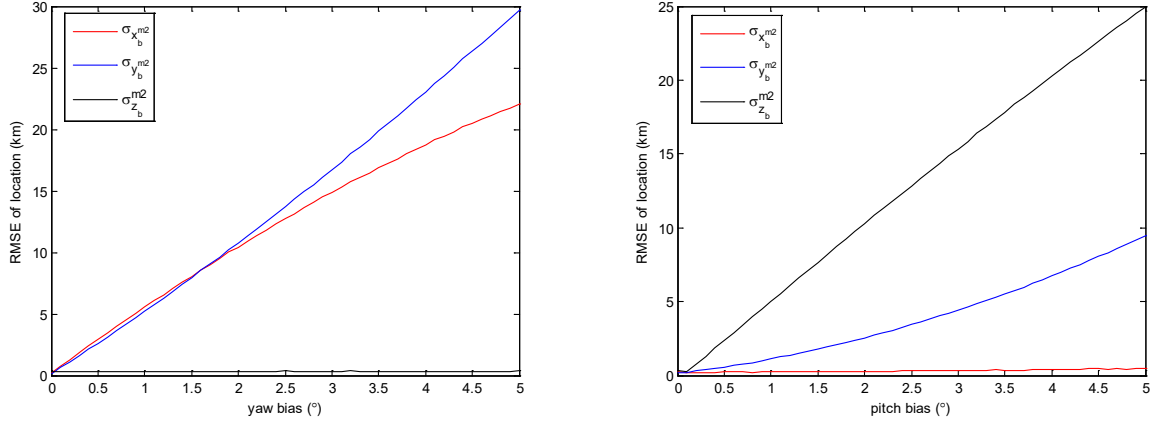


FIGURE 8: Second kind of mobile radar steady state RMSEs of location estimate versus varying attitude bias. (a) change yaw bias (b) change pitch bias.

The third experiment is based on the second experiment scenario, the difference is that radar belongs to the second kind of mobile radar. The platform attitudes make simple harmonic motions at origin with 20 s motion periods, all the magnitudes of attitude angles are same as 10° . The initial attitude angles are random generated. Two situations are tested respectively as: (1) $\Delta\phi = (0^\circ \sim 5^\circ)$, $\Delta\eta = \Delta\psi = 0^\circ$; (2) $\Delta\eta = (0^\circ \sim 5^\circ)$, $\Delta\phi = \Delta\psi = 0^\circ$. Figure 8 depicts the steady-state (at time instance 200 s) Root Mean Square Error (RMSE) curves versus varying attitude biases.

The location and velocity errors between (45a) and (44) are given in Figure 6(a) and Figure 6(b), respectively. The solid, dashed, and dotted lines represent stationary, first kind and second kind of mobile radars, respectively. For the second kind of mobile radar, the errors will be bigger with shorter platform attitude harmonic period. Simulation results manifest that the errors are small and the *proposition* can be applied well.

The RMSE of the target state estimations for biased and bias free measurements using the same filter are given in Figure 7. Figure 7(b) represents the location estimations and Figure 7(c) represents the velocity estimations. The solid lines (using subscript t in the legend) represent the RMSEs of no biased location estimation of radar A, and the dashed lines (using subscript b in the legend) represent the biased estimations of radar B.

In the calculations of the RMSE for the biased measurement, the “true” target states are the states obtained from (12). Figure 7 shows that the corresponding curves of radar A and B measurements are almost coinciding, which proves the *inference*.

Figure 8 manifests that for the second kind of mobile radar,

the usage of first-order approximation makes quaternion method have big errors. Furthermore, more than nine variables included in RM terms which complicate the analyses of TKM in MF, for length limitation, the detailed error analyses of this kind of mobile radar will be made in the future [45].

5. Conclusions

In this paper, it is proved that the biased measurements (contain radar systematic biases and random noises simultaneously) can be used directly as the inputs of Kalman filter to obtain the “biased” target track. And the “biased” track is in the sense of MMSE. Here, “biased” means that the estimates contain the influences of all the systematic biases. This result applies for any form of target kinematic model with the prerequisite that the target displacements between two successive observation instances are much smaller (an order of magnitude or more) compared with its range. Fortunately, this prerequisite is usually satisfied. The conclusion above provide theory basis for all the registration methods using local tracks. And also tell us that for single radar, all the data processing methods such as track initiation, association, and tracking *et al.* which are usually used for bias free sensors can be used in the same way for biased sensors just as no biases existing.

It should be noted that for the second kind mobile radar (rigidly connected and sways with platform), the first-order approximate rotation matrix has large errors and complicated nonlinear functions which makes it difficult to apply the proposed model without further investigation, modification, and analysis.

Another contribution of this paper is the introduction of quaternionic analysis to radar data processing field. The convenient angular rotation operation of quaternion is offer as a useful mathematic tool for radar data processing.

6. Acknowledgement

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Appendix A: Quaternion component rotation matrix $\mathbf{M}$ conversion from ENU to MF

In (10) the elements of the quaternion component RM $\mathbf{M}$ are computed.

The inverse of $\mathbf{M}$ can be written as

$$\mathbf{M}^{-1} = \begin{bmatrix} m_{11}^{-1} & m_{12}^{-1} & m_{13}^{-1} \\ m_{21}^{-1} & m_{22}^{-1} & m_{23}^{-1} \\ m_{31}^{-1} & m_{32}^{-1} & m_{33}^{-1} \end{bmatrix}, \quad (\text{A2a})$$

$$m_{11}^{-1} = \begin{vmatrix} \cos\Delta\theta\cos^2\theta_b + \cos\Delta\epsilon\cos\Delta\theta\sin^2\theta_b \\ +\cos\theta_b\sin\Delta\theta\sin\theta_b \\ -\cos\Delta\epsilon\cos\theta_b\sin\Delta\theta\sin\theta_b \end{vmatrix}, \quad (\text{A2b})$$

$$m_{12}^{-1} = \begin{vmatrix} -\sin\Delta\theta\sin^2\theta_b - \cos\Delta\epsilon\sin\Delta\theta\cos^2\theta_b \\ -\cos\Delta\theta\cos\theta_b\sin\theta_b \\ +\cos\Delta\epsilon\cos\Delta\theta\cos\theta_b\sin\theta_b \end{vmatrix}, \quad (\text{A2c})$$

$$m_{13}^{-1} = \sin\tilde{\theta}_b\sin\Delta\epsilon, \quad (\text{A2d})$$

$$m_{21}^{-1} = \begin{vmatrix} \sin\Delta\theta\cos^2\theta_b + \cos\Delta\epsilon\sin\Delta\theta \\ -\cos\Delta\epsilon\sin\Delta\theta\cos^2\theta_b \\ -\cos\theta_b\cos\Delta\theta\sin\theta_b \\ +\cos\Delta\epsilon\cos\Delta\theta\sin\theta_b\cos\theta_b \end{vmatrix}, \quad (\text{A2e})$$

$$m_{22}^{-1} = \begin{vmatrix} \cos\Delta\theta\sin^2\theta_b + \cos\Delta\epsilon\cos\Delta\theta\cos^2\theta_b \\ -\sin\Delta\theta\cos\theta_b\sin\theta_b \\ +\cos\Delta\epsilon\sin\Delta\theta\cos\theta_b\sin\theta_b \end{vmatrix}, \quad (\text{A2f})$$

$$m_{23}^{-1} = \cos\tilde{\theta}_b\sin\Delta\epsilon, \quad (\text{A2g})$$

$$m_{31}^{-1} = -\sin\theta_b\sin\Delta\epsilon, \quad (\text{A2h})$$

$$m_{32}^{-1} = -\cos\theta_b\sin\Delta\epsilon, \quad (\text{A2i})$$

$$m_{33}^{-1} = \cos\Delta\epsilon. \quad (\text{A2j})$$

Appendix B: Quaternion component rotation matrix conversion from PF to ENU

$\mathbf{T}_{p2ENU_b}$ in (35) can be written as

$$\mathbf{T}_{p2ENU_b} = \mathbf{T}_{ENU2p_b}^\dagger = \mathbf{T}_{\phi_b}^\dagger \mathbf{T}_{\eta_b}^\dagger \mathbf{T}_{\psi_b}^\dagger, \quad (\text{B1})$$

where

$$\mathbf{T}_{\phi_b}^\dagger = \begin{bmatrix} \cos(\phi_t + \Delta\phi) & \sin(\phi_t + \Delta\phi) & 0 \\ -\sin(\phi_t + \Delta\phi) & \cos(\phi_t + \Delta\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\text{B2})$$

$$\mathbf{T}_{\eta_b}^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\eta_t + \Delta\eta) & -\sin(\eta_t + \Delta\eta) \\ 0 & \sin(\eta_t + \Delta\eta) & \cos(\eta_t + \Delta\eta) \end{bmatrix}, \quad (\text{B3})$$

$$\mathbf{T}_{\psi_b}^\dagger = \begin{bmatrix} \cos(\psi_t + \Delta\psi) & 0 & -\sin(\psi_t + \Delta\psi) \\ 0 & 1 & 0 \\ \sin(\psi_t + \Delta\psi) & 0 & \cos(\psi_t + \Delta\psi) \end{bmatrix}. \quad (\text{B4})$$

Using the first-order Taylor series expansion about the true attitude angles, (B2)-(B4) can be approximated as

$$\mathbf{T}_{\phi_b}^\dagger \approx \mathbf{T}_{\phi_t}^\dagger + (\mathbf{T}_{\phi}^\dagger)'|_{\phi=\phi_t} \Delta\phi, \quad (\text{B5})$$

$$\begin{aligned} (\mathbf{T}_{\phi}^\dagger)'|_{\phi=\phi_t} &= \begin{bmatrix} -\sin\phi_t & \cos\phi_t & 0 \\ -\cos\phi_t & -\sin\phi_t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ &= \mathbf{T}_{\phi_t}^\dagger \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\text{B6}) \end{aligned}$$

$$\mathbf{T}_{\eta_b}^\dagger \approx \mathbf{T}_{\eta_t}^\dagger + (\mathbf{T}_{\eta}^\dagger)'|_{\eta=\eta_t} \Delta\eta, \quad (\text{B7})$$

$$\begin{aligned} (\mathbf{T}_{\eta}^\dagger)'|_{\eta=\eta_t} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\eta_t & -\cos\eta_t \\ 0 & \cos\eta_t & -\sin\eta_t \end{bmatrix} \\ &= \mathbf{T}_{\eta_t}^\dagger \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\text{B8}) \end{aligned}$$

$$\mathbf{T}_{\psi_b}^\dagger \approx \mathbf{T}_{\psi_t}^\dagger + (\mathbf{T}_{\psi}^\dagger)'|_{\psi=\psi_t} \Delta\psi, \quad (\text{B9})$$

$$\begin{aligned} (\mathbf{T}_{\psi}^\dagger)'|_{\psi=\psi_t} &= \begin{bmatrix} -\sin\psi_t & 0 & -\cos\psi_t \\ 0 & 0 & 0 \\ \cos\psi_t & 0 & -\sin\psi_t \end{bmatrix} \\ &= \mathbf{T}_{\psi_t}^\dagger \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (\text{B10}) \end{aligned}$$

Substituting (B5)-(B10) into (B1), and omitting the higher-order terms, we can obtain

$$\mathbf{T}_{p2ENU_b} = \left[\begin{array}{c} \mathbf{T}_{\eta_t}^\dagger \mathbf{T}_{\phi_t}^\dagger \mathbf{T}_{\psi_t}^\dagger + \mathbf{T}_{\phi_t}^\dagger (\mathbf{T}_{\eta_t}^\dagger)'|_{\eta=\eta_t} \mathbf{T}_{\psi_t}^\dagger \Delta\eta + (\mathbf{T}_{\phi_t}^\dagger)'|_{\phi=\phi_t} \mathbf{T}_{\eta_t}^\dagger \mathbf{T}_{\psi_t}^\dagger \Delta\phi + \mathbf{T}_{\phi_t}^\dagger \mathbf{T}_{\eta_t}^\dagger (\mathbf{T}_{\psi_t}^\dagger)'|_{\psi=\psi_t} \Delta\psi \\ \left(\mathbf{T}_{\phi_t}^\dagger + (\mathbf{T}_{\phi_t}^\dagger)'|_{\phi=\phi_t} \Delta\phi \right) \left(\mathbf{T}_{\eta_t}^\dagger + (\mathbf{T}_{\eta_t}^\dagger)'|_{\eta=\eta_t} \Delta\eta \right) \left(\mathbf{T}_{\psi_t}^\dagger + (\mathbf{T}_{\psi_t}^\dagger)'|_{\psi=\psi_t} \Delta\psi \right) = \mathbf{T}_{ENU2p_t}^{-1} \tilde{\mathbf{A}}. \end{array} \right. \quad (\text{B11})$$

Equation (B11) manifests the relational expression between the true RM and the biased RM from PF to the ENU frame.

Appendix C: Quaternionic analysis of the linearized biased TKM

Since the biases $\Delta\theta$ and $\Delta\varepsilon$ are usually on the order of 10^{-2} (rad) (e.g. $\Delta\theta = 2^\circ = 0.035$ rad), the first-order Maclaurin series expansion about them can be used to analyze the model.

In view of this, according to (10) and (A2a), we can obtain

$$\mathbf{M}(k) \approx \mathbf{I} + \underbrace{\left[\begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \Delta\theta + \left[\begin{array}{ccc} 0 & 0 & -\sin\theta_b(k) \\ 0 & 0 & -\cos\theta_b(k) \\ \sin\theta_b(k) & \cos\theta_b(k) & 0 \end{array} \right] \Delta\varepsilon}_{\Delta\mathbf{M}(k)}, \quad (\text{C1a})$$

$$\mathbf{M}^{-1}(k) \approx \mathbf{I} - \Delta\mathbf{M}(k), \quad (\text{C1b})$$

where $\mathbf{I}$ denotes the corresponding unit matrix. We find

$$\begin{aligned} & \mathbf{M}(k+1)\mathbf{M}^{-1}(k) \\ & \approx [\mathbf{I} + \Delta\mathbf{M}(k+1)][\mathbf{I} - \Delta\mathbf{M}(k)] \approx \mathbf{I} + \boldsymbol{\lambda}(k+1), \end{aligned} \quad (\text{C2})$$

where

$$\boldsymbol{\lambda}(k+1) = \left[\begin{array}{ccc} 0 & 0 & \lambda_1(k+1) \\ 0 & 0 & \lambda_2(k+1) \\ -\lambda_1(k+1) & -\lambda_2(k+1) & 0 \end{array} \right], \quad (\text{C3a})$$

$$\begin{aligned} \lambda_1(k+1) &= \{-\sin[\theta_b(k+1)] + \sin[\theta_b(k)]\} \Delta\varepsilon \\ \lambda_2(k+1) &= \{-\cos[\theta_b(k+1)] + \cos[\theta_b(k)]\} \Delta\varepsilon. \end{aligned} \quad (\text{C3b})$$

According to (42a),

$$\begin{aligned} & \mathbf{x}_b(k+1) - \mathbf{x}_b(k) \\ &= \mathbf{M}(k+1)\mathbf{x}_t(k+1) - \mathbf{M}(k)\mathbf{x}_t(k) \\ & \quad + \Delta\mathbf{x}(k+1) - \Delta\mathbf{x}(k) \\ &= \mathbf{M}(k+1)[\mathbf{x}_t(k+1) - \mathbf{M}^{-1}(k+1)\mathbf{M}(k)\mathbf{x}_t(k)] \\ & \quad + \Delta\mathbf{x}(k+1) - \Delta\mathbf{x}(k). \end{aligned} \quad (\text{C4})$$

Substituting (C1a)-(C3a) into (44), (C4) can be approximated as,

$$\begin{aligned} & \mathbf{x}_b(k+1) - \mathbf{x}_b(k) \\ & \approx \mathbf{M}(k+1)[\mathbf{x}_t(k+1) - (\mathbf{I} - \boldsymbol{\lambda}(k+1))\mathbf{x}_t(k)] \\ & \quad + \Delta\mathbf{x}(k+1) - \Delta\mathbf{x}(k) \\ &= \mathbf{M}(k+1)[\mathbf{x}_t(k+1) - \mathbf{x}_t(k) + \boldsymbol{\lambda}(k+1)\mathbf{x}_t(k)] \\ & \quad + \Delta\mathbf{x}(k+1) - \Delta\mathbf{x}(k) \\ &= \mathbf{M}(k+1)[T\mathbf{v}_t(k) + 0.5T^2\mathbf{a}_t(k)] + \\ & \quad + \mathbf{M}(k+1)\boldsymbol{\lambda}(k+1)\mathbf{x}_t(k) + \Delta\mathbf{x}(k+1) - \Delta\mathbf{x}(k). \end{aligned} \quad (\text{C5})$$

If the approximate kinematic equation (C5) has the same form with (41) which is the true TKE in ENU, or (C5) can be approximated to (41), then the *proposition* can be proved.

Let

$$\boldsymbol{\chi}(k) = \boldsymbol{\lambda}(k+1)\mathbf{x}_t(k). \quad (\text{C6})$$

Substituting (C3a) into (C6), each term of the vector $\boldsymbol{\chi}(k)$ can be written respectively as

$$\begin{aligned} \chi_x(k) &= \Delta\varepsilon \left[\frac{x_b(k)}{r_b(k) \cos[\varepsilon_b(k)]} - \frac{x_b(k+1)}{r_b(k+1) \cos[\varepsilon_b(k+1)]} \right] z_t(k), \\ |\chi_x(k)| &\ll |x_b(k+1) - x_b(k)| \end{aligned} \quad (\text{C7})$$

$$\begin{aligned} \chi_y(k) &= \Delta\varepsilon \left[\frac{y_b(k)}{r_b(k) \cos[\varepsilon_b(k)]} - \frac{y_b(k+1)}{r_b(k+1) \cos[\varepsilon_b(k+1)]} \right] z_t(k), \\ |\chi_y(k)| &\ll |y_b(k+1) - y_b(k)|, \end{aligned} \quad (\text{C8})$$

$$\begin{aligned} \chi_{zx}(k) &= \Delta\varepsilon \left[\frac{x_b(k+1)}{r_b(k+1) \cos \varepsilon_b(k+1)} - \frac{x_b(k)}{r_b(k) \cos \varepsilon_b(k)} \right] x_t(k) \\ \chi_{zy}(k) &= \Delta\varepsilon \left[\frac{y_b(k+1)}{r_b(k+1) \cos[\varepsilon_b(k+1)]} - \frac{y_b(k)}{r_b(k) \cos[\varepsilon_b(k)]} \right] y_t(k) \\ \chi_z(k) &= \chi_{zx}(k) + \chi_{zy}(k) \end{aligned}$$

$$|\chi_z(k)| \ll |x_b(k+1) - x_b(k)| + |y_b(k+1) - y_b(k)|. \quad (\text{C9})$$

Since $\mathbf{M}(k+1)$ is a unit RM, then, in (C5)

$$\mathbf{M}(k+1)\boldsymbol{\lambda}(k+1)\mathbf{x}_t(k) \ll \mathbf{x}_b(k+1) - \mathbf{x}_b(k). \quad (\text{C10})$$

Here $\mathbf{A} \ll \mathbf{B}$ denotes that the elements in vector $\mathbf{A}$ are far less than the corresponding elements in vector $\mathbf{B}$.

Similar, according to (38b)

$$\begin{aligned} \Delta\mathbf{x}(k+1) - \Delta\mathbf{x}(k) &= k_r[r_t(k+1)\mathbf{m}(k+1) - r_t(k)\mathbf{m}(k)] \\ & \quad + \Delta r[\mathbf{m}(k+1) - \mathbf{m}(k)] \\ & \ll \mathbf{x}_b(k+1) - \mathbf{x}_b(k). \end{aligned} \quad (\text{C11})$$

According to (C10) and (C11), (C4) can be written approximately as,

$$\mathbf{x}_b(k+1) \approx \mathbf{x}_b(k) + T\mathbf{v}_b(k) + 0.5T^2\mathbf{a}_b(k), \quad (\text{C12})$$

where

$$\mathbf{v}, \mathbf{a}_b(k) = \mathbf{M}(k+1)\mathbf{v}, \mathbf{a}_t(k), \quad (\text{C13})$$

Equation (C12) has the same form with (41), which proves the *proposition*.

In order to prove the *proposition* in another perspective, we will prove below that the biased velocity variables keep invariant. In view of this, according to (C13) and (C1a), the variation of the “biased” velocity between two successive observation instances can be written as

$$\Delta\mathbf{v}_b(k) = \mathbf{v}_b(k) - \mathbf{v}_b(k-1)$$

$$= \mathbf{v}_t(k) - \mathbf{v}_t(k-1) + \Delta \mathbf{M}(k+1) \mathbf{v}_t(k) - \Delta \mathbf{M}(k) \mathbf{v}_t(k-1). \quad (\text{C14})$$

Similarly to (C6)~(C9), we have

$$\Delta \mathbf{M}(k+1) \mathbf{v}_t(k) - \Delta \mathbf{M}(k) \mathbf{v}_t(k-1) \ll \mathbf{v}_t(k) - \mathbf{v}_t(k-1). \quad (\text{C15})$$

The result of $\mathbf{v}_t(k) - \mathbf{v}_t(k-1)$ of the right hand side of (C14) is the variation of the “biased” velocity caused by the “biased” acceleration. And the result of $\Delta \mathbf{M}(k+1) \mathbf{v}_t(k) - \Delta \mathbf{M}(k) \mathbf{v}_t(k-1)$ is caused by SBs which is “abnormal and redundant” for the variation of the velocity, however, it is small in magnitude and can be omitted. This manifests that the influences of SBs on the target velocities almost keep invariant and the “biased” TKM is approximately a CV model.

According to (10) and (A2a), the elements of the quaternion RM $\mathbf{M}$ and its inverse matrix are both infinitely differentiable functions. Since $\Delta \theta$ and $\Delta \varepsilon$ are usually on the order of 10^{-2} [rad], the Taylor series for these elements are convergent, and their second-order remainders are much smaller than their corresponding first-order terms. According to theorem of Taylor's mean, for the first-order approximation model, the effects of SBs on the variations of the target velocities are very small and can be omitted, as for the even smaller higher-order terms, their effects can also be omitted or viewed as minor process noises.

In all, though the “biased” target locations, velocities, and accelerations are different from the true situation, when the measurements of the target at any two successive observation instances change small compared with the range, the TKM in MF approximates to the same as that in the ENU frame.

Appendix D: Special cases of quaternionic analysis

In (25) if we set $\eta_t = \psi_t = \phi_t = 0$ and $\theta_t \neq 0$ we have

$$\tilde{m}_{11} = \tilde{m}_{22} = \tilde{m}_{33} = 1 \quad (\text{D1a})$$

$$\tilde{m}_{12} = \Delta \theta - \Delta \phi \quad (\text{D1b})$$

$$\tilde{m}_{13} = \Delta \psi - \Delta \varepsilon \sin \theta_t \quad (\text{D1c})$$

$$\tilde{m}_{23} = \Delta \eta - \Delta \varepsilon \cos \phi_t \quad (\text{D1d})$$

It appears that this result is the same as (15)

$$\mathbf{M}_{q1} \approx \begin{bmatrix} 1 & \Delta \theta - \Delta \phi & \Delta \psi - \sin \theta_t \Delta \varepsilon \\ \Delta \phi - \Delta \theta & 1 & \Delta \eta - \cos \theta_t \Delta \varepsilon \\ \sin \theta_t \Delta \varepsilon - \Delta \psi & \cos \theta_t \Delta \varepsilon - \Delta \eta & 1 \end{bmatrix} \quad (\text{D2})$$

According to Appendix A, in (10) $\mathbf{M}$ can be first-order approximated as $\tilde{\mathbf{M}}$:

$$\tilde{m}_{11} = \sin \tilde{\theta}_b \sin \theta_b [\cos \Delta \varepsilon - 1] + \cos \Delta \theta \approx 1, \quad (\text{D3a})$$

$$\tilde{m}_{12} = \sin \theta_b \cos \Delta \varepsilon \cos \tilde{\theta}_b - \cos \theta_b \sin \tilde{\theta}_b \approx \sin \theta_b \cos \tilde{\theta}_b - \cos \theta_b \sin \tilde{\theta}_b = \sin \Delta \theta \approx \Delta \theta, \quad (\text{D3b})$$

$$\tilde{m}_{13} = -\sin \theta_b \sin \Delta \varepsilon \approx -\sin \theta_t \Delta \varepsilon, \quad (\text{D3c})$$

$$m_{21} = \cos \Delta \varepsilon \cos \theta_b \sin \tilde{\theta}_b - \sin \theta_b \cos \tilde{\theta}_b \approx \cos \theta_b \sin \tilde{\theta}_b - \sin \theta_b \cos \tilde{\theta}_b = -\sin \Delta \theta \approx -\Delta \theta, \quad (\text{D3d})$$

$$\tilde{m}_{22} = \cos \theta_b \cos \Delta \varepsilon \cos \tilde{\theta}_b + \sin \theta_b \sin \tilde{\theta}_b \approx \cos \theta_b \cos \tilde{\theta}_b + \sin \theta_b \sin \tilde{\theta}_b = \cos \Delta \theta \approx 1, \quad (\text{D3e})$$

$$\tilde{m}_{23} = -\cos \theta_b \sin \Delta \varepsilon \approx -\cos \theta_t \Delta \varepsilon, \quad (\text{D3f})$$

$$\tilde{m}_{31} = \sin \tilde{\theta}_b \sin \Delta \varepsilon \approx \sin \tilde{\theta}_b \Delta \varepsilon, \quad (\text{D3g})$$

$$\tilde{m}_{32} = \cos \tilde{\theta}_b \sin \Delta \varepsilon \approx \cos \tilde{\theta}_b \Delta \varepsilon, \quad (\text{D3h})$$

$$\tilde{m}_{33} = \cos \Delta \varepsilon \approx 1, \quad (\text{D3i})$$

In (D2), if we set $\Delta \phi = \Delta \eta = \Delta \psi = 0$ we have

$$\mathbf{M}_{q1} \approx \tilde{\mathbf{M}} \quad (\text{D4})$$

For special cases, we have (D2) and (D4), which exemplifies the correctness of (15) and (25).

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ⁱ This conclusion is right for this special case. Though the physical processes of the measurements are little different, the formulas of both situations are the same because the sequence of attitude biases and offset biases can be changed. In fact, this has already been proven. Only the special situation when attitude angles are zeroes, this equivalence can be accepted. Otherwise, the sequence cannot be changed. This is also the reason why the second kind of mobile radar has different results from the first kind. In rectangular coordinates, though the offset biases are the same, different target coordinates have different errors in x, y, and z directions, respectively such as the more pronounced differences of target coordinates, the larger the differences of errors in x, y, and z directions.

ⁱⁱ Equation (2) can be also written by means of complex numbers. A complex number is a number that can be expressed in the form $a + bi$, where a and

b are real numbers and i is the imaginary unit, that satisfies the equation $x^2 = -1$, that is, $i^2 = -1$. In this expression, a is the real part and b is the imaginary part of the complex number. A feature of complex numbers is that multiplication of two complex numbers is commutative. (For more information, please visit [10])

ⁱⁱⁱ In mathematics, the quaternions are a number system that extends the complex numbers; i.e., for the quaternion $a + bi + cj + dk$ the following holds $i^2 = j^2 = k^2 = ijk = -1$. They were first described by Irish mathematician William Rowan Hamilton in 1843 and applied to mechanics in three-dimensional space. (For more information, please visit [11]).

^{iv} Psiaki [21], Um et al. [23], Bachmann et al. [27], Elkaim [29], Gebre-Egziabher et al. [28], Shin et al. [31], Tome et al. [34], Hirokawa et al. [35], Jun [36] use $\otimes$ to denote quaternion product or spectrum convolution Soloviev

et al. [32]. We believe it is incorrect to use $\otimes$ to denote quaternion product or multiplication or spectrum convolution as this symbol is used to denote Kronecker product [46] or Tensor product [12]. Shin [17] (which is the same author Shin et al. [31] which used $\otimes$ in [31]) uses $\star$ to denote the quaternion product which is a symbol used for Hodge star operator [47] by most scholars. Stuart [36] uses just $\cdot$ product to denote quaternion product which is a symbol used for vector dot product. Bosse et al. [44] uses $\oplus$ to denote a quaternion product. This is the symbol used for N-ary circled plus operator or Kronecker sum [46]. Moreover, most references do not even mention the word Hamilton product [11], with the exception of one reference which uses the word Grassman [42]. While we do not know why this an ongoing issue in the literature because other authors do not even use any symbol at all or might use other symbols, one thing is clear that most of authors are not even clear as to the correct name and use of the quaternion product or multiplication operator. However, we believe that both the symbol $\ast_{\mathbb{H}}$ [12] and the word Hamilton [11] might be the correct use to denote quaternion product or multiplication because $\times$ is used for vector product and quaternion is a generalized form of a vector; hence, $\ast_{\mathbb{H}} \equiv \cdot + \times_{\mathbb{H}}$ will denote a generalized form of vector addition, dot product, and vector product that of quaternion or Hamilton product or multiplication.

^v A feature of quaternions is that multiplication of two quaternions is noncommutative (or anti-commutative) in contrast to the multiplication of two complex numbers which is commutative; however, both are associative and left/right distributive over addition. Hamilton defined a quaternion as the quotient of two directed lines in a three-dimensional space or equivalently as the quotient of two vectors (For more information please visit [11]).

^{vi} Hamilton product is a generalized form of a vector product because in (8b) if we assume $q_{\Delta\epsilon 0} = q_{\Delta\theta 0} = 0$ and $\vec{q}_{\Delta\epsilon 12}$ orthogonal with $\vec{q}_{\Delta\theta 3}$ or $\vec{q}_{\Delta\epsilon 12} \vec{q}_{\Delta\theta 3} = 0$ then the Hamilton product is equal to the vector product $\vec{q} = \vec{q}_{\Delta\epsilon 12} \times \vec{q}_{\Delta\theta 3}$ or $\vec{q} = \vec{q}_{\Delta\epsilon 12} \ast_{\mathbb{H}} \vec{q}_{\Delta\theta 3} \equiv \vec{q}_{\Delta\epsilon 12} \times \vec{q}_{\Delta\theta 3}$ or the vector product is a special case of the Hamilton product which is another reason why it is more appropriate to use $\cdot + \times_{\mathbb{H}}$ to denote a Hamilton product. Furthermore if we also assume that $q_{\Delta\epsilon 0} = q_{\Delta\theta 0} \equiv q_{\Delta}$ and $\vec{q}_{\Delta\epsilon 12} = \vec{q}_{\Delta\theta 3} \equiv$

$\vec{q}_{\Delta}$ then $\vec{q} = q_{\Delta\epsilon 0} q_{\Delta\theta 0} - \vec{q}_{\Delta\epsilon 12} \cdot \vec{q}_{\Delta\theta 3} + q_{\Delta\epsilon 0} \vec{q}_{\Delta\theta 3} + q_{\Delta\theta 0} \vec{q}_{\Delta\epsilon 12} + \vec{q}_{\Delta\epsilon 12} \times \vec{q}_{\Delta\theta 3} = 2\vec{q}_{\Delta}$. This shows that the Hamilton product of a quaternion with itself is twice that quaternion hence Hamilton product is also a generalized form of a vector sum. Furthermore, if we assume that $q_{\Delta\epsilon 0} = q_{\Delta\theta 0} = 0$ and $\vec{q}_{\Delta\epsilon 12}$ co-linear with $\vec{q}_{\Delta\theta 3}$ then $\vec{q} = q_{\Delta\epsilon 0} q_{\Delta\theta 0} - \vec{q}_{\Delta\epsilon 12} \cdot \vec{q}_{\Delta\theta 3} + q_{\Delta\epsilon 0} \vec{q}_{\Delta\theta 3} + q_{\Delta\theta 0} \vec{q}_{\Delta\epsilon 12} + \vec{q}_{\Delta\epsilon 12} \times \vec{q}_{\Delta\theta 3} = -\vec{q}_{\Delta\epsilon 12} \cdot \vec{q}_{\Delta\theta 3}$; hence, the Hamilton product is a generalized form of a dot product. Hence, a Hamilton product symbol should be able to use all these symbols $\ast_{\mathbb{H}}$ and nothing more or less. Until that symbol is created, we will propose to use $\ast_{\mathbb{H}}$ to denote a Hamilton product instead.

^{vii} Since no transition exist and rotations are about rectangular axes here, this rotation matrix can be called as quaternion components matrix even it is not obtained by quaternion method. However, traditionally, it is better to call this as a rotation matrix.

^{viii} One might use a more elegant, compact form via tensor notation as discussed in "Historical impact on physics" paragraph in [11]. A quaternion can be represented as the sum of a scalar and a vector. It can also be represented as the product of its tensor and its versor. Each quaternion has a tensor, which is a measure of its magnitude (in the same way as the length of a vector is a measure of a vector's magnitude). When a quaternion is defined as the quotient of two vectors, its tensor is the ratio of the lengths of these vectors. A versor is a quaternion with a tensor of 1. Alternatively, a versor can be defined as the quotient of two equal-length vectors [13].

^{ix} The noncommutativity of multiplication has some unexpected consequences, among them that polynomial equations over the quaternions can have more distinct solutions than the degree of the polynomial [11].

^x The IMM is an estimator which can either be used by multiple hypothesis tracker (MHT) or Joint Probabilistic Data Association Filter (JPDAF). IMM uses two or more Kalman filters which run in parallel, each using a different model for target motion or errors. The IMM forms an optimal weighted sum of the output of all the filters and is able to rapidly adjust to target maneuvers. While MHT or JPDAF handles the association and track maintenance, an IMM helps MHT or JPDAF in obtaining a filtered estimate of the target position.